Existence and uniqueness of solution for Cahn-Hilliard hyperbolic phase field system with Dirichlet boundary conditions and polynomial potential

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Abstract

Our aim in this article is to study the existence and the uniqueness of solution for Cahn-Hilliard hyperbolic phase-field system, with initial conditions, homogeneous Dirichlet boundary conditions, polynomial potential in a bounded and smooth domain.

Keywords: Cahn-Hilliard hyperbolic phase-field system, polynomial potential, Dirichlet boundary conditions.

1. Introduction

G. Caginalp introduced in [3] the following phase-field system

$$\frac{\partial u}{\partial t} - \Delta^2 u - \Delta f(u) = -\Delta \theta \quad (1)$$

$$\frac{\partial \theta}{\partial t} - \Delta \theta = -\frac{\partial u}{\partial t} \quad (2)$$

where u is the order parameter and θ is the (relative) temperature. These systems model phase transition processus such as melting solidification processes and have studied (see [1] and [9]) for a similar phase-field model with a nonlinear term.

These Cahn-Hilliard phase-field systems are known as conserved phase-field system (see [7] and [14]) based on type III heat conduction and with two temperatures (see [13]), the authors have proven the existence and the uniqueness of the solutions, the existence of global attractor and exponential attractors.

In [18], Ntsokongo and Batangouna have studied the following Cahn-Hillard hyperbolic phase-field system

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta\left(\frac{\partial \alpha}{\partial t} - \beta \Delta \frac{\partial \alpha}{\partial t}\right) \quad (3)$$

$$\frac{\partial^2 \alpha}{\partial^2 t} - \Delta \frac{\partial^2 \alpha}{\partial^2 t} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t} \quad (4)$$

where $\beta = 1$, u is the order parameter and α is the temperature. They have proven the existence and the uniqueness of solution with Dirichlet boundary condition and the regular potential $f(s) = s^3 - s$. In [13], Jean De Dieu Mangoubi and al. have studied the following Cahn-Hillard hyperbolic phase field system

$$\varepsilon(-\Delta)\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta \frac{\partial \alpha}{\partial t} \quad in \ \mathbb{R}_+ \times \Omega \ (5)$$

$$\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t} \quad in \ \mathbb{R}_+ \times \Omega \ (6)$$

$$u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = \alpha|_{\partial\Omega} = 0$$

$$u(0,x) = u_0(x), \ \frac{\partial u}{\partial t}(0,x) = u_1(x) \ \forall \, x \in \Omega$$

$$\alpha(0,x) = \alpha_0(x), \ \frac{\partial \alpha}{\partial t}(0,x) = \alpha_1(x) \ \forall \, x \in \Omega.$$

They have proven the existence and the uniqueness of the solution with Dirichlet boundary condition and the potential $f(s) = s^3 - s$. In this paper, we consider the following Cahn-Hilliard hyperbolic phase-field system

$$\varepsilon(-\Delta)\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta\left(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}\right) \quad in \ \mathbb{R}_+^* \times \Omega (7)$$

$$\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -u - \frac{\partial u}{\partial t} \quad in \ \mathbb{R}_+^* \times \Omega \quad (8)$$

$$u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = \alpha|_{\partial\Omega} = 0 \quad (9)$$

$$u(0,x) = u_0(x), \ \frac{\partial u}{\partial t}(0,x) = u_1(x), \ \forall \, x \in \Omega \quad (10)$$

$$\alpha(0,x) = \alpha_0(x), \ \frac{\partial \alpha}{\partial t}(0,x) = \alpha_1(x), \ \forall \, x \in \Omega \quad (11)$$

where ε is a relaxation parameter and Ω is a bounded and regular domain of $\mathbb{R}^n$ with $n = 1, 2$ or 3 and f is a polynomial potential of order $2p - 1$.

