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The variational homotopy perturbation method for solving the K(2,2)equations

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Abstract

This paper deals with implementation of the variational homotopy pertubation method (VHPM) for solving the K(2,2) compacton equation. The numerical results show that the approach is easy to implement and accurate when it is compared with the exact solution. The suggested algorithm is quite efficient and is practically well suited for use in the nonlinear problems. The fact that the proposed technique solves nonlinear problems without using the Adomian's polynomials can be considered as a clear advantage of this algorithm over the decomposition method.

Keywords: *variational homotopy, perturbation method, variational iteration method.*

1 Introduction

Nonlinear differential equations are encountered in various fields in physics, chemistry, biology, mathematics and engineering. For example, Burgers' equation is used to describe various kinds of phenomena such as turbulence and the approximation theory of flow through a shock wave traveling in a viscous fluid [2]. Numerical methods which are commonly used such as finite difference [4], finite element or characteristics method need large size of computational works and usually the effect of round-off error causes loss of accuracy in the results. Most nonlinear models of real-life problems are still very difficult to solve either numerically or theoretically.

In recent years, several methods have drawn particular attention, such as the Adomian decomposition method [1], the variational iteration method [5], the homotopy analysis method [13], and the homotopy perturbation method [6, 7, 8, 9].

In this paper we consider the following nonlinear dispersive $K(m, n)$ equation:

$$u_t + a(u^m)_x + (u^n)_{xxx} = 0 \tag{1}$$

developed in [10] for describing the compacton ($m > 0,\ 1 < n \leq 3$) which is a compact wave that preserves its shape after the interaction with another compact wave.

In the case $m = n = 1$, this equation becomes

$$u_t + (u^2)_x + (u^2)_{xxx} = 0, \tag{2}$$

has

$$u(x, t) = (4c/3)\cos^2((x - ct)/4) \tag{3}$$

as exact solution and it is developed in [4] for describing the compacton solution. While taking $u(x, 0) = (4/3)\cos^2(x/4)$, and considering Eq. (2), it derives the initial value problem

$$\begin{cases} u_t + (2u + 6u_{xx})u_x + 2uu_{xxx} = 0, & t > 0 \\ u(x, 0) = (4/3)\cos^2(x/4) \end{cases}. \tag{4}$$

In what follows, the variational iteration method is modified by introducing a transformation such that the solution is expressed by the series approximation. Precisely, we couple the classical variational iteration method with He's polynomials [7, 9] and construct a new homotopy to solve (4). Our modification proposed in Section 4 extends the variational iteration method with He's polynomials. This modification provides an accurate approximation for the $K(2,2)$ equation. This implies that our method provides a new idea of the variational iteration method with He's polynomials for finding an approximation of the nonlinear differential equations. Observing the numerical results, and comparing our approximation with the exact solution, the proposed method reveals to be very close to the exact solution. The details of the comparison results are displayed in Table 1 and Table 2 in Section 5.

2 Variational iteration method

To illustrate the basic concepts of the VIM, we consider the following differential equation

$$L(u(x,t)) + N(u(x,t) = g(x,t) \quad , \tag{5}$$

where L is a linear operator, N is a nonlinear operator and $g(x,t)$ is an inhomogeneous term. Then we can construct a correction functional as follows

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \{L(u(x,\tau)) + N(\tilde{u}(x,\tau)) - g(x,\tau)\} d\tau \tag{6}$$

where λ is a general Lagrange multiplier, which can be identified optimally via variational theory. The second term on the right hand side is called the correction and is considered as a restricted variation, i.e., $\delta\tilde{u}_n = 0$. By this method, it is required first to determine the Lagrangian multiplier λ that will be identified optimally. The successive approximations $u_{n+1}(x,t)$, $n \geq 0$ of the solution $u(x,t)$ will be readily obtained upon using the determined Lagrangian multiplier and any selective function $u_0(x,t)$. Consequently, the solution is given by

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{7}$$

3 Homotopy perturbation method

To illustrate the basic ideas of the HPM, we consider the following nonlinear differential equation

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{8}$$

with the boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{9}$$

where A, is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω.
Generally speaking, the operator A can be divided into two parts L and N where L is the linear part, and N the nonlinear part. Therefore Eq. (8) can be rewritten as

$$L(u) + N(u) - f(r) = 0. \tag{10}$$

By the homotopy perturbation technique, we construct a homotopy $v(r,p) : \Omega \times [0,1] \to R$ which satisfies

$$H(v,p) = (1-p)\left[L(v) - L(u_0)\right] + p\left[A(v) - f(r)\right] = 0, \tag{11}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p\left[N(v) - f(r)\right] = 0, \tag{12}$$

where $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation of equation (8) which satisfies the boundary conditions. Considering equation (12) we will have

$$H(v,0) = L(v) - L(u_0) = 0 \tag{13}$$

and

$$H(v,1) = A(v) - f(r) = 0. \tag{14}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology this is called deformation and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopy. According to the homotopy perturbation theory, we can first use the embedding parameter p as a small parameter and assume that the solution of equation (11) can be written as a power series in p

$$v = v_0 + pv_1 + p^2 v_2 + \dots \tag{15}$$

Setting $p = 1$ one have the approximation solution of equation (8) as the following

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \dots \tag{16}$$

The convergence of series (16) is discussed in [3].

