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Application of a modified Adomian decomposition method to solving a kind of wave equation

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Abstract

In this paper, a modified Adomian decomposition method is used to construct the solution of the initial value problem of a wave equation.

Keywords: *Modified Adomian decomposition method, wave equation.*

1 Introduction

The wave equation is used in many areas among them: electronic, physics, mechanics. The Adomian decomposition method (ADM) is very useful to get an approximation of the solution of an equation. In the last years, the ADM has been applied to get the solutions of various kinds of problems of EDEs and PDEs [1]-[10]. Here, we use a modified Adomian decomposition method proposed in [11], to investigate a wave equation. Using this last method, we get the exact solution.

2 About modified Adomian decomposition method

In [11], a new modified Adomian decomposition method is proposed for solving a kind of the evolution equation. Here, we apply this method to wave equation.

2.1 The standard Adomian decomposition method

Let's consider the following initial value problem of a wave equation:

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial u}{\partial t} + \sum_{m=0}^{+\infty} b_m \frac{\partial^m u}{\partial x^m} + \sum_{m=0}^{N} c_m \frac{\partial^m u^{k+1}}{\partial x^m} + d\,\frac{\partial^{i+1}}{\partial x^i \partial t} + f\left(x,t\right), -\infty < x < +\infty, t > 0 \quad (2.1)$$

$$u\left(x,0\right) = h\left(x\right); \frac{\partial u\left(x,0\right)}{\partial t} = g\left(x\right) \quad (2.2)$$

where a, b_m, c_m and d are real constants, k and i are positive integers, M and N are nonnegative integers, $f\left(x,t\right)$ is given function and $u\left(x,t\right)$ the unknown.
We denote by:

$$L_{tt}u = \frac{\partial^2 u}{\partial t^2}, Ru = a\frac{\partial u}{\partial t} + \sum_{m=0}^{M} b_m \frac{\partial^m u}{\partial x^m} + d\,\frac{\partial^{i+1}}{\partial x^i \partial t}, Nu = \sum_{m=0}^{N} c_m \frac{\partial^m u^{k+1}}{\partial x^m} \quad (2.3)$$

and $L_{tt}^{-1}=\int_0^t\int_0^s dzds$. Thus we can rewrite (2.1) as

$$L_{tt}u=Ru+Nu+f\left(x,t\right) \tag{2.4}$$

From (2.4), we have

$$u=h\left(x\right)+tg\left(x\right)+L_{tt}^{-1}Ru+L_{tt}^{-1}Nu+L_{tt}^{-1}f\left(x,t\right) \tag{2.5}$$

According to the standard Adomian decomposition method, we suppose that the solution of (2.1) has the following form:

$$u\left(x\right)=\sum_{n=0}^{+\infty}u_n\left(x\right) \tag{2.6}$$

and

$$N\left(u\left(x\right)\right)=\sum_{n=0}^{+\infty}A_n\left(x\right) \tag{2.7}$$

where A_n are special polynomials of variables $u_0,u_1,...,u_n$ called Adomian polynomials and defined by [4],[5], [6]:

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{+\infty}\lambda^i u_i\right)\right]_{\lambda=0};n=0,1,2,... \tag{2.8}$$

From (2.5); (2.6) and (2.7) we have the following Adomian algorithm:

$$\begin{cases}u_0=h\left(x\right)+tg\left(x\right)+L_{tt}^{-1}f\left(x,t\right)\\ u_{n+1}=L_{tt}^{-1}Ru_n+L_{tt}^{-1}A_n;\forall n\geq 0\end{cases} \tag{2.9}$$

2.2 The modified Adomian decomposition method

According to the modified Adomian decomposition method [11], the first term u_0 in (2.6), is to be determined as follows:

$$u_0=h\left(x\right)+tg\left(x\right)+L_{tt}^{-1}f\left(x,t\right)+\alpha L_{tt}^{-1}G\left(u_0\right)$$

