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Research paper

Approximate analytic solutions of fractional Zakharov-Kuznetsov equations by fractional complex transform

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Abstract

In this paper, fractional complex transform (FCT) with help of variational iteration method (VIM) is used to obtain numerical and analytical solutions for the fractional Zakharov–Kuznetsov equations. Fractional complex transform (FCT) is proposed to convert fractional Zakharov–Kuznetsov equations to its differential partner and then applied VIM to the new obtained equations. Several examples are given and the results are compared to exact solutions. The results reveal that the method is very effective and simple.

Keywords: *Fractional complex transform; Variational iteration method; Fractional Zakharov–Kuznetsov equations.*

1. Introduction

Fractional models have been shown by many scientists to adequately describe the operation of variety of physical and biological processes and systems. Consequently, considerable attention has been given to the solution of fractional ordinary differential equations, integral equations and fractional partial differential equations of physical interest. Since most fractional differential equations do not have exact analytic solutions, approximation and numerical techniques, therefore, are used extensively [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. Numerical and analytical methods have included finite difference method [12, 13, 14], Adomian decomposition method [15, 16, 17, 18, 19], variational iteration method [20, 21, 22, 23], homotopy perturbation method [24, 25, 26, 27], and homotopy analysis method [28, 29].

Transform is an important method to solve mathematical problems. Many useful transforms for solving various problems were appeared in open literature, such as the travelling wave transform [30] , the Laplace transform [31], the Fourier transform [32], the Bücklund transformation [33], the integral transform [34], and the local fractional integral transforms [35]. Very recently the fractional complex transform [36, 37, 38, 39] was suggested to convert fractional order differential equations with modified Riemann-Liouville derivatives into integer order differential equations, and the resultant equations can be solved by advanced calculus.

In this paper, we consider the fractional version of the Zakharov–Kuznetsov equations as studied in [40, 41]. The fractional Zakharov-Kuznetsov equations (shortly called FZK(p,q,r)) are of the form:

$$D_t^{\alpha}u + a(u^p)_x + b(u^q)_{xxx} + c(u^r)_{yyx} = 0, \tag{1}$$

where $u = u(x,y,t)$, α is a parameter describing the order of the fractional derivative ($0 < \alpha \leq 1$), a, b and c are arbitrary constants and p, q, and r are integers and $p, q, r \neq 0$ governs the behavior of weakly nonlinear ion acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [42, 43].

We aim in this paper to solve the fractional Zakharov–Kuznetsov equations by FCT with help of VIM, and to determine the effectiveness of FCT in solving these kinds of problems.

2. Fractional Complex Transform (FCT)

Consider the following general fractional differential equation

$$f\left(u, u_t^{(\alpha)}, u_x^{(\beta)}, u_y^{(\gamma)}, u_z^{(\lambda)}, u_t^{(2\alpha)}, u_x^{(2\beta)}, u_y^{(2\gamma)}, u_z^{(2\lambda)}, \dots\right) = 0, \tag{2}$$

where $u_t^{(\alpha)} = \partial^{\alpha}u(x,y,z,t)/\partial^{\alpha}$ denotes the modified Riemann–Liouville derivative. $0 < \alpha \leq 1$, $0 < \beta \leq 1$, $0 < \gamma \leq 1$, $0 < \lambda \leq 1$.

Introducing the following transforms

$$T = \frac{qt^{\alpha}}{\Gamma(1+\alpha)}, \tag{3}$$

$$X = \frac{px^{\beta}}{\Gamma(1+\beta)}, \tag{4}$$

$$Y = \frac{ky^{\gamma}}{\Gamma(1+\gamma)}, \tag{5}$$

$$Z = \frac{lz^{\lambda}}{\Gamma(1+\lambda)}, \tag{6}$$

where p, q, k and l are constants.

