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Common coupled fixed point theorems in fuzzy metric spaces

Mahmoud Bousselsal [1,2], **Anwar Jabor Fawakhreh** [2,*]

[1] *Laboratoire d'Analyse Nonlineaire et H.M. Dept. de Maths, Ecole Normale Suprieure, 16050, Vieux-Kouba, Algiers, Algeria*
[2] *Department of Mathematics, College of Science, Qassim University, P.O. Box 6644, Buraydah 51402, Saudi Arabia*
**Corresponding author E-mail: abuanas7@hotmail.com*

Abstract

In this work, we prove the existence and uniqueness of a common fixed point for self-maps in M-complete fuzzy metric spaces and we apply these results on maps satisfying a contractive condition of an integral type.

Keywords: *M-Cauchy sequence, M-complete fuzzy metric space, coupled fixed point of a mapping.*

1 Introduction

In 1965, the concept of fuzzy sets was introduced by Zadeh [17]. Since then many authors have expansively developed the theory of fuzzy sets and applications. In 1975, Kramosil and Michalek [9] first introduced the concept of a fuzzy metric space, which can be regarded as a generalization of the statistical (probabilistic) metric space and it provides an important basis for the construction of fixed point theory in fuzzy metric spaces. After that, Wenzhi [16] and many others initiated the study of probabilistic 2-metric spaces which is a real valued function of a point triples on a set X, whose abstract properties were suggested by the area function in Euclidean spaces.

Afterwards, Grabiec [7] defined the completeness of the fuzzy metric space or what is known as a G-complete fuzzy metric space in [8], and extended the Banach contraction theorem to G-complete fuzzy metric spaces. Following Grabiec's work, Fang [3] further established some new fixed point theorems for contractive type mappings in G-complete fuzzy metric spaces. Soon after, Mishra et al. [10] also obtained several common fixed point theorems for asymptotically commuting maps in the same space, which generalize several fixed point theorems in metric, fuzzy, Menger and uniform spaces. Besides these works based on the G-complete fuzzy metric space, George and Veeramani [5] modified the definition of the Cauchy sequence introduced by Grabiec [7] because even R is not complete with Grabiec's completeness definition. George and Veeramani [5] slightly modified the notion of a fuzzy metric space introduced by Kramosil and Michalek [9] and then defined a Hausdorff and first countable topology on this fuzzy metric space which has important applications in quantum particle physics in connection with string and E-infinity theory.

Since then, the notion of a complete fuzzy metric space presented by George and Veeramani [5], which is now known as an M-complete fuzzy metric space (as in [15]) has emerged as another characterization of completeness, and some fixed point theorems have also been constructed on the basis of this metric space. Recently, Fang [4] gave some common fixed point theorems under ϕ-contractions for compatible and weakly compatible mappings in Menger probabilistic metric spaces. Moreover, Rao et al. [11] have proved two unique common coupled fixed point theorems for self maps in symmetric G-fuzzy metric spaces. Recently, Shen et al. [14] have proposed a new class of self-maps by altering the distance between two points in fuzzy environment, in which the φ-function was used, and on the basis of this kind of self-map, they have proved some fixed point theorems in M-complete fuzzy metric spaces and compact fuzzy metric spaces. From the above analysis, we can see that there are many studies related to fixed point theory based on the above two kinds of complete fuzzy metric spaces, namely: G-complete and M-complete fuzzy metric spaces. Note that every G-complete fuzzy metric space is M-complete; and the construction of fixed point theorems in M-complete fuzzy metric spaces seems to be more valuable.

The purpose of this work is to propose a new class of self-maps by using a φ-function. More importantly, we prove the existence and uniqueness of a common fixed point for these self-maps in M-complete fuzzy metric spaces and we apply these results on maps satisfying a contractive condition of an integral type.

2 Preliminaries

We begin with some basic concepts on fuzzy metric spaces.