3 Application

3.1 The modified Adomian decomposition method

Let’s use the modified Adomian decomposition method (2.11) and (2.12) to solve the following problem

$$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}+\dfrac{\partial u}{\partial t}+\dfrac{\partial^2 u}{\partial x^2}-2u^3+u^2=0\\ u\left(x,0\right)=\dfrac{1}{x}\\ \dfrac{\partial u\left(x,0\right)}{\partial t}=-\dfrac{1}{x^2}\end{cases} \tag{3.1}$$

we denote

$$L_{tt}u = \frac{\partial^2 u}{\partial t^2}, L_t u = \frac{\partial u}{\partial t}, L_{xx} u = \frac{\partial^2 u}{\partial x^2}$$

$$Ru = -\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = -L_t u - L_{xx} u, Nu = 4u^3 - u^2 \tag{3.2}$$

From (3.1), we have

$$L_{tt}u = Ru + Nu \tag{3.3}$$

(3.3) gives us

$$u = u(x,0) + \frac{\partial u(x,0)}{\partial t} t + L_{tt}^{-1} Ru + L_{tt}^{-1} Nu \tag{3.4}$$

From (2.10), (2.11), (2.12) and (2.13) we have following modified Adomian algorithm

$$\begin{cases} u_0 = \frac{1}{x} - \frac{t}{x^2} + L_{tt}^{-1} L_{xx} u_0 \\ u_1 = L_{tt}^{-1} Ru_0 + L_{tt}^{-1} A_0 - L_{tt}^{-1} L_{xx} u_0 \\ u_{n+1} = L_{tt}^{-1} Ru_n + L_{tt}^{-1} L_{xx} u_n ; \forall n \geq 1 \end{cases} \tag{3.5}$$

Where u_0 is the solution of the following system:

$$\begin{cases} \frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2} \\ v(x,0) = \frac{1}{x} \\ \frac{\partial v(x,0)}{\partial t} = -\frac{1}{x^2} \end{cases} \tag{3.6}$$

To solve (3.6), we use the standard Adomian algorithm (2.9). From (3.6), we have

$$v(x,t) = v(x,0) + \frac{\partial v(x,0)}{\partial t} t + L_{tt}^{-1} L_{xx} v(x,t) \tag{3.7}$$

We suppose that the solution of (3.6) has the following form

$$v(x,t) = \sum_{n=0}^{+\infty} v_n(x,t) \tag{3.8}$$

Thus we have the following Adomian algorithm

$$\begin{cases} v_0(x,t) = v(x,0) + \frac{\partial v(x,0)}{\partial t} t \\ v_{n+1}(x,t) = L_{tt}^{-1} L_{xx} v_n(x,t); n \geq 1 \end{cases} \tag{3.9}$$

This gives as

$$\begin{cases} v_0(x,t) = \frac{1}{x} - \frac{1}{x^2} \\ v_1(x,t) = \frac{t^2}{x^3} - \frac{t^3}{x^4} \\ v_2(x,t) = \frac{t^4}{x^5} - \frac{t^5}{x^6} \\ v_3(x,t) = \frac{t^6}{x^7} - \frac{t^7}{x^8} \\ \ldots \end{cases} \tag{3.10}$$

Therefore

$$\begin{cases} u_0(x,t) = v(x,t) = \frac{1}{x} - \frac{t}{x^2} + \frac{t^2}{x^3} - \frac{t^3}{x^4} + \frac{t^4}{x^5} - \frac{t^5}{x^6} + \frac{t^6}{x^7} - \frac{t^7}{x^8} + \ldots \\ \qquad = \frac{1}{x}\left(1 - \frac{t}{x} + \frac{t^2}{x^2} - \frac{t^3}{x^3} + \frac{t^4}{x^4} - \frac{t^5}{x^5} + \ldots\right) \\ \qquad = \frac{1}{x}\sum_{n=0}^{+\infty}\left(\frac{t}{x}\right)^n \\ \qquad = \frac{1}{x+t} \end{cases} \tag{3.11}$$

with $\left|\frac{t}{x}\right| < 1.$

From (2.8) and (3.5), we have

$$A_0 = N(u_0) = 4\left(\frac{1}{x+t}\right)^3 - \left(\frac{1}{x+t}\right)^2$$

$$u_1 = L_{tt}^{-1}\left(Ru_0 + A_0 - L_{xx}u_0\right)$$

$$= L_{tt}^{-1}\left[\frac{\partial\left(\frac{1}{x+t}\right)}{\partial t} - \frac{\partial^2\left(\frac{1}{x+t}\right)}{\partial x^2} + 4\left(\frac{1}{x+t}\right)^3 - \left(\frac{1}{x+t}\right)^2 - \frac{\partial^2\left(\frac{1}{x+t}\right)}{\partial x^2}\right]$$