$$f(s) = \sum_{i=1}^{2p-1} a_i s^i, \, a_{2p-1} > 0, \, p \geq 2.$$

In this paper we prove the existence and the uniqueness of solution of the hyperbolic system (7)-(14).
The potential f satisfies the following properties

$$\frac{1}{2}a_{2p-1}s^{2p}-c_1 \le f(s)s \le \frac{3}{2}a_{2p-1}s^{2p}$$
$$+c_1, \; c_1 > 0, \; \forall s \in \mathbb{R}, \tag{12}$$
$$-\kappa \le \frac{2p-1}{2}a_{2p-1}s^{2p-2}-c_2 \le f'(s) \le 3pa_{2p-1}s^{2p-2}$$
$$+c_2; \kappa > 0, \; c_2 > 0 \; \forall s \in \mathbb{R}, \tag{13}$$
$$\frac{1}{4p}a_{2p-1}s^{2p}-c_3 \le F(s) \le \frac{3}{4p}a_{2p-1}s^{2p}$$
$$+c_3, \; c_3 > 0, \forall s \in \mathbb{R} \tag{14}$$

where $\qquad\qquad F(s) = \int_0^s f(\tau)d\tau$.

2. Notations

We denote by $(.,.)$ the scalar product in $L^2(\Omega)$, $\|.\|$ the associated norm usual and $\|.\|_{-1} = \|(-\Delta)^{\frac{-1}{2}}.\|$, where $-\Delta$ denotes the minus Laplace operator Dirichlet boundary conditions. More generally, $\|.\|_X$ denote the norm of Banach space X.
Throughout this paper, the letter $C_i > 0$, denote (generally positive) constants which may change from line to line, or ever or same line.

3. A priori estimates

Multiplying (7) by $(-\Delta)^{-1}\frac{\partial u}{\partial t}$ and (8) by $\left(\frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\right)$, integrating over Ω and adding the two resulting differential equalities, we find

$$\frac{d}{dt}E_1 + 2\|\frac{\partial u}{\partial t}\|_{-1}^2 + 2\|\frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\|^2 = -2\left(u, \frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\right)$$
$$\le 2\|u\|\|\frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\|$$
$$\le \|u\|^2 + \|\frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\|^2$$
$$\frac{d}{dt}E_1 + 2\|\frac{\partial u}{\partial t}\|_{-1}^2 + \|\frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\|^2 \le C\|\nabla u\|^2$$

where

$$E_1 = \varepsilon\|\frac{\partial u}{\partial t}\|^2 + \|\nabla u\|^2 + 2(F(u)+c_3, 1) + \|\frac{\partial \alpha}{\partial t}\|^2$$
$$+\|\nabla\frac{\partial \alpha}{\partial t}\|^2 + \|\nabla\alpha\|^2 + \|\Delta\alpha\|^2.$$

Thanks to the propertie (14), we have

$$(F(u)+c_3, 1) \geqslant 0,$$

which implies

$$\frac{d}{dt}E_1 + 2\|\frac{\partial u}{\partial t}\|_{-1}^2 + \|\frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\|^2 \leqslant K_1E_1.$$

Applying the Gronwall's lemma, we obtain

$$E_1(t) + 2\int_0^t\|\frac{\partial u}{\partial t}\|_{-1}^2 d\tau + \int_0^t\|\frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\|^2 d\tau \leqslant E_1(0)e^{K_1T}$$

for all $t \in [0, T]$.
Using again the propertie (14), E_1 satisfies

$$E_1 \ge$$
$$C\left(\varepsilon\|\frac{\partial u}{\partial t}\|^2 + \|u\|_{H^1}^2 + \|u\|_{L^{2p}}^{2p} + \|\nabla\frac{\partial \alpha}{\partial t}\|^2 + \|\Delta\alpha\|^2\right) + C', \; C > 0.$$

Finally, we deduce that

$$u \in L^\infty\left(0,T;L^{2p}(\Omega) \cap H_0^1(\Omega)\right), \alpha \in L^\infty\left(0,T;H^2(\Omega) \cap H_0^1(\Omega)\right)$$
$$\frac{\partial u}{\partial t} \in L^\infty\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H^{-1}(\Omega)\right)$$

and

$$\frac{\partial\alpha}{\partial t} \in L^\infty\left(0,T;H_0^1(\Omega)\right) \cap L^2\left(0,T;H^2(\Omega) \cap H_0^1(\Omega)\right), \; \forall T > 0.$$