Definition 2.1 *([12], [5]). A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is called a* continuous t-norm *if it satisfies the following conditions:*
(TN-1) $*$ *is commutative and associative;*
(TN-2) $*$ *is continuous;*
(TN-3) $a * 1 = a$ *for every* $a \in [0,1]$*;*
(TN-4) $a * b \leq c * d$ *whenever* $a \leq c, b \leq d$ *and* $a, b, c, d \in [0,1]$.

Definition 2.2 *([5]). A* fuzzy metric space *is an ordered triple* $(X, M, *)$ *such that* X *is a nonempty set,* $*$ *is a continuous* $t-norm$ *and* M *is a fuzzy set on* $X \times X \times (0,\infty)$ *satisfying the following conditions, for all* $x, y, z \in X, s, t > 0$*:*
(FM-1) $M(x,y,t) > 0$*;*
(FM-2) $M(x,y,t) = 1$ *if and only if* $x = y$*;*
(FM-3) $M(x,y,t) = M(y,x,t)$*;*
(FM-4) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$*;*
(FM-5) $M(x,y,\cdot) : (0,\infty) \to (0,1]$ *is continuous.*

Note that $M(x,y,t)$ denotes the degree of nearness between x and y with respect to t.

Definition 2.3 *Let* $(X, M, *)$ *be a fuzzy metric space. Then:*
(i) A sequence $\{x_n\}$ *in* X *is said to be convergent ([5], [7]) to a point* x *in* X*, denoted by* $lim_{n\to\infty} x_n = x$ *(or* $x_n \to x$*), if and only if* $lim_{n\to\infty} M(x_n, x, t) = 1$ *for all* $t > 0$*, i.e. for each* $r \in (0,1)$ *and* $t > 0$*, there exists* $n_0 \in N$ *such that* $M(x_n, x, t) > 1 - r$ *for all* $n \geq n_0$.
(ii) A sequence $\{x_n\}$ *in* X *is called a Cauchy sequence [7] if and only if*
$lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ *for all* $t > 0$ *and* $p > 0$.
(iii) A sequence $\{x_n\}$ *in* X *is called an M-Cauchy sequence ([5], [8]) if and only if for each* $\epsilon \in (0,1), t > 0$*, there exists* $n_0 \in N$ *such that* $M(x_m, x_n, t) > 1 - \epsilon$ *for any* $m, n > n_0$.
(iv) The fuzzy metric space $(X, M, *)$ *is called complete ([5], [7]) if every Cauchy sequence is convergent.*
(v) The fuzzy metric space $(X, M, *)$ *is called M-complete ([5], [8]) if every M-Cauchy sequence is convergent.*

Lemma 2.4 *([7]). For all* $x, y \in X$*,* $M(x, y, \cdot)$ *is nondecreasing.*

Remark 2.5 *Since* $*$ *is continuous, it follows from (FM-4) that the limit of a sequence in a fuzzy metric space is uniquely determined.*

Definition 2.6 *([13]). A function* M *is continuous in fuzzy metric spaces if whenever* $x_n \to x, y_n \to y$*, then* $lim_{n\to\infty} M(x_n, y_n, t) = M(x, y, t)$ *for all* $t > 0$.

Lemma 2.7 *([6]). Let* $M(x, y, *)$ *be a fuzzy metric space. Then* M *is a continuous function on* $X \times X \times (0,\infty)$.

Definition 2.8 *Let* X *be a nonempty set. An element* $(x, y) \in X \times X$ *is called*
(i) a coupled coincidence point [1] of mappings $F : X \times X \longrightarrow X$ *and* $g : X \longrightarrow X$ *if* $gx = F(x,y)$ *and* $gy = F(y,x)$.
(ii) a coupled fixed point [2] of the mapping $F : X \times X \longrightarrow X$ *if* $x = F(x,y)$ *and* $y = F(y,x)$.
(iii) a common coupled fixed point [4] of mappings $F : X \times X \longrightarrow X$ *and* $g : X \longrightarrow X$ *if* $x = gx = F(x,y)$ *and* $y = gy = F(y,x)$.