4 Variational homotopy perturbation method

In the homotopy perturbation method [11], the basic assumption is that the solutions can be written as a power series in p

$$u = \sum_{i=0}^{+\infty} p^i u_i = u_0 + pu_1 + p^2 u_2 + \dots \tag{17}$$

To illustrate the concept of the variational homotopy perturbation method [14] [matinfar2] we consider the general differential equation (5). We construct the correction functional (6) and apply the homotopy perturbation method to obtain [12, 14].

$$\sum_{r=1}^{+\infty} p^i u_i(x,t) = u_0(x,t) + p\int_0^t \lambda \left\{ N(\sum_{r=1}^{+\infty} p^i u_i(x,\tau)) - g(x,\tau) \right\} d\tau. \tag{18}$$

As we see, the procedure is formulated by the coupling of variational iteration method and homotopy perturbation method. A comparison of like powers of p gives solutions of various orders.

5 Numerical results

In this section we will examine the nonlinear dispersive equation $K(2,2)$ defined in Eq. (2) and expressed in the form of the initial value problem (4). We apply the VHPM developed in Section 4, construct the correction functional and calculate the Lagrange multipliers optimally via variational theory.
The correction functional for (4) reads

$$u_{n+1} = u_n + \int_0^t \lambda(\tau) \left\{ (u_n)_\tau + ((\tilde{u}_n)^2)_x + ((\tilde{u}_n)^2)_{xxx} \right\} d\tau, \tag{19}$$

or

$$u_{n+1} = u_n + \int_0^t \lambda(\tau) \left\{ \frac{\partial u_n}{\partial \tau} + \left(2u_n + 6\frac{\partial^2 u_n}{\partial x^2}\right) \frac{\partial u_n}{\partial x} + 2u_n \frac{\partial^3 u_n}{\partial x^3} \right\} d\tau \tag{20}$$

and which yields the stationary conditions

$$\left\{ \begin{array}{l} \lambda = 0 \\ \lambda + 1 = 0\,. \end{array} \right\} \tag{21}$$

Therefore, the general Lagrange multiplier can be readily identified as $\lambda = -1$.

Substituting this value of the Lagrangian multiplier into functional (19) or its equivalent equation (20) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left\{ \begin{array}{l} \frac{\partial u_n(x,\tau)}{\partial \tau} + \left(2u_n(x,\tau) + 6\frac{\partial^2 u_n(x,\tau)}{\partial x^2}\right) \frac{\partial u_n(x,\tau)}{\partial x} \\ +2u_n(x,\tau)\frac{\partial^3 u_n(x,\tau)}{\partial x^3} \end{array} \right\} d\tau \quad . \tag{22}$$

Applying the variational homotopy perturbation method, one obtains

$$\sum_{i=0}^{+\infty} p^i u_i(x,t) = u_0(x,t) - p\int_0^t \left\{ \begin{array}{l} \frac{\partial \sum_{i=0}^{+\infty} p^i u_i(x,\tau)}{\partial \tau} + \left(\begin{array}{c} 2\sum_{i=0}^{+\infty} p^i u_i(x,\tau) + \\ 6\frac{\partial^2 \sum_{i=0}^{+\infty} p^i u_i(x,\tau)}{\partial x^2} \end{array} \right) \frac{\partial \sum_{i=0}^{+\infty} p^i u_i(x,\tau)}{\partial x} \\ +2\sum_{i=0}^{+\infty} p^i u_i(x,\tau) \frac{\partial^3 \sum_{i=0}^{+\infty} p^i u_i(x,\tau)}{\partial x^3} \end{array} \right\} d\tau. \tag{23}$$