Using the above transforms, we can convert fractional derivatives into classical derivatives:

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = q\frac{\partial u}{\partial T}, \quad (7)$$

$$\frac{\partial^{\beta}u}{\partial x^{\beta}} = p\frac{\partial u}{\partial X}, \quad (8)$$

$$\frac{\partial^{\gamma}u}{\partial y^{\gamma}} = k\frac{\partial u}{\partial Y}, \quad (9)$$

$$\frac{\partial^{\lambda}u}{\partial z^{\lambda}} = l\frac{\partial u}{\partial Z}. \quad (10)$$

Therefore, we can easily covert the fractional differential equations into partial differential equations, so that everyone familiar with advanced calculus can deal with fractional calculus without any difficulty. For example, consider a fractional differential equation

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}+2u\frac{\partial^{\beta}u}{\partial x^{\beta}}+4\frac{\partial^{\gamma}u}{\partial y^{\gamma}}+5\frac{\partial^{\lambda}u}{\partial z^{\lambda}}=0. \quad (11)$$

By using the above transformations we get:

$$q\frac{\partial u}{\partial T}+2pu\frac{\partial u}{\partial X}+4k\frac{\partial u}{\partial Y}+5l\frac{\partial u}{\partial Z}=0, \quad (12)$$

which can be solved by variational iteration method.

3. Variational iteration method (VIM)

To illustrate the basic concepts of VIM [44], we consider the following general nonlinear functional equation:

$$Lu(x,y,t)+Nu(x,y,t)=g(x,y,t), \quad (13)$$

where L is a linear operator and N is a nonlinear operator, and $g(x,y,t)$ is an inhomogeneous term.
VIM is based on the general Lagrange multiplier method [45]. The main feature of the method is that the solution of a mathematical problem with linearization assumption is used as initial approximation or trial function. Then a more highly precise approximation at some special point can be obtained. According to VIM , we can construct a correction functional for Eq. (13) as follows:

$$u_{k+1}(x,y,t)=u_k(x,y,t) + \int_0^t \lambda(\xi)\left[Lu_k(x,y,s)+N\tilde{u}_k(x,y,s)-g(x,y,s)\right]\mathrm{d}s, \quad (14)$$

where λ, a general Lagrange multiplier, can be identified optimally via the variational theory. The subscript k indicates the kth approximation and $\tilde{u}_k$ is considered as a restricted variation [46, 47], i.e. $\delta\tilde{u}_k=0$.

4. Applications

In this section, the applicability of FCT shall be demonstrated by two test examples.

4.1. Example 1

First, we consider the time-fractional FZK(2,2,2) in the form:

$$D_t^{\alpha}u+(u^2)_x+\frac{1}{8}(u^2)_{xxx}+\frac{1}{8}(u^2)_{xyy}=0, \quad (15)$$

where $0<\alpha\le 1$ is a parameter describing the order of the fractional time derivative. The exact solution to Eq. (15) when $\alpha=1$ and subject to the initial condition

$$u(x,y,0)=\frac{4}{3}\rho\sinh^2(x+y), \quad (16)$$

where ρ is an arbitrary constant, was derived in [48] and is given as:

$$u(x,y,t)=\frac{4}{3}\rho\sinh^2(x+y-\rho t). \quad (17)$$

To apply FCT to Eq.(15), we use the above transformations, so we have the following partial differential equation:

$$q\frac{\partial u}{\partial T}+\frac{\partial u^2}{\partial x}+\frac{1}{8}\left(\frac{\partial^3u^2}{\partial x^3}\right)+\frac{1}{8}\left(\frac{\partial^3u^2}{\partial y^2x}\right). \quad (18)$$

For simplicity we set $q=1$, so we get

$$\frac{\partial u}{\partial T}+\frac{\partial u^2}{\partial x}+\frac{1}{8}\left(\frac{\partial^3u^2}{\partial x^3}\right)+\frac{1}{8}\left(\frac{\partial^3u^2}{\partial y^2x}\right). \quad (19)$$

Now, we solve Eq. (19) by means of VIM. To apply VIM to (19), we construct the correction functional as follows:

$$u_{k+1}=u_k+\int_0^T\lambda(s)\left[\frac{\partial u_k}{\partial s}+\left(\frac{\partial\tilde{u}_k^2}{\partial x}\right)+\frac{1}{8}\left(\frac{\partial^3\tilde{u}_k^2}{\partial x^3}\right)+\frac{1}{8}\left(\frac{\partial^3\tilde{u}_k^2}{\partial y^2\partial x}\right)\right]\mathrm{d}s. \quad (20)$$