Definition 2.9 *Let* X *be a nonempty set. An element* $x \in X$ *is called a common fixed point [4] of mappings* $F : X \times X \longrightarrow X$ *and* $g : X \longrightarrow X$ *if* $x = gx = F(x,x)$.

Definition 2.10 *([4]). Let* X *be a nonempty set. The mappings* $F : X \times X \longrightarrow X$ *and* $g : X \longrightarrow X$ *are called commutative if* $g(F(x,y)) = F(gx, gy)$ *for all* $y, x \in X$.

3 Main results

In this section, we will establish common (coupled) fixed point theorems for a mapping $F : X \times X \longrightarrow X$ of an M-complete fuzzy metric space. In these metric spaces, a function $\varphi : [0,1] \to [0,1]$ which is used by altering the distance between two points satisfies the following properties:
(P1) φ is strictly decreasing and left continuous;
(P2) $\varphi(\lambda) = 0$ if and only if $\lambda = 1$.
Obviously, $lim_{\lambda \to 1^-} \varphi(\lambda) = \varphi(1) = 0$.

Theorem 3.1 *Let $(X, M, *)$ be an M-complete fuzzy metric space (With $a * b = min\{a, b\}$ for all $a, b \in [0,1]$). Let $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ be two functions such that*

$$\varphi\big(M(F(x,y), F(u,v), t)\big) \le k(t) \cdot \varphi\big(M(gx, gu, t) * M(gy, gv, t)\big) \tag{1}$$

for all $t > 0$ and for all $(x,y), (u,v) \in X \times X$ and $(x,y) \neq (u,v)$ where $k : (0, +\infty) \longrightarrow [0,1)$ and $\varphi : [0,1] \longrightarrow [0,1]$ satisfy the foregoing properties: (P1) and (P2), $F(X \times X) \subset g(X)$ and g is continuous and commutative with F. Then there exists a unique common fixed point $x \in X$ of the mappings F and g such that $x = gx = F(x,x)$.

Proof. Let x_0, y_0 be two arbitrary points of X. Since $F(X \times X) \subset g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Again, from $F(X \times X) \subset g(X)$, we can choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that, for all $n \in N$

$$gx_{n+1} = F(x_n, y_n) \text{ and } gy_{n+1} = F(y_n, x_n). \tag{2}$$

Now, let

$$\tau_n(t) = M(gx_n, gx_{n+1}, t),$$
$$\theta_n(t) = M(gy_n, gy_{n+1}, t),$$

and

$$\delta_n(t) = \tau_n(t) * \theta_n(t)$$

for all $n \in N \cup \{0\}$ and $t > 0$.
Then we have two cases:

Case 1. If there exists $n_0 \in N \cup \{0\}$ such that $\tau_{n_0}(t) = \theta_{n_0}(t) = 1$, that is, $gx_{n_0} = gx_{n_0+1}$ and $gy_{n_0} = gy_{n_0+1}$. Then $gx_{n_0} = F(x_{n_0}, y_{n_0})$ and $gy_{n_0} = F(y_{n_0}, x_{n_0})$, then it follows that (x_{n_0}, y_{n_0}) is a coupled coincidence point of F and g.

Case 2. For any $n \in N \cup \{0\}$, $0 < \tau_n(t) < 1$ or $0 < \theta_n(t) < 1$, that is, $gx_n \neq gx_{n+1}$ or $gy_n \neq gy_{n+1}$. Then we might take the following three cases:

(i) If $gx_n \neq gx_{n+1}$and $gy_n = gy_{n+1}$, then $\theta_n(t) = 1$. Using (1), we obtain

$$\varphi\big(\tau_n(t)\big) = \varphi\big(M(gx_n, gx_{n+1}, t)\big) = \varphi\big(M\big(F(x_{n-1}, y_{n-1}), F(x_n, y_n), t\big)\big)$$
$$\le k(t) \cdot \varphi\big(M(gx_{n-1}, gx_n, t) * M(gy_{n-1}, gy_n, t)\big) \le k(t) \cdot \varphi\big(\delta_{n-1}(t)\big) < \varphi\big(\delta_{n-1}(t)\big).$$

Since φ is strictly decreasing, we have, $\tau_n(t) > \delta_{n-1}(t)$ for all $t > 0$. Since $\theta_n(t) = 1$, $\delta_n(t) > \delta_{n-1}(t)$ for all $t > 0$.