4. Existence and uniqueness of the solution

Theorem 4.1. *(Existence) We assume that* $(u_0, u_1, \alpha_0, \alpha_1) \in \left(L^{2p}(\Omega) \cap H_0^1(\Omega)\right) \times L^2(\Omega) \times \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega)$. *Then the system (7)-(14)*
possesses at least one solution (u, α) *such that* $u \in L^\infty\left(0,T;L^{2p}(\Omega) \cap H_0^1(\Omega)\right)$,
$\alpha \in L^\infty\left(0,T;H^2(\Omega) \cap H_0^1(\Omega)\right)$, $\frac{\partial u}{\partial t} \in L^\infty\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H^{-1}(\Omega)\right)$
and
$\frac{\partial\alpha}{\partial t} \in L^\infty\left(0,T;H_0^1(\Omega)\right) \cap L^2\left(0,T;H^2(\Omega) \cap H_0^1(\Omega)\right), \forall T > 0.$

The proof is based on a priori estimates obtained in the previous section and on the standard Galerkin scheme.

Theorem 4.2. *(Uniqueness) Let the assumptions of theorem 4.1 hold. The system (7)-(14) possesses a unique solution* (u, α) *such that* $u \in L^\infty\left(0,T;L^{2p}(\Omega) \cap H_0^1(\Omega)\right)$,
$\alpha \in L^\infty\left(0,T;H^2(\Omega) \cap H_0^1(\Omega)\right)$, $\frac{\partial u}{\partial t} \in L^\infty\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H^{-1}(\Omega)\right)$
and
$\frac{\partial\alpha}{\partial t} \in L^\infty\left(0,T;H_0^1(\Omega)\right) \cap L^2\left(0,T;H^2(\Omega) \cap H_0^1(\Omega)\right) \forall T > 0.$

Proof. Let (v, α^1) and (w, α^2) be two solutions of the system (7)-(14), with initial data $(v_0, v_1, \alpha_0^1, \alpha_1^1)$ and $(w_0, w_1, \alpha_0^2, \alpha_1^2) \in \left(L^{2p}(\Omega) \cap H_0^1(\Omega)\right) \times L^2(\Omega) \times \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega)$, respectively.
We set $u = v - w$ and $\alpha = \alpha^1 - \alpha^2$, then (u, α) is a solution of the following system

$$\varepsilon(-\Delta)\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(v) - \Delta f(w) = -\Delta\left(\frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\right) \tag{15}$$
$$\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t} - \Delta\alpha = -u - \frac{\partial u}{\partial t} \tag{16}$$
$$u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = \alpha|_{\partial\Omega} = 0$$
$$u|_{t=0} = v_0 - w_0, \; \frac{\partial u}{\partial t}|_{t=0} = v_1 - w_1$$
$$\alpha|_{t=0} = \alpha_0^1 - \alpha_0^2, \; \frac{\partial \alpha}{\partial t}|_{t=0} = \alpha_1^1 - \alpha_1^2.$$

Multiply (15) by $(-\Delta)^{-1}\frac{\partial u}{\partial t}$ and (16) by $\frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}$, integrate over Ω and add the two resulting differential equalities. We find

$$\frac{d}{dt}E_2 + 2\|\frac{\partial u}{\partial t}\|_{-1}^2 + 2\int_\Omega(f(v)-f(w))\frac{\partial u}{\partial t}dx + 2\|\frac{\partial \alpha}{\partial t}$$
$$-\Delta\frac{\partial \alpha}{\partial t}\|^2 = -2\left(u, \frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\right).$$

where

$$E_2 = \varepsilon\|\frac{\partial u}{\partial t}\|^2 + \|\nabla u\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2 + \|\nabla\frac{\partial \alpha}{\partial t}\|^2 + \|\nabla\alpha\|^2 + \|\Delta\alpha\|^2.$$

Applying Hölder and Young inequalities, we get

$$\frac{d}{dt}E_2 + 2\|\frac{\partial u}{\partial t}\|_{-1}^2$$
$$+\|\frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\|^2 \leqslant 2\int_\Omega|f(v)-f(w)||\frac{\partial u}{\partial t}|dx + \|u\|^2 \tag{17}$$