For $\alpha=1$, we have

$$\delta u_{k+1}=\delta u_k+\delta\int_0^T\lambda(s)\left(\frac{\partial u_k}{\partial s}\right)\mathrm{d}s. \quad (21)$$

The general Lagrange multiplier can be identified as:

$$\lambda(s)=-1. \quad (22)$$

Substituting (22) into the correction functional (20), we obtain the following iteration formula:

$$u_{k+1}=u_k-\int_0^T\left[\frac{\partial u_k}{\partial s}+\left(\frac{\partial u_k^2}{\partial x}\right)+\frac{1}{8}\left(\frac{\partial^3u_k^2}{\partial x^3}\right)+\frac{1}{8}\left(\frac{\partial^3u_k^2}{\partial y^2\partial x}\right)\right]\mathrm{d}s. \quad (23)$$

The iteration starts with an initial approximation as given in (16). The iteration formula (23) now yields

$$u_1(x,y,t) = \frac{4}{3}\rho\sinh w^2-\frac{224}{9}\rho^2\sinh w^3\cosh wT -\frac{32}{3}\rho^2\sinh w\cosh w^3T \quad (24)$$

$$\begin{aligned}u_2(x,y,t) = {} & \frac{4}{3}\rho\sinh w^2-2\left(\frac{4}{3}\rho\sinh w^2-\frac{224}{9}\rho^2\sinh w^3\cosh wT\right.\\
&\left.-\frac{32}{3}\rho^2\sinh w\cosh w^3T\right)\left(\frac{8}{3}\rho\sinh w\cosh w\right.\\
&-\frac{320}{3}\rho^2\sinh w^2\cosh w^2T\\
&\left.-\frac{224}{9}\rho^2\sinh w^4T-\frac{32}{3}\rho^2\cosh w^4T\right)T\\
&-\frac{3}{2}\left(\frac{8}{3}\rho\sinh w\cosh w-\frac{320}{3}\rho^2\sinh w^2\cosh w^2T\right.\\
&\left.-\frac{224}{9}\rho^2\sinh w^4T-\frac{32}{3}\rho^2\cosh w^4T\right)\\
&\left(\frac{8}{3}\rho\cosh w^2+\frac{8}{3}\rho\sinh w^2\right.\\
&\left.-256\rho^2\sinh w\cosh w^3T-\frac{2816}{9}\rho^2\sinh w^3\cosh wT\right)T\\
&-\frac{1}{2}\left(\frac{4}{3}\rho\sinh w^2-\frac{224}{9}\rho^2\sinh w^3\cosh wT\right.\\
&\left.-\frac{32}{3}\rho^2\sinh w\cosh w^3T\right)\\
&\left(\frac{32}{3}\rho\sinh w\cosh w-256\rho^2\cosh w^4T\right.\\
&\left.-\frac{5120}{3}\rho^2\sinh w^2\cosh w^2T-\frac{2816}{9}\rho^2\sinh w^4T\right)T\end{aligned} \quad (25)$$

and so on, where $w = x + y$. The remaining components of $u_k(x,y,t)$ can be completely determined such that each term is determined by using (23).
By the fractional complex transform

$$T = \frac{t^\alpha}{\Gamma(1+\alpha)}, \tag{26}$$

we have

$$u_1(x,y,t) = \frac{4}{3}\rho \sinh w^2 - \frac{224}{9}\frac{\rho^2 \sinh w^3 \cosh w t^\alpha}{\Gamma(1+\alpha)} - \frac{32}{3}\frac{\rho^2 \sinh w \cosh w^3 t^\alpha}{\Gamma(1+\alpha)}, \tag{27}$$
$$\vdots$$

and so on, where $w = x + y$.
Table 1 shows the approximate solutions of Eq. (15) for different values of α: $\alpha = 0.67$, $\alpha = 0.75$ and $\alpha = 1.0$ using only three iterations of the VIM solution.