(ii) If $gx_n = gx_{n+1}$ and $gy_n \neq gy_{n+1}$, then we have also $\delta_n(t) > \delta_{n-1}(t)$ for all $t > 0$, as in (i) above.

(iii) If $gx_n = gx_{n+1}$ and $gy_n \neq gy_{n+1}$, that is $0 < \tau_n(t) < 1$ and $0 < \theta_n(t) < 1$. Using similar ways as in (i) and (ii) above, we obtain that:
$\tau_n(t) > \delta_{n-1}(t)$ and $\theta_n(t) > \delta_{n-1}(t)$ for all $t > 0$. Thus, $\tau_n(t) * \theta_n(t) > \delta_{n-1}(t)$ for all $t > 0$. Hence, $\delta_n(t) > \delta_{n-1}(t)$ for all $t > 0$.

Now in the three cases (i), (ii) and (iii) above and since $\delta_n(t)$ is bounded and increasing, it converges to some $\delta(t)$, and we write

$$lim_{n \to +\infty} \delta_n(t) = \delta(t) \tag{3}$$

Suppose that $\delta(t) \in (0,1)$. Since $\theta_n(t) \in (0,1]$ and $\tau_n(t) \in (0,1]$ are bounded and increasing, there exist subsequences $\{\theta_{n_k}(t)\}$ and $\{\tau_{n_k}(t)\}$ of $\{\theta_n(t)\}$ and $\{\tau_n(t)\}$ respectively, such that $\theta_{n_k}(t) \to \theta(t)$ and $\tau_{n_k}(t) \to \tau(t)$ as $k \to \infty$.

By the continuity of the operation $*$, we obtain: $\tau_{n_k}(t) * \theta_{n_k}(t) \to \tau(t) * \theta(t)$ as $k \to \infty$, and since $\delta_{n_k}(t) \to \delta(t)$, by the uniqueness of the limit, we obtain that $\tau(t) * \theta(t) = \delta(t)$.

Note that $\varphi\big(\tau_{n_{k+1}}(t)\big) \le k(t) \cdot \varphi\big(\delta_{n_k}(t)\big)$ and by taking the limit as $k \to \infty$, we get $\varphi\big(\tau(t)\big) \le k(t) \cdot \varphi\big(\delta(t)\big) < \varphi\big(\delta(t)\big)$. Hence $\tau(t) > \delta(t)$ and with the same way we also obtain that $\theta(t) > \delta(t)$. Thus $\tau(t) * \theta(t) > \delta(t)$. Hence $\delta(t) > \delta(t)$, which is a contradiction. Therefore, in all cases above we have

$$\delta(t) \equiv 1 \ \text{ and } \ lim_{n\to\infty}\delta_n(t) = 1. \tag{4}$$

Thus by the uniqueness of the limit, $\tau_n(t) = \theta_n(t) = 1$.

Next, we show that the sequences $\{gx_n\}$ and $\{gy_n\}$ are M-Cauchy sequences. Suppose, on the contrary, that at least one of $\{gx_n\}$ or $\{gy_n\}$ is not an M-Cauchy sequence, then there exists $\epsilon \in (0,1)$ and subsequences $\{gx_{p(n)}\}$, $\{gx_{q(n)}\}$ of $\{gx_n\}$ and $\{gy_{p(n)}\}$, $\{gy_{q(n)}\}$ of $\{gy_n\}$ with $p(n) > q(n) \ge n$ and

$$M\big(gx_{p(n)}, gx_{q(n)}, t\big) * M\big(gy_{p(n)}, gy_{q(n)}, t\big) \le 1-\epsilon, \ \text{ for all } \ t > 0. \tag{5}$$