We know that

$$
\begin{aligned}
f(v) - f(w) &= \sum_{k=1}^{2p-1} a_k(v^k - w^k) \\
&= (v-w)\left(a_1 + a_2(v+w) + \sum_{k=3}^{2p-1}a_k\sum_{i=0}^{k-1}v^{k-i-1}w^i\right)
\end{aligned}
$$

which implies

$$
\begin{aligned}
|f(v) - f(w)| &\le |u|\Big(|a_1| + |a_2|(|v|+|w|) \\
&\quad + \sum_{k=3}^{2p-1}|a_k|\sum_{i=0}^{k-1}|v|^{k-i-1}|w|^i\Big).
\end{aligned}
$$

Applying Young's inequality, we obtain

$$
\begin{aligned}
|v|^{k-1-i}|w|^i &\le \frac{k-1-i}{k-1}(|v|^{k-1-i})^{\frac{k-1}{k-1-i}} + \frac{i}{k-1}(|w|^i)^{\frac{k-1}{i}} \\
&\le \frac{k-1-i}{k-1}|v|^{k-1} + \frac{i}{k-1}|w|^{k-1}
\end{aligned}
$$

which implies

$$
\begin{aligned}
\sum_{i=0}^{k-1}(|v|^{k-i-1}|w|^i) &\le \sum_{i=0}^{k-1}\frac{k-i-1}{k-1}|v|^{k-1} + \sum_{i=0}^{k-1}\frac{i}{k-1}|w|^{k-1} \\
&\le \frac{k}{2}\left(|v|^{k-1} + |w|^{k-1}\right).
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
|f(v)-f(w)| &\le |u|\Big(|a_1| + |a_2|(\frac{1}{2p-2}|v|^{2p-2} \\
&\quad + \frac{1}{2p-2}|w|^{2p-2} + C) + \sum_{k=3}^{2p-1}\frac{k}{2}|a_k||v|^{k-1} \\
&\quad + \sum_{k=3}^{2p-1}\frac{k}{2}|a_k||w|^{k-1}\Big) \\
&\le C|u|\Big(|a_1| + \frac{|a_2|}{2p-2}|v|^{2p-2} \\
&\quad + \frac{|a_2|}{2p-2}|w|^{2p-2} \\
&\quad + \sum_{k=3}^{2p-1}|a_k|\frac{(k-1)k}{4(p-1)}(|v|^{2p-2} \\
&\quad + |w|^{2p-2}) + 1\Big) \\
&\le C|u|\Big(\frac{1}{2p-2}(|v|^{2p-2} + |w|^{2p-2}) \\
&\quad + \frac{|v|^{2p-2} + |w|^{2p-2}}{4(p-1)}\sum_{k=3}^{2p-1}k(k-1) + 1\Big),
\end{aligned}
$$

which implies

$$
|f(v) - f(w)| \le C|u|(|v|^{2p-2} + |w|^{2p-2} + 1).
$$

Hence

$$
\int_\Omega |f(v) - f(w)||\frac{\partial u}{\partial t}|dx \le C\int_\Omega |u|(|v|^{2p-2} + |w|^{2p-2} + 1)|\frac{\partial u}{\partial t}|dx.
$$

In order to continue we consider two following cases.
If $n = 1$.
We know that $H^1(\Omega) \subset L^\infty(\Omega)$. Since $v, w \in L^\infty\left(0, T; H_0^1(\Omega)\right)$, then $v, w \in L^\infty((0,T) \times \Omega)$, there exists $C_1, C_2 > 0$ such that $\sup_{(t,x)\in(0,T)\times\Omega}|v(t,x)| \le C_1$ and

$$
\sup_{(t,x)\in(0,T)\times\Omega}|w(t,x)| \le C_2,
$$

then

$$
\begin{aligned}
\int_\Omega |f(v) - f(w)||\frac{\partial u}{\partial t}|dx &\le C_3\Big(\|v\|_{L^\infty}^{2p-2} \\
&\quad + \|w\|_{L^\infty}^{2p-2} + 1\Big)\int_\Omega |u||\frac{\partial u}{\partial t}|dx \\
&\le C\|u\|\|\frac{\partial u}{\partial t}\| \\
&\le C\|u\|_{H^1}\|\frac{\partial u}{\partial t}\|.
\end{aligned}
$$