Comparing the coefficient of like powers of p we obtain the following set of linear partial differential equations

$$u_1(x,t) = -\int_0^t \left\{ 2u_0(x,\tau)\frac{\partial^3 u_0(x,\tau)}{\partial x^3} + \left(2u_0(x,\tau) + 6\frac{\partial^2 u_0(x,\tau)}{\partial x^2}\right)\frac{\partial u_0(x,\tau)}{\partial x} \right\} d\tau \tag{24}$$

$$u_2(x,t) = -\int_0^t \left\{ \begin{array}{l} 6\frac{\partial^2 u_0(x,\tau)}{\partial x^2}\frac{\partial u_1(x,\tau)}{\partial x} + 2\frac{\partial^3 u_0(x,\tau)}{\partial x^3} u_1(x,\tau) \\ +6\frac{\partial u_0(x,\tau)}{\partial x}\frac{\partial^2 u_1(x,\tau)}{\partial x^2} + 2u_0(x,\tau)\frac{\partial^3 u_1(x,\tau)}{\partial x^3} \\ +2u_0(x,\tau)\frac{\partial u_1(x,\tau)}{\partial x} + 2\frac{\partial u_0(x,\tau)}{\partial x} u_1(x,\tau) \end{array} \right\} d\tau \tag{25}$$

$$u_3(x,t) = -\int_0^t \left\{ \begin{array}{l} 6\frac{\partial^2 u_0(x,\tau)}{\partial x^2}\frac{\partial u_2(x,\tau)}{\partial x} + 2u_1(x,\tau)\frac{\partial^3 u_1(x,\tau)}{\partial x^3} \\ +2u_0(x,\tau)\frac{\partial u_2(x,\tau)}{\partial x} + 2u_1(x,\tau)\frac{\partial u_1(x,\tau)}{\partial x} \\ +2u_2(x,\tau)\frac{\partial^3 u_0(x,\tau)}{\partial x^3} + 2u_2(x,\tau)\frac{\partial u_0(x,\tau)}{\partial x} \\ +6\frac{\partial u_0(x,\tau)}{\partial x}\frac{\partial^2 u_2(x,\tau)}{\partial x^2} + 2u_0(x,\tau)\frac{\partial^3 u_2(x,\tau)}{\partial x^3} \\ +6\frac{\partial u_1(x,\tau)}{\partial x}\frac{\partial^2 u_1(x,\tau)}{\partial x^2} \end{array} \right\} d\tau \tag{26}$$

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and so on, in the same manner the rest of components can be obtained using the Maple package.

Consequently, while taking the initial value $u(x,0) = \frac{4}{3}\cos^2(\frac{x}{4})$, and according to Eqs. (24)-(26), the first few components of the variational homotopy perturbation solution for Eq. (19) are derived as follows

$$u_0(x,t) = u(x,0) = \tfrac{4}{3}\cos^2(\tfrac{x}{4})$$

$$u_1(x,t) = \tfrac{2}{3}\cos(\tfrac{x}{4})\sin(\tfrac{x}{4})t$$

$$u_2(x,t) = \tfrac{-1}{12}\left(-1 + 2\cos^2(\tfrac{x}{4})\right)t^2$$

$$u_3(x,t) = \tfrac{-1}{36}\left(\cos(\tfrac{x}{4})\sin(\tfrac{x}{4})\right)t^3$$

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The other components of the VHPM can be determined in a similar way. Finally, the approximate solution of Eq. (19) in a series form is

$$u(x,t) \simeq u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots \tag{27}$$

Consequently, the third-order term approximate solution for Eq. (19) is given by

$$u(x,t) = \tfrac{4}{3}\cos^2(\tfrac{x}{4}) + \tfrac{2}{3}\cos(\tfrac{x}{4})\sin(\tfrac{x}{4})t \\ -\tfrac{1}{12}\left(-1+2\cos^2(\tfrac{x}{4})\right)t^2 - \tfrac{1}{36}\left(\cos(\tfrac{x}{4})\sin(\tfrac{x}{4})\right)t^3 \tag{28}$$

and this will, in the limit of infinitely many terms, yield the closed form solution

$$u(x,t) = (\frac{4}{3})\cos^2(\frac{x-t}{4}) \tag{29}$$

which is represented in Fig. 1.
On the other hand, a development of the exact solution (29) in Taylor series over t=0 to order 3 gives:

$$u(x,t) = \tfrac{4}{3}\cos^2(\tfrac{x}{4}) + \tfrac{2}{3}\cos(\tfrac{x}{4})\sin(\tfrac{x}{4})t \\ -\tfrac{1}{12}\left(-1+2\cos^2(\tfrac{x}{4})\right)t^2 - \tfrac{1}{36}\left(\cos(\tfrac{x}{4})\sin(\tfrac{x}{4})\right)t^3 + \mathrm{O}(t^4) \tag{30}$$

which confirms our result.