Furthermore, corresponding to $q(n)$, we can choose $p(n)$ in such away that it is the smallest integer with $p(n) > q(n) \ge n$ and

$$M\big(gx_{p(n)-1}, gx_{q(n)-1}, t\big) * M\big(gy_{p(n)-1}, gy_{q(n)-1}, t\big) > 1-\epsilon, \ \text{ for all } t > 0, \tag{6}$$

and

$$M\big(gx_{p(n)}, gx_{q(n)-1}, t\big) * M\big(gy_{p(n)}, gy_{q(n)-1}, t\big) > 1-\epsilon, \ \text{ for all } \ t > 0. \tag{7}$$

For each $n \in N \cup \{0\}$, let

$$\delta_{n(t)} = M\big(gx_{p(n)}, gx_{q(n)}, t\big) * M\big(gy_{p(n)}, gy_{q(n)}, t\big). \tag{8}$$

From (3) and (FM-4), we obtain that

$$1-\epsilon \ge \delta_n(t) \ge M\big(gx_{p(n)}, gx_{q(n)-1}, \frac{t}{2}\big) * M\big(gx_{q(n)-1}, gx_{q(n)}, \frac{t}{2}\big) *$$

$$M\big(gy_{p(n)}, gy_{q(n)-1}, \frac{t}{2}\big) * M\big(gy_{q(n)-1}, gy_{q(n)}, \frac{t}{2}\big) \ge \delta_{q(n)-1}(\frac{t}{2}) * (1-\epsilon).$$

Since $lim_{n\to\infty}\delta_{q(n)-1}(\frac{t}{2}) = 1$, for every $t > 0$, then by taking the limit as $n \to \infty$ it follows that

$$lim_{n\to\infty}\delta_n(t) = 1-\epsilon, \ \text{ for every } \ t > 0. \tag{9}$$

Moreover, by (1),

$$\begin{aligned}\varphi\big(M(gx_{p(n)}, gx_{q(n)}, t)\big) &= \varphi\big(M\big(F(gx_{p(n)-1}, gy_{p(n)-1}), F(gx_{q(n)-1}, gy_{q(n)-1}), t\big)\big) \\ &\le k(t) \cdot \varphi\big(M(gx_{p(n)-1}, gx_{q(n)-1}, t) * M(gy_{p(n)-1}, gy_{q(n)-1}, t)\big) \\ &< \varphi\big(M(gx_{p(n)-1}, gx_{q(n)-1}, t) * M(gy_{p(n)-1}, gy_{q(n)-1}, t)\big).\end{aligned}$$

By monotonicity of φ, we obtain that

$$M\big(gx_{p(n)}, gx_{q(n)}, t\big) > M\big(gx_{p(n)-1}, gx_{q(n)-1}, t\big) * M\big(gy_{p(n)-1}, gy_{q(n)-1}, t\big).$$

Similarly,

$$M\big(gy_{p(n)}, gy_{q(n)}, t\big) > M\big(gx_{p(n)-1}, gx_{q(n)-1}, t\big) * M\big(gy_{p(n)-1}, gy_{q(n)-1}, t\big).$$

Thus

$$M\big(gx_{p(n)}, gx_{q(n)}, t\big) * M\big(gy_{p(n)}, gy_{q(n)}, t\big) >$$
$$M\big(gx_{p(n)-1}, gx_{q(n)-1}, t\big) * M\big(gy_{p(n)-1}, gy_{q(n)-1}, t\big).$$

Therefore, by (5), (6), (8) and (9) we obtain that

$$1-\epsilon \geq \delta_n(t) > M\big(gx_{p(n)-1}, gx_{q(n)-1}, t\big) * M\big(gy_{p(n)-1}, gy_{q(n)-1}, t\big) > 1-\epsilon,$$

for every $t > 0$. Which leads to a contradiction.