If $n = 2$ or 3.
We have

$$
\int_\Omega |u||v|^{2p-2}|\frac{\partial u}{\partial t}|dx \le \|u\|_{L^6}\||v|^{2p-2}\|_{L^3}\|\frac{\partial u}{\partial t}\|
$$

and

$$
\||v|^{2p-2}\|_{L^3} = \|v\|_{L^{3(2p-2)}}^{2p-2}.
$$

Since $3(2p-2) < 6p$, $\|v\|_{L^{6p}} = \||v|^p\|_{L^6}^{\frac{1}{p}}$ and $H^1(\Omega) \subset L^6(\Omega)$ (the continuous embeding), we have $\|v\|_{L^{6p}} \le C\||v|^p\|^{\frac{1}{p}} \le C'\|v\|_{H^1}$. Since $v \in L^\infty(0,T;H_0^1(\Omega))$, we have

$$
\begin{aligned}
\int_\Omega |u||v|^{2p-2}|\frac{\partial u}{\partial t}|dx &\le C\|v\|_{H^1}^{2p-2}\|u\|_{L^6}\|\frac{\partial u}{\partial t}\| \\
&\le C\|u\|_{L^6}\|\frac{\partial u}{\partial t}\| \\
&\le C\|u\|_{H^1}\|\frac{\partial u}{\partial t}\|.
\end{aligned}
$$

We have the same estimate for w.
Finally, we have for $n = 1$, 2 or 3

$$
\begin{aligned}
\int_\Omega |f(v) - f(w)||\frac{\partial u}{\partial t}|dx &\le C\|u\|_{H^1}\|\frac{\partial u}{\partial t}\| \\
&\le C\left(\|u\|_{H^1}^2 + \|\frac{\partial u}{\partial t}\|^2\right).
\end{aligned}
$$

Inserting the above estimate into (17) we have

$$
\frac{d}{dt}E_2 + 2\|\frac{\partial u}{\partial t}\|_{-1}^2 + \|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\|^2 \le KE_2, \quad K > 0.
$$

Applying Gronwall's lemma, we obtain for all $t \in [0,T]$

$$
E_2(t) \le E_2(0)e^{KT}.
$$

Then we deduce the continuous dependence of solution which respect to the initial conditions, and the uniqueness of the solution is proven. $\qquad\square$

The existence and the uniqueness of the solution of problem (7)-(14) being proven in a larger space, we now establish the solution with more regularity.

Theorem 4.3. *Assume* $(u_0, u_1, \alpha_0, \alpha_1) \in (L^{2p}(\Omega) \cap H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega) \times (H^3(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)).$
Then the system (7)-(14) possesses a unique solution (u, α) *such that*
$u \in L^\infty(0,T;(L^{2p}(\Omega) \cap H_0^1(\Omega) \cap H^2(\Omega)), \ \alpha \in L^\infty(0,T;H^3(\Omega) \cap H_0^1(\Omega)), \ \frac{\partial u}{\partial t} \in L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;L^2(\Omega)),$

$\frac{\partial \alpha}{\partial t} \in L^\infty(0,T;H^2(\Omega) \cap H_0^1(\Omega)) \cap L^2(0,T;H^3(\Omega) \cap H_0^1(\Omega)),$

$\frac{\partial^2 \alpha}{\partial t^2} \in L^2(0,T;L^2(\Omega))$ *and* $\frac{\partial^2 u}{\partial t^2} \in L^2(0,T;L^2(\Omega)) \quad \forall \ T > 0.$

Proof. According to the theorems 4.1 and 4.2, the hyperbolic system (7)-(14) possesses the unique solution (u, α) such that
$u \in L^\infty(0,T;L^{2p}(\Omega) \cap H_0^1(\Omega) \cap H^2(\Omega))$, $\alpha \in L^\infty(0,T;H^2(\Omega) \cap H_0^1(\Omega))$, $\frac{\partial u}{\partial t} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^{-1}(\Omega))$ and
$\frac{\partial \alpha}{\partial t} \in L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega))$, $\forall \ \ T > 0$.