$$= L_{tt}^{-1}\left[\left(\frac{1}{x+t}\right)^2 - 2\left(\frac{1}{x+t}\right)^3 + 4\left(\frac{1}{x+t}\right)^3 - \left(\frac{1}{x+t}\right)^2 - 2\left(\frac{1}{x+t}\right)^3\right] = 0$$

$$A_1 = 12u_0^2u_1 - 2u_0u_1 = 0$$

$$u_2(x,t) = L_{tt}^{-1}\left(Ru_1 + A_1\right) = 0$$

$$A_2 = 12u_0u_1^2 + 6u_0^2 - u_1^2 - u_0u_2 = 0$$

$$u_3(x,t) = L_{tt}^{-1}\left(Ru_2 + A_0\right) = 0$$

We can easily get: $A_n = 0$ and $u_n = 0$ for $n \geq 1$.
Thus the solution of (3.1) is:

$$u(x,t) = \sum_{n=0}^{+\infty} u_n(x,t) = u_0(x,t) = \frac{1}{x+t} \tag{3.12}$$

3.2 The standard Adomian decomposition method

From (3.1) we have

$$\frac{\partial u(x,t)}{\partial t}-\frac{\partial u(x,0)}{\partial t}+u(x,t)-u(x,0)+\int_0^t \frac{\partial^2 u(x,s)}{\partial x^2}ds+\int_0^t N\left(u(x,s)\right)ds=0 \tag{3.13}$$

From (3.13), we have

$$u(x,t)-u(x,0)-\frac{\partial u(x,0)}{\partial t}t+\int_0^t u(x,s)ds-u(x,0)t+\int_0^t\int_0^z \frac{\partial^2 u(x,s)}{\partial x^2}dsdz+\int_0^t\int_0^z N\left(u(x,s)\right)dsdz=0 \tag{3.14}$$

From (2.9) and (3.14), we have the following standard Adomian algorithm

$$\begin{cases} u_0(x,t)=\frac{1}{x}-\frac{t}{x^2}+\frac{t}{x} \\ u_{n+1}(x,t)=-\int_0^t u_n(x,s)ds-\int_0^t\int_0^z \frac{\partial^2 u(x,s)}{\partial x^2}dsdz-\int_0^t\int_0^z A_n(x,s)dsdz\,;n\geq 0 \end{cases} \tag{3.15}$$

which gives us

$$u_1=\left(\frac{1}{5x^3}-\frac{3}{5x^4}+\frac{3}{5x^5}-\frac{1}{5x^6}\right)t^5+\left(\frac{7}{6x^3}-\frac{1}{12x^2}-\frac{25}{12x^4}+\frac{1}{x^5}\right)t^4+\left(\frac{2}{x^3}-\frac{1}{3x^2}-\frac{1}{x^4}\right)t^3+\left(\frac{1}{x^3}-\frac{1}{2x}\right)t^2+\left(-\frac{1}{x}\right)t$$

$$\begin{aligned} u_2=&\left(\frac{1}{30x^5}-\frac{1}{6x^6}+\frac{1}{3x^7}-\frac{1}{3x^8}+\frac{1}{6x^9}-\frac{1}{30x^{10}}\right)t^9+\left(\frac{2}{5x^5}-\frac{1}{40x^4}-\frac{27}{20x^6}+\frac{19}{10x^7}-\frac{49}{40x^8}+\frac{3}{10x^9}\right)t^8+\\ &\left(\frac{1}{252x^3}-\frac{89}{420x^4}+\frac{697}{420x^5}-\frac{4337}{1260x^6}+\frac{284}{105x^7}-\frac{5}{7x^2}\right)t^7+\left(\frac{83}{36x^2}-\frac{1}{60x^4}-\frac{37}{180x^3}-\frac{247}{90x^6}+\frac{3}{5x^7}\right)t^6+\\ &\left(\frac{1}{3x^2}-\frac{37}{12x^3}+\frac{25}{12x^4}\right)t^5+\left(\frac{1}{6x}+\frac{1}{3x^2}-\frac{2}{x^3}\right)t^4+\frac{1}{2x}t^2 \end{aligned}$$