In particular, for each $\epsilon \in (0,1)$ there exists $n_0 \in N \cup \{0\}$ such that

$$M\big(gx_m, gx_n, t\big) * M\big(gy_m, gy_n, t\big) \leq 1-\epsilon, \qquad \text{for all} \;\; m,n \geq n_0.$$

Obviously, for any $p \in N$, with $m = n_0 + p$ and $n = n_0 + p + 1$ we obtain,

$$M\big(gx_{n_0+p}, gx_{n_0+p+1}, t\big) * M\big(gy_{n_0+p}, gy_{n_0+p+1}, t\big) \leq 1-\epsilon.$$

Thus the sequence $\{\delta_{n_0+p}(t)\}_{p\geq 1}$ is monotone and bounded with respect to p and $0 < \delta_{n_0+p}(t) \leq 1-\epsilon$. Thus $lim_{p\to\infty}\delta_{n_0+p}(t) = \delta(t) \leq 1-\epsilon$, for all $t > 0$. So $0 < \delta(t) \leq 1-\epsilon$. But by (4), $\delta(t) = 1$, therefore $0 < 1 \leq 1-\epsilon$, which is a contradiction since $\epsilon > 0$

Hence, the sequences $\{gx_n\}$ and $\{gy_n\}$ are M-Cauchy sequences in the M-complete fuzzy metric space X. Therefore, we conclude that there exist points $x, y \in X$ such that

$$lim_{n\to\infty} gx_n = x \text{ and } lim_{n\to\infty} gy_n = y. \tag{10}$$

From (10) and continuity of g, $lim_{n\to\infty} ggx_n = gx$ and $lim_{n\to\infty} ggy_n = gy$.

But $ggx_{n+1} = gF(x_n, y_n) = F(gx_n, gy_n)$ and $ggy_{n+1} = gF(y_n, x_n) = F(gy_n, gx_n)$, since g commutes with F.

Now we show that $gx = F(x,y)$ and $gy = F(y,x)$, that is (x,y) is a coupled coincidence point of F and g. Note that

$$\varphi\Big(M\big(ggx_{n+1}, F(x,y), t\big)\Big) = \varphi\Big(M\big(gF(x_n,y_n), F(x,y), t\big)\Big)$$

$$= \varphi\Big(M\big(F(gx_n, gy_n), F(x,y), t\big)\Big) \leq k(t)\cdot\varphi\Big(M\big(gx_n, x, t\big) * M\big(gy_n, y, t\big)\Big).$$

Again, by the monotonicity of φ, we obtain that

$$M\big(ggx_{n+1}, F(x,y), t\big) > M\big(gx_n, x, t\big) * M\big(gy_n, y, t\big), \qquad as \;\; n \to \infty.$$

By taking the limit as $n \to \infty$, we obtain that

$$M\big(gx, F(x,y), t\big) \geq M\big(x,x,t\big) * M\big(y,y,t\big) \geq 1.$$

So $gx = F(x,y)$, and with the same way we get $gy = F(y,x)$.
Now, we show that $gx = y$ and $gy = x$:
I) If $gx = gy_n$, by the uniqueness of the limit and since $lim_{n\to\infty} gy_n = y$, then we have $gx = y$.
II) We assume that $gx \neq gy_n$, hence by inequality (1) above,

$$\begin{aligned}\varphi\big(M(gx, gy_n, t)\big) &= \varphi\big(M(F(x,y), F(y_{n-1}, x_{n-1}), t)\big)\\ &\leq k(t)\cdot\varphi\big(M(gx, gy_{n-1}, t) * M(gy, gx_{n-1}, t)\big)\\ &< \varphi\big(M(gx, gy_{n-1}, t) * M(gy, gx_{n-1}, t)\big),\end{aligned} \tag{11}$$

for all $t > 0$.
Then by the monotonicity of φ, we get

$$M(gx, gy_n, t) > M(gx, gy_{n-1}, t) * M(gy, gx_{n-1}, t). \tag{12}$$

In the same way, we obtain

$$M(gy, gx_n, t) > M(gx, gy_{n-1}, t) * M(gy, gx_{n-1}, t). \tag{13}$$

From (12) and (13), it follows that

$$M(gx, gy_n, t) * M(gy, gx_n, t) > M(gx, gy_{n-1}, t) * M(gy, gx_{n-1}, t). \tag{14}$$