Multiplying (7) by $\frac{\partial u}{\partial t}$ and integrating over Ω, we have

$$\frac{d}{dt}(\varepsilon \|\bigtriangledown \frac{\partial u}{\partial t}\|^2 + \|\Delta u\|^2) + 2\|\frac{\partial u}{\partial t}\|^2 = -2(\nabla f(u), \nabla \frac{\partial u}{\partial t})$$
$$+ 2(\nabla \frac{\partial u}{\partial t}, \nabla(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t})). \tag{18}$$

Multiplying (8) by $-\Delta\left(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}\right)$ and integrating over Ω, we get

$$\frac{d}{dt}(\|\nabla \frac{\partial \alpha}{\partial t}\|^2 + \|\Delta \frac{\partial \alpha}{\partial t}\|^2 + \|\Delta \alpha\|^2 + \|\nabla \Delta \alpha\|^2)$$
$$+ 2\|\nabla(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t})\|^2 = -2(\nabla u, \nabla(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}))$$
$$- 2(\nabla \frac{\partial u}{\partial t}, \nabla(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t})). \tag{19}$$

Now summing (18), (19) and applying Hölder and Young inequalities, we get

$$\frac{d}{dt}E_3 + 2\|\frac{\partial u}{\partial t}\|^2 + \|\nabla(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t})\|^2 \leqslant \|\nabla u\|^2$$
$$+ 2\int_\Omega |\nabla f(u)||\nabla \frac{\partial u}{\partial t}|dx \tag{20}$$

where

$$E_3 = \varepsilon \|\bigtriangledown \frac{\partial u}{\partial t}\|^2 + \|\Delta u\|^2 + \|\nabla \frac{\partial \alpha}{\partial t}\|^2 + \|\Delta \frac{\partial \alpha}{\partial t}\|^2 + \|\Delta \alpha\|^2$$
$$+ \|\nabla \Delta \alpha\|^2.$$

We know that

$$\int_\Omega |\nabla f(u)||\nabla \frac{\partial u}{\partial t}|dx = \int_\Omega |f'(u)\nabla u||\nabla \frac{\partial u}{\partial t}|dx$$

Using (13), we have

$$\int_\Omega |f'(u)\nabla u||\nabla \frac{\partial u}{\partial t}|dx \ \leqslant \ \int_\Omega (3pa_{2p-1}|u|^{2p-2} + c_2)|\nabla u||\nabla \frac{\partial u}{\partial t}|dx$$
$$\leqslant \ C_4 \int_\Omega |u|^{2p-2}|\nabla u||\nabla \frac{\partial u}{\partial t}|dx$$
$$+ c_2 \int_\Omega |\nabla u||\nabla \frac{\partial u}{\partial t}|dx$$
$$\leqslant \ C_4 \int_\Omega |u|^{2p-2}|\nabla u||\nabla \frac{\partial u}{\partial t}|dx + c_2 \|\nabla u\|^2$$
$$+ c_2 \|\nabla \frac{\partial u}{\partial t}\|^2$$
$$\leqslant \ C_4 \int_\Omega |u|^{2p-2}|\nabla u||\nabla \frac{\partial u}{\partial t}|dx + c_2 \|\Delta u\|^2$$
$$+ c_2 \|\nabla \frac{\partial u}{\partial t}\|^2.$$

Now, we need the estimate of

$$\int_\Omega |u|^{2p-2}|\nabla u||\nabla \frac{\partial u}{\partial t}|dx$$

Similarly to the proof theorem 4.2 we have

if $n = 1$,

$$\int_\Omega |u|^{2p-2}|\nabla u||\nabla \frac{\partial u}{\partial t}|dx \ \leqslant \ \|u\|_{L^\infty}^{2p-2}\|\nabla u\|\|\nabla \frac{\partial u}{\partial t}\|$$
$$\leqslant \ C\|\nabla u\|\|\nabla \frac{\partial u}{\partial t}\|$$
$$\leqslant \ C\|\Delta u\|^2 + C\|\nabla \frac{\partial u}{\partial t}\|^2$$