4.2. Example 2

Now, we consider FZK(3,3,3) in the form:

$$D_t^\alpha u + (u^3)_x + 2(u^3)_{xxx} + 2(u^3)_{xyy} = 0, \tag{28}$$

where $0 < \alpha \le 1$.
The exact solution to Eq. (28) when $\alpha = 1$ and subject to the initial condition

$$u(x,y,0) = \frac{3}{2}\rho \sinh\left[\frac{1}{6}(x+y)\right]. \tag{29}$$

where ρ is an arbitrary constant, was derived in [48] and is given by

$$u(x,y,t) = \frac{3}{2}\rho \sinh\left[\frac{1}{6}(x+y-\rho t)\right]. \tag{30}$$

To apply FCT to Eq.(28), we use the above transformations, so we have the following partial differential equation:

$$q\frac{\partial u}{\partial T} + \frac{\partial u^3}{\partial x} + 2\frac{\partial^3 u^3}{\partial x^3} + 2\frac{\partial^3 u^3}{\partial y^2 x}. \tag{31}$$

For simplicity we set $q = 1$, so we get

$$\frac{\partial u}{\partial T} + \frac{\partial u^3}{\partial x} + 2\frac{\partial^3 u^3}{\partial x^3} + 2\frac{\partial^3 u^3}{\partial y^2 x}. \tag{32}$$

Now, we solve Eq. (32) by means of VIM.
To apply VIM, we construct the following correction functional for Eq. (32):

$$u_{k+1} = u_k + \int_0^T \lambda(s)\left[\frac{\partial u_k}{\partial s} + \left(\frac{\partial \widetilde{u_k^3}}{\partial x}\right) + 2\left(\frac{\partial^3 \widetilde{u_k^3}}{\partial x^3}\right) + 2\left(\frac{\partial^3 \widetilde{u_k^3}}{\partial y^2 \partial x}\right)\right] ds. \tag{33}$$

The general Lagrange multiplier can be identified as:

$$\lambda(s) = -1. \tag{34}$$

Hence we obtain the following iteration formula:

$$u_{k+1} = u_k - \int_0^T \left[\frac{\partial u_k}{\partial s} + \left(\frac{\partial u_k^3}{\partial x}\right) + 2\left(\frac{\partial^3 u_k^3}{\partial x^3}\right) + \left(\frac{\partial^3 u_k^3}{\partial y^2 \partial x}\right)\right] ds. \tag{35}$$

Using Eq. (29) as an initial condition yields the following:

$$u_1(x,y,t) = \frac{3}{2}\rho \sinh w - 3\rho^3 \sinh w^2 \cosh w T - \frac{3}{8}\rho^3 \cosh w^3 T \tag{36}$$

$$\begin{aligned} u_2(x,y,t) = {} & \frac{3}{2}\rho \sinh w - 3\left(\frac{3}{2}\rho \sinh w - 3\rho^3 \sinh w^2 \cosh w T - \frac{3}{8}\rho^3 \cosh w^3 T\right)^2 \\ & \left(\frac{1}{4}\rho \cosh w - \frac{19}{16}\rho^3 \sinh w \cosh w^2 T - \frac{1}{2}\rho^3 \sinh w^3 T\right) T \\ & -24\left(\frac{1}{4}\rho \cosh w - \frac{19}{16}\rho^3 \sinh w \cosh w^2 T - \frac{1}{2}\rho^3 \sinh w^3 T\right)^3 T \\ & -72\left(\frac{3}{2}\rho \sinh w - 3\rho^3 \sinh w^2 \cosh w T - \frac{3}{8}\rho^3 \cosh w^3 T\right) \\ & \left(\frac{1}{4}\rho \cosh w - \frac{19}{16}\rho^3 \sinh w \cosh w^2 T - \frac{1}{2}\rho^3 \sinh w^3 T\right) \\ & \left(\frac{1}{24}\rho \sinh w - \frac{19}{96}\rho^3 \cosh w^3 T - \frac{31}{48}\rho^3 \sinh w^2 \cosh w T\right) T \\ & -12\left(\frac{3}{2}\rho \sinh w - 3\rho^3 \sinh w^2 \cosh w T - \frac{3}{8}\rho^3 \cosh w^3 T\right)^2 \\ & \left(\frac{1}{144}\rho \cosh w - \frac{181}{576}\rho^3 \sinh w \cosh w^2 T - \frac{31}{288}\rho^3 \sinh w^3 T\right) T \end{aligned} \tag{37}$$