Now, let $\beta_n(t) = M(gx, gy_n, t) * M(gy, gx_n, t)$, then the sequence $\{\beta_n(t)\}$ is increasing and bounded, thus there exists $\beta(t) \in (0, 1]$ such that
$lim_{n\to\infty}\beta_n(t) = \beta(t)$. Since M and $*$ are continuous, then by the uniqueness of the limit, we get $\beta(t) = M(gx, y, t) * M(gy, x, t)$.
If $\beta(t) \in (0, 1)$, then from (11) above, we obtain that

$$\varphi\big(M(gx, gy_n, t)\big) \leq k(t) \cdot \varphi\big(M(gx, gy_{n-1}, t) * M(gy, gx_{n-1}, t)\big).$$

By taking the limit, as $n \to \infty$, we get

$$\varphi\big(M(gx, y, t)\big) \leq k(t) \cdot \varphi\big(M(gx, y, t) * M(gy, x, t)\big)$$
$$< \varphi\big(M(gx, y, t) * M(gy, x, t)\big). \tag{15}$$

Which implies that,

$$M(gx, y, t) > M(gx, y, t) * M(gy, x, t). \tag{16}$$

In the same way, we obtain that

$$M(gy, x, t) > M(gx, y, t) * M(gy, x, t). \tag{17}$$

From (16) and (17), we get

$$M(gx, y, t) * M(gy, x, t) > M(gx, y, t) * M(gy, x, t).$$

That is $\beta(t) > \beta(t)$, which is a contradiction. Therefore, $lim_{n\to\infty}\beta_n(t) = 1$, and by the uniqueness of the limit, we obtain $M(gx, y, t) * M(gy, x, t) = 1$, thus $M(gx, y, t) = 1$ and $M(gy, x, t) = 1$. That is $gx = y$ and $gy = x$. But, $gx = F(x, y)$ and $gy = F(y, x)$, so $y = gx = F(x, y)$ and $x = gy = F(y, x)$.
Finally, we prove that $x = y$. If, on the contrary $x \neq y$, then

$$\varphi\big(M(x, y, t)\big) = \varphi\big(M(F(x, y), F(y, x), t)\big)$$
$$\leq k(t) \cdot \varphi\big(M(x, y, t) * M(y, x, t)\big) \leq \varphi\big(M(x, y, t)\big)$$

for all $t > 0$. Since φ is strictly decreasing, $M(x, y, t) > M(x, y, t)$, which is a contradiction. Thus, $x = y$. Therefore, $x = gx = F(x, x)$, that is x is a common fixed point of the mappings F and g.

To prove the uniqueness of the common fixed point x of F and g, suppose that $z \neq x$ is another common fixed point of F and g. Then

$$M(z, x, t) = M\big(F(z, x), F(x, z), t\big) = M\big(gz, gx, t\big).$$

Since $z \neq x$, we have

$$\varphi\big(M(z, x, t)\big) = \varphi\big(M(F(z, x), F(x, z), t)\big) \leq k(t) \cdot \varphi\big(M(gz, gx, t) * M(gx, gz, t)\big)$$
$$< \varphi\big(M(gz, gx, t) * M(gx, gz, t)\big) = \varphi\big(M(z, x, t) * M(x, z, t)\big) < \varphi\big(M(x, z, t)\big),$$

for all $t > 0$. Since φ is strictly decreasing, $M(z, x, t) > M(z, x, t)$, which is a contradiction, thus $x = z$. Therefore, F and g have a unique common fixed point $x \in X$.

The next theorem is similar to Theorem 3.1 above, but with the operation $a * b = ab$.