If $n = 2$ **or** 3,

$$\int_\Omega |u|^{2p-2}|\nabla u||\nabla \frac{\partial u}{\partial t}|dx \ \leqslant \ \|\nabla u\|_{L^6}\||u|^{2p-2}\|_{L^3}\|\nabla \frac{\partial u}{\partial t}\|$$
$$\leqslant \ C\|\nabla u\|_{L^6}\|\nabla \frac{\partial u}{\partial t}\|$$
$$\leqslant \ C\|\Delta u\|\|\nabla \frac{\partial u}{\partial t}\|$$
$$\leqslant \ C(\|\Delta u\|^2 + C\|\nabla \frac{\partial u}{\partial t}\|^2).$$

Then for $n = 1, 2, 3$, (20) can be written as

$$\frac{d}{dt}E_3 + 2\|\frac{\partial u}{\partial t}\|^2 + \|\nabla\left(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}\right)\|^2 \leqslant K_1 E_3, \ \ K_1 > 0.$$

Applying the Gronwall's lemma, we deduce that
$u \in L^\infty(0,T;L^{2p}(\Omega) \cap H^2(\Omega) \cap H_0^1(\Omega))$, $\alpha \in L^\infty(0,T;H^3(\Omega) \cap H_0^1(\Omega))$, $\frac{\partial u}{\partial t} \in L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;L^2(\Omega))$ and $\frac{\partial \alpha}{\partial t} \in L^\infty(0,T;H^2(\Omega) \cap H_0^1(\Omega)) \cap L^2(0,T;H^3(\Omega) \cap H_0^1(\Omega))$.

Multiplying (8) by $\frac{\partial^2 \alpha}{\partial t^2}$ and integrating over Ω, we get

$$\frac{d}{dt}(\|\frac{\partial \alpha}{\partial t}\|^2 + \|\nabla \frac{\partial \alpha}{\partial t}\|^2) + 2\|\frac{\partial^2 \alpha}{\partial t^2}\|^2 = 2(\Delta \alpha, \frac{\partial^2 \alpha}{\partial t^2})$$
$$+ 2(-u, \frac{\partial^2 \alpha}{\partial t^2}) + 2(-\frac{\partial u}{\partial t}, \frac{\partial^2 \alpha}{\partial t^2})$$

Applying Hölder and Young inequalities, we find the following estimate

$$\frac{d}{dt}\left(\|\frac{\partial \alpha}{\partial t}\|^2 + \|\nabla \frac{\partial \alpha}{\partial t}\|^2\right) + \|\frac{\partial^2 \alpha}{\partial t^2}\|^2 \leqslant C(\|\alpha\|_{H^2}^2 + \|u\|_{H^1}^2 + \|\frac{\partial u}{\partial t}\|^2)$$

which implies

$$\frac{\partial^2 \alpha}{\partial t^2} \in L^2\left(0,T;L^2(\Omega)\right).$$

Multiplying (7) by $(-\Delta)^{-1}\frac{\partial^2 u}{\partial t^2}$ and integrating over Ω, we obtain

$$\frac{d}{dt}\|\frac{\partial u}{\partial t}\|_{-1}^2 + 2\varepsilon\|\frac{\partial^2 u}{\partial t^2}\|^2 \ = \ 2(\Delta u, \frac{\partial^2 u}{\partial t^2}) + 2(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}, \frac{\partial^2 u}{\partial t^2})$$
$$- 2(f(u), \frac{\partial^2 u}{\partial t^2})$$
$$\leqslant \ C(\|\Delta u\|^2 + \|f(u)\|^2 + \|\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}\|^2)$$
$$+ \varepsilon\|\frac{\partial^2 u}{\partial t^2}\|^2,$$

which yields, using the fact that $u \in L^\infty(0,T;H^2(\Omega))$ and $H^2(\Omega) \subset L^\infty(\Omega)$,

$$\frac{\partial^2 u}{\partial t^2} \in L^2(0,T;L^2(\Omega)).$$

Then the proof of theorem 4.3 is complete.

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