k-cordial Labeling of Fan and Double Fan

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Abstract: We discuss here k-cordial labeling of fans. We prove that fans f_n are k-cordial for all k. We divide the proof of the result into two parts namely odd k and even k. Moreover we prove that double fans Df_n are k-cordial for all k and $n = \frac{k+1}{2}$. The present authors are motivated by the research article entitled as 'A-cordial graphs' by A Hovey.
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1 Introduction

Throughout this work, by a graph we mean finite, connected, undirected, simple graph $G = (V(G), E(G))$ of order $|V(G)|$ and size $|E(G)|$.

Definition 1.1. A *graph labeling* is an assignment of integers to the vertices or edges or both subject to certain condition(s).
A latest survey on various graph labeling problems can be found in Gallian[2].

Definition 1.2. Let $< A, * >$ be any Abelian group. A graph $G = (V(G), E(G))$ is said to be *A-cordial* if there is a mapping $f : V(G) \to A$ which satisfies the following two conditions when the edge $e = uv$ is labeled as $f(u) * f(v)$
(i) $|v_f(a) - v_f(b)| \leq 1$; for all $a, b \in A$,
(ii) $|e_f(a) - e_f(b)| \leq 1$; for all $a, b \in A$.
Where
$v_f(a)$=the number of vertices with label a;
$v_f(b)$=the number of vertices with label b;
$e_f(a)$=the number of edges with label a;
$e_f(b)$=the number of edges with label b.

We note that if $A =< Z_k, +_k >$, that is additive group of modulo k then the labeling is known as k-cordial labeling.
The concept of A-cordial labeling was introduced by Hovey[4] and proved the following results.

- All the connected graphs are 3-cordial.
- All the trees are $3, 4, 5$-cordial.
- Cycles are k-cordial for all odd k.

Here we consider the following definitions of standard graphs.

- The *fan* f_n is $P_n + K_1$.
- The *double fan* Df_n is obtained by $P_n + 2K_1$.

2 Main Results

Theorem 2.1: Fans f_n are k-cordial for all odd k.

Proof: Let f_n be the fan and v_0 be the apex vertex. Let $n = mk + j$, where $m \geq 0$ and $1 \leq j \leq k - 1$. We divide the n path vertices of the fan f_n into two blocks of mk and j vertices namely $v_1, v_2, .., v_{mk}, v_1^{'}, v_2^{'}, .., v_j^{'}$. We note that $|V(G)| = n + 1$ and $|E(G)| = 2n - 1$.

Define k-cordial labeling $f : V(G) \rightarrow Z_k$ as follows.

$f(v_0) = 0$;

For the first block of mk vertices, if $m \geq 1$,

$$f(v_i) = \tfrac{k+p_i}{2}; \qquad i \equiv p_i (mod\, k),\ p_i \text{ is odd,}$$

$$= \tfrac{p_i}{2}; \qquad i \equiv p_i (mod\, k),\ p_i \text{ is even, } 1 \leq i \leq mk.$$

The labeling pattern of second block of j vertices, where $1 \leq j \leq k-1$ is divided into following two cases.

Case 1: $\frac{k+1}{2}$ is odd.

Sub-case I: $1 \leq j \leq \frac{k+1}{2}$.

$$f(v_i') = \tfrac{k+i}{2}; \qquad i \text{ is odd,}$$

$$= \tfrac{k-1}{4} + \tfrac{i}{2};\ i \text{ is even, } 1 \leq i \leq j.$$

Sub-case II: $\frac{k+3}{2} \leq j \leq k-2$.

$$f(v_i') = \tfrac{k+i}{2}; \qquad i \text{ is odd,}$$

$$= \tfrac{k-l}{4} + \tfrac{i}{2};\ i \text{ is even, } 1 \leq i \leq j,\ \tfrac{k+l-2}{2} \leq j \leq \tfrac{k+l}{2},$$
$$\text{where } l = 5, 9, ..., k-4.$$

Sub-case III: $j = k-1$.

$$f(v_i') = \tfrac{k+i}{2}; \qquad i \text{ is odd,}$$

$$= \tfrac{i}{2}; \qquad i \text{ is even,} 1 \leq i \leq j.$$

Case 2: $\frac{k+1}{2}$ is even.

Sub-case I: $1 \leq j \leq \frac{k+3}{2}$.

$$f(v_i') = \tfrac{k+i}{2}; \qquad i \text{ is odd,}$$

$= \frac{k-1}{4} + \frac{i}{2};\ i$ is even, $1 \leq i \leq j$.