Theorem 3.2 *Let $(X, M, *)$ be an M-complete fuzzy metric space (With $a * b = ab$ for all $a, b \in [0, 1]$). Let $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ be two functions such that*

$$\varphi\big(M(F(x, y), F(u, v), t)\big) \leq k(t) \cdot \varphi\big(\sqrt{M(gx, gu, t)} * \sqrt{M(gy, gv, t)}\big), \tag{18}$$

for all $t > 0$ and for all $(x, y), (u, v) \in X \times X$ and $(x, y) \neq (u, v)$ where $k : (0, +\infty) \longrightarrow (0, 1)$ and $\varphi : [0, 1] \longrightarrow [0, 1]$ satisfy the foregoing properties: (P1) and (P2) above, $F(X \times X) \subset g(X)$ and g is continuous and commutative with F. Then there exists a unique common fixed point $x \in X$ such that $x = gx = F(x, x)$.

Proof. The proof of Theorem 3.2 is similar to the proof of Theorem 3.1 above.

Later, from the previous obtained results, we deduce some coupled fixed point results for mapping satisfying a contraction of an integral type, as an application of Theorem 3.1 above. For this purpose, let $Y = \{\chi : [0, 1] \to [0, 1]$, where χ is a summable and nonpositive Lebesgue integrable mapping satisfies $\int_{1-\epsilon}^{1} \chi(t)dt > 0$ for each $0 < \epsilon < 1\}$.

Theorem 3.3 *Let $(X, M, *)$ be an M-complete fuzzy metric space (With $a * b = min\{a, b\}$ for all $a, b \in [0, 1]$). Let $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ be two functions such that*

$$\int_{1-\varphi\left(M(F(x,y),F(u,v),t)\right)}^{1} \chi(s)ds \leq k(t) \int_{1-\varphi\left(M(gx,gu,t)*M(gy,gv,t)\right)}^{1} \chi(s)ds, \tag{19}$$

for all $t > 0$ and for all $(x, y), (u, v) \in X \times X$ and $(x, y) \neq (u, v)$ where $k : (0, +\infty) \longrightarrow [0, 1)$ and $\varphi : [0, 1] \longrightarrow [0, 1]$ satisfy the foregoing properties: (P1) and (P2), $F(X \times X) \subset g(X)$ and g is continuous and commutative with F. Then there exists a unique common fixed point $x \in X$ of the mappings F and g such that $x = gx = F(x, x)$.

Proof. For $\chi \in Y$, consider the function $\wedge : [0, 1] \to [0, 1]$ defined by $\wedge(\epsilon) = \int_{1-\epsilon}^{1} \chi(s)ds$. We note that $\wedge$ is continuous, $\wedge(0) = 0$ and $\wedge$ is strictly increasing. Thus the Inequality (19) becomes:

$$\wedge\Bigg(\varphi\big(M(F(x,y),F(u,v),t)\big)\Bigg) \leq k(t) \wedge \Bigg(\varphi\big(M(gx,gu,t) * M(gy,gv,t)\big)\Bigg).$$

Setting $\varphi_1 = \wedge \circ \varphi$, we note that φ_1 is strictly decreasing and left continuous and

$$\varphi_1(\lambda) = 0 \Leftrightarrow \wedge\big(\varphi(\lambda)\big) = 0 \Leftrightarrow \varphi(\lambda) = 0 \Leftrightarrow \lambda = 1.$$

Thus $\varphi_1 : [0, 1] \to [0, 1]$ satisfies the forgoing properties (P_1) and (P_2). Therefore, by Theorem 3.1 above, there exists a unique common fixed point $x \in X$ of the mappings F and g such that $x = gx = F(x, x)$.

4 Conclusion

In this work, we proposed a new class of self-maps by altering the distance between two points in fuzzy metric spaces. On this kind of self-map, we proved the existence and uniqueness of a common fixed point in M-complete fuzzy metric spaces and we applied the results on maps satisfying a contractive condition of an integral type.

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