$$\begin{aligned} u_3=&\left(\frac{17}{3900x^7}-\frac{119}{3900x^8}+\frac{119}{1300x^9}-\frac{119}{780x^{10}}+\frac{119}{780x^{11}}-\frac{119}{1300x^{12}}+\frac{119}{3900x^{13}}-\frac{17}{3900x^{14}}\right)t^{13}+\\ &\left(\frac{17}{200x^7}-\frac{17}{3600x^6}-\frac{493}{1200x^8}+\frac{17}{18x^9}-\frac{289}{240x^{10}}+\frac{527}{600x^{11}}-\frac{1241}{3600x^{12}}+\frac{17}{300x^{13}}\right)t^{12}+\\ &\left(\frac{139}{92400x^5}-\frac{1977}{30800x^6}+\frac{849}{1400x^7}-\frac{95083}{46200x^8}+\frac{12535}{3696x^9}-\frac{91141}{30800x^{10}}+\frac{4351}{3600x^{11}}-\frac{1366}{5775x^{12}}\right)t^{11}+\\ &\left(\frac{94901}{56700x^7}-\frac{1997}{15120x^6}+\frac{6469}{226800x^5}-\frac{11}{90720x^4}+\frac{1972183}{453600x^8}-\frac{355927}{75600x^9}+\frac{84487}{37800x^{10}}-\frac{38}{105x^{11}}\right)t^{10}+\\ &\left(\frac{613}{22680x^4}-\frac{43919}{90720x^5}+\frac{85003}{90720x^6}-\frac{9479}{9072x^7}-\frac{152587}{45380x^8}+\frac{8453}{3780x^9}-\frac{173}{420x^{10}}\right)t^9+\\ &\left(\frac{2603}{10080x^4}-\frac{1}{315x^3}-\frac{1579}{672x^5}+\frac{42577}{10080x^6}-\frac{8357}{5040x^7}-\frac{461}{560x^8}+\frac{99}{140x^9}\right)t^8+ \end{aligned}$$

$$\left(\frac{13}{126x^3}+\frac{81}{140x^4}-\frac{1651}{360x^5}+\frac{12991}{2520x^6}-\frac{67}{70x^7}-\frac{6}{35x^8}\right)t^7$$
$$\left(\frac{313}{360x^3}-\frac{1}{40x^2}-\frac{2}{9x^4}-\frac{239}{72x^5}+\frac{55}{36x^6}+\frac{2}{5x^7}\right)t^6+\left(\frac{259}{120x^3}-\frac{3}{20x^2}-\frac{67}{60x^4}-\frac{3}{5x^5}\right)t^5+$$
$$\left(\frac{5}{3x^3}-\frac{5}{24x^2}-\frac{1}{24x}\right)t^4+\left(-\frac{1}{6x}\right)t^3$$