<u>Sub-case II</u>: $\frac{k+5}{2} \leq j \leq k-2$.

$f(v_i') = \frac{k+i}{2};\quad i$ is odd,

$= \frac{k-l}{4} + \frac{i}{2};\ i$ is even, $1 \leq i \leq j,\ \frac{k+l-2}{2} \leq j \leq \frac{k+l}{2}$,

where $l = 7, 11, ..., k-4$.

<u>Sub-case III</u>: $j = k-1$.

$f(v_i') = \frac{k+i}{2};\quad i$ is odd,

$= \frac{i}{2};\quad i$ is even,$1 \leq i \leq j$.

The labeling pattern defined above covers all possible arrangement of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions for k-cordial labeling. Hence fans are k-cordial for all odd k.

Illustration 2.2 The fan graph f_{23} and its 17-cordial labeling is shown in *Figure 1*.

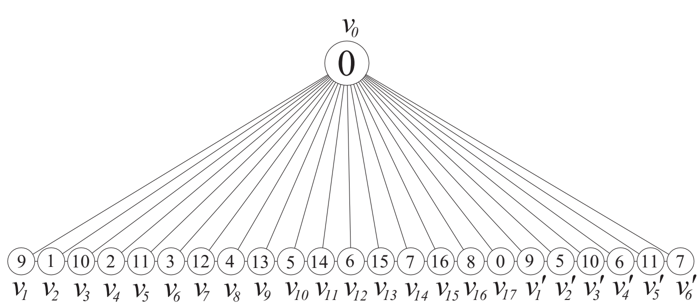

Figure 1 17-Cordial labeling of fan graph f_{23}.

Theorem 2.3: Fans f_n are k-cordial for all even k.

Proof: Let f_n be the fan and v_0 be the apex vertex. Let $n = mk + j$, where $m \geq 0$ and $1 \leq j \leq k-1$. We divide the n path vertices of the fan f_n into two blocks of mk and j vertices namely $v_1, v_2, .., v_{mk}, v_1', v_2', .., v_j'$. We note that $|V(G)| = n+1$ and $|E(G)| = 2n-1$.

To define k-cordial labeling $f : V(G) \to Z_k$ we consider the following four cases.

Case 1: m is odd and $\frac{k}{2}$ is odd.

$$f(v_0) = 0;$$

For the first block of mk vertices, if $m \geq 1$,

$$\begin{aligned} f(v_i) &= \tfrac{p_i+1}{2}; \quad i \equiv p_i (mod\ k),\ p_i \text{ is odd},\\ &= \tfrac{k}{2} + \tfrac{p_i}{2}; \quad i \equiv p_i (mod\ k),\ p_i \text{ is even},\\ &\qquad (t-1)k + 1 \leq i \leq tk,\ 1 \leq t \leq m, \text{ if } t \text{ is odd.} \end{aligned}$$

$$\begin{aligned} f(v_i) &= \tfrac{k}{2} + \tfrac{p_i+1}{2}; \quad i \equiv p_i (mod\ k),\ p_i \text{ is odd},\\ &= \tfrac{p_i}{2}; \quad i \equiv p_i (mod\ k),\ p_i \text{ is even},\\ &\qquad (t-1)k + 1 \leq i \leq tk,\ 1 \leq t \leq m, \text{ if } t \text{ is even.} \end{aligned}$$

For the second block of j vertices, where $1 \leq j \leq k-1$,

Sub-case I: $1 \leq j \leq \frac{k+4}{2}$.

$$\begin{aligned} f(v_i') &= \tfrac{k}{2} + \tfrac{i+1}{2};\ i \text{ is odd},\\ &= \tfrac{k-2}{4} + \tfrac{i}{2};\ i \text{ is even},\ 1 \leq i \leq j. \end{aligned}$$

Sub-case II: $\frac{k+6}{2} \leq j \leq k-2$.

$$\begin{aligned} f(v_i') &= \tfrac{k}{2} + \tfrac{i+1}{2};\ i \text{ is odd},\\ &= \tfrac{k-l}{4} + \tfrac{i}{2};\ i \text{ is even},\ 1 \leq i \leq j,\ \tfrac{k+l}{2} \leq j \leq \tfrac{k+l+2}{2},\\ &\qquad \text{where } l = 6, 10, ..., k-4. \end{aligned}$$

Sub-case III: $j = k-1$.

$f(v_i') = \frac{k}{2} + \frac{i+1}{2}$; i is odd, $1 \le i \le j-2$,

$= 1$; i is odd, $i = j$.

$= \frac{