and so on, where $w = \frac{1}{6}(x+y)$.
By the fractional complex transform

$$T = \frac{t^\alpha}{\Gamma(1+\alpha)}, \tag{38}$$

we have

$$u_1(x,y,t) = \frac{3}{2}\rho \sinh w - \frac{3\rho^3 \sinh w^2 \cosh w t^\alpha}{\Gamma(1+\alpha)} - \frac{3}{8}\frac{\rho^3 \cosh w^3 t^\alpha}{\Gamma(1+\alpha)}, \quad \vdots \tag{39}$$

and so on, where $w = \frac{1}{6}(x+y)$.
Table 2 show the solutions obtained using the three-iterates of VIM for different values of α when $\rho = 0.001$.

5. Conclusion

In this paper, we have successfully developed FCT with help of VIM to obtain approximate solution of the fractional Zakharov–Kuznetsov equation. The fractional complex transform can easily convert a fractional differential equation to its differential partner, so that its variational iteration algorithm can be simply constructed. The fractional complex transform is extremely simple but effective for solving fractional differential equations. The method is accessible to all with basic knowledge of Advanced Calculus and with little Fractional Calculus. It may be concluded that FCT–VIM is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of fractional differential equations.

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Table 1: Solutions using the 3-iterates for different values of α when $\rho = 0.001$.

			VIM for Eq. (19)			
x	y	t	$\alpha = 0.67$	$\alpha = 0.75$	$\alpha = 1$	Exact ($\alpha = 1$)
0.1	0.1	0.2	5.312992862E-5	5.325267164E-5	5.355612471E-5	5.393877159E-5
		0.3	5.285029317E-5	5.297615384E-5	5.331384269E-5	5.388407669E-5
		0.4	5.260303851E-5	5.272490734E-5	5.307396595E-5	5.382941057E-5
0.6	0.6	0.2	2.953543396E-3	2.964363202E-3	2.991347666E-3	3.036507411E-3
		0.3	2.928652795E-3	2.939926307E-3	2.969760240E-3	3.035778955E-3
		0.4	2.905913439E-3	2.917239345E-3	2.948601126E-3	3.035050641E-3
0.9	0.9	0.2	1.045289537E-2	1.064555345E-2	1.102746671E-2	1.153697757E-2
		0.3	0.990546789E-2	1.017186398E-2	1.073227877E-2	1.153454074E-2
		0.4	0.927982231E-2	0.960539982E-2	1.035600465E-2	1.153210438E-2

Table 2: Solutions using the 3-iterates for different values of α when $\rho = 0.001$.

			VIM for Eq.(32)			
x	y	t	$\alpha = 0.67$	$\alpha = 0.75$	$\alpha = 1$	Exact ($\alpha = 1$)
0.1	0.1	0.2	5.000911707E-5	5.000913646E-5	5.000918398E-5	4.995923204E-5
		0.3	5.000907252E-5	5.000909264E-5	5.000914609E-5	4.993421817E-5
		0.4	5.000903274E-5	5.000905240E-5	5.000910820E-5	4.990920434E-5
0.6	0.6	0.2	3.020038072E-4	3.020038341E-4	3.020038992E-4	3.019530008E-4
		0.3	3.020037458E-4	3.020037735E-4	3.020038472E-4	3.019274992E-4
		0.4	3.020036910E-4	3.020037181E-4	3.020037950E-4	3.019019978E-4
0.9	0.9	0.2	4.567801693E-4	4.567802061E-4	4.567802964E-4	4.567281735E-4
		0.3	4.567800847E-4	4.567801231E-4	4.567802242E-4	4.567020404E-4
		0.4	4.567800092E-4	4.567800467E-4	4.567801525E-4	4.566759074E-4

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