$$u_4=\left(\frac{397}{442000x^9}-\frac{3573}{442000x^{10}}+\frac{3573}{110500x^{11}}-\frac{8337}{110500x^{12}}+\frac{25011}{221000x^{13}}-\frac{25011}{221000x^{14}}\right)t^{17}+$$
$$\left(\frac{8337}{110500x^{15}}-\frac{3573}{11500x^{16}}+\frac{3573}{442000x^{17}}-\frac{397}{442000x^{18}}\right)t^{17}+$$
$$\left(\frac{397}{15600x^9}-\frac{397}{312000x^8}-\frac{12307}{78000x^{10}}+\frac{19453}{39000x^{11}}-\frac{147287}{156000x^{12}}+\frac{5558}{4875x^{13}}\right)t^{16}+$$
$$\left(-\frac{2779}{3120x^{14}}+\frac{17071}{39000x^{15}}-\frac{38509}{312000x^{16}}+\frac{397}{26000x^{17}}\right)t^{16}+$$
$$\left(\frac{661}{1078000x^7}-\frac{29611}{1201200x^8}+\frac{1028273}{382200x^9}-\frac{1563901}{1274000x^{10}}+\frac{126828329}{42042000x^{11}}-\frac{186596959}{42042000x^{12}}\right)t^{15}+$$
$$\left(\frac{6810649}{1681680x^{13}}-\frac{31603181}{14014000x^{14}}+\frac{1233377}{1751750x^{15}}-\frac{995191}{10510500x^{16}}\right)t^{15}+$$
$$\left(\frac{921551}{764400x^9}-\frac{349}{112112x^7}-\frac{31819}{2866550x^8}-\frac{2837}{25225200x^6}-\frac{38124529}{8408400x^{10}}+\frac{73032979}{8408400x^{11}}\right)t^{14}+$$
$$\left(-\frac{23824531}{2522520x^{12}}+\frac{16534753}{2802800x^{13}}-\frac{24815149}{12612600x^{14}}+\frac{567311}{2102100x^{15}}\right)t^{14}+$$
$$\left(\frac{41}{7076160x^5}+\frac{4116053}{389188800x^6}-\frac{35900477}{194594400x^7}+\frac{69251453}{194594400x^8}+\frac{135703553}{77837760x^9}-\frac{2966586941}{389188800x^{10}}\right)t^{13}+$$
$$\left(\frac{767076589}{64864800x^{11}}-\frac{193484789}{21621600x^{12}}+\frac{8823523}{2702700x^{13}}-\frac{133543}{300300x^{14}}\right)t^{13}+$$
$$\left(\frac{272743}{1663200x^6}-\frac{11747}{3742200x^5}-\frac{8096521}{4989600x^7}+\frac{43899193}{9979200x^8}-\frac{44782379}{14968800x^9}\right)t^{12}+$$
$$\left(-\frac{23354071}{5987520x^{10}}+\frac{35038397}{4989600x^{11}}-\frac{592574}{155925x^{12}}+\frac{16987}{23100x^{13}}\right)t^{12}+$$
$$\left(\frac{1027}{4989600x^4}+\frac{1579}{90720x^5}+\frac{721307}{997920x^6}-\frac{7495991}{1247400x^7}+\frac{68582693}{4989600x^8}\right)t^{11}+$$
$$\left(-\frac{136787}{118820x^9}+\frac{1418161}{831600x^{10}}+\frac{38191}{17325x^{11}}-\frac{3923}{4620x^{12}}\right)t^{11}+$$
$$\left(\frac{95233}{181440x^5}-\frac{5773}{226800x^4}+\frac{172993}{302400x^6}-\frac{4641457}{453600x^7}+\frac{149713}{8100x^8}-\frac{2401079}{226800x^9}+\frac{289}{378x^{10}}+\frac{713}{2100x^{11}}\right)t^{10}+$$
$$\left(\frac{13}{6480x^3}-\frac{22219}{90720x^4}+\frac{247763}{90720x^5}-\frac{124807}{45360x^6}-\frac{683647}{90720x^7}+\frac{976351}{90720x^8}-\frac{2843}{945x^9}-\frac{89}{210x^{10}}\right)t^9+$$
$$\left(\frac{9407}{1440x^5}-\frac{173}{210x^4}-\frac{179}{5040x^3}-\frac{3131}{480x^6}-\frac{24889}{10080x^7}+\frac{4709}{1689x^8}-\frac{17}{70x^9}\right)t^8+$$

$$\left(\frac{1}{105x^2}-\frac{85}{168x^3}-\frac{271}{315x^4}+\frac{2597}{360x^5}-\frac{4831}{1260x^6}-\frac{43}{35x^7}\right)t^7+\left(\frac{1}{15x^2}-\frac{1019}{720x^3}+\frac{2}{15x^4}+\frac{5}{2x^5}\right)t^6+$$
$$\left(\frac{1}{120x}+\frac{13}{120x^2}-\frac{73}{60x^3}\right)t^5+\frac{1}{24x}t^4$$

etc …, and further terms can easily be calculated using a symbolic computation package.

3.3 Numerical analysis

Here, we compare the exact solution obtained by the standard Adomian decomposition method. We take

$$U_N(x,t)=\sum_{n=0}^{N}u_n(x,t).$$

(x,t)	U_0	U_1	U_3	U_4	U_{exact}
$(0.3,0.1)$	2.5556	2.5416	2.5009	2.5007	2.5
$(0.4,0.1)$	2.125	2.0121	2.0001	2.0001	2.0
$(0.5,0.1)$	1.8	1.6694	1.6667	1.6666	1.6667

4 Conclusion

In this paper, we showed through this example that the modified Adomian decomposition method can be used to solve the wave equation and give us the exact, on the other hand, the standard Adomian decomposition method gives the approached solution, we remark too that the approached solution rapidly converges to the exact solution.

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