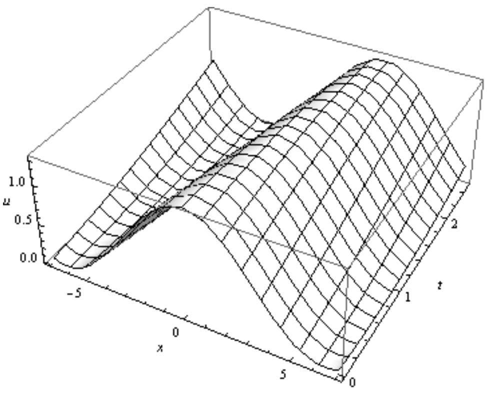

Figure 1: Graphic representation of the exact solution (29) of the initial value problem (4).

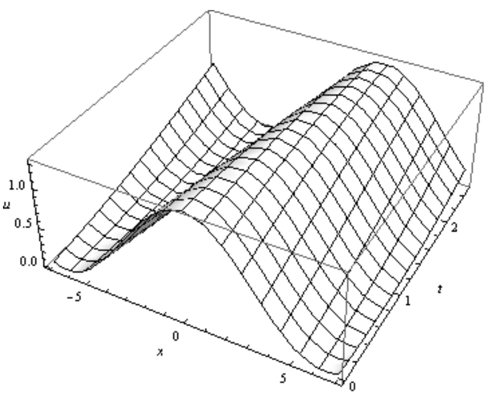

Figure 2: Approximate solution (27) of the Eq. (19) given by the VHPM method with third order.

In figure 1, we have represented the graph of the exact solution of Eq. (20). As we see, there is practically no difference between the graph of the approximate series solution in Fig. 2 and the exact solution in Fig. 1. Additionally, we see in table 1 and table 2, that the calculation of error between the exact solution and that obtained by the VHPM method shows that the resulting value is very close to the exact solution.

6 Conclusion

In this paper, we have studied the one-dimensional K(2,2) equation by using the variational homotopy perturbation method. The results show that the proposed method is powerful for finding the numerical solutions and can be used to obtain the series solution for the general case K(m,n) equations, where m and n can be different from

Table 1: The VHPM results for 3 iterations in comparison with the exact solution of the K(2,2) equation with initial conditions of Eq. (4).

t/x	0.1	0.2	0.3	0.4	0.5
0.1	$1.73466 * 10^{-7}$	$1.72903 * 10^{-7}$	$1.71907 * 10^{-7}$	$1.70481 * 10^{-7}$	$1.68629 * 10^{-7}$
0.2	$2.77616 * 10^{-6}$	$2.76852 * 10^{-6}$	$2.75397 * 10^{-6}$	$2.73253 * 10^{-6}$	$2.70427 * 10^{-6}$
0.3	0.0000140555	0.0000140239	0.0000139572	0.0000138556	0.0000137194
0.4	0.0000444185	0.0000443408	0.0000441522	0.0000438533	0.0000434448
0.5	0.000108417	0.000108281	0.000107875	0.000107199	0.000106255

Table 2: The VHPM results for 6 iterations in comparison with the exact solution of the K(2,2) equation with initial conditions of Eq. 4.

t/x	0.1	0.2	0.3	0.4	0.5
0.1	$1.44542 * 10^{-11}$	$1.44051 * 10^{-11}$	$1.43199 * 10^{-11}$	$1.41993 * 10^{-11}$	$1.40423 * 10^{-11}$
0.2	$9.25265 * 10^{-10}$	$9.22456 * 10^{-10}$	$9.17342 * 10^{-10}$	$9.09935 * 10^{-10}$	$9.00253 * 10^{-10}$
0.3	$1.05408 * 10^{-8}$	$1.05125 * 10^{-8}$	$1.0458 * 10^{-8}$	$1.03774 * 10^{-8}$	$1.02708 * 10^{-8}$
0.4	$5.92275 * 10^{-8}$	$5.909 * 10^{-8}$	$5.88049 * 10^{-8}$	$5.83727 * 10^{-8}$	$5.77947 * 10^{-8}$
0.5	$2.25925 * 10^{-7}$	$2.25481 * 10^{-7}$	$2.24474 * 10^{-7}$	$2.22906 * 10^{-7}$	$2.2078 * 10^{-7}$

2. We have seen that the VHPM method requires the evaluation of the Lagrangian multiplier λ, while the ADM method requires the evaluation of the Adomian polynomials. As the evaluation of the Adomian polynomials for every nonlinear terms requires more and more algebraic calculations, it should be better to used the VHPM method to overcome this difficulty. We can integrate the equation directly without use calculation of Adomian polynomials. Moreover, an observation of the error analysis in Table 1 and 2 shows that more accuracy can be obtained by adding terms in the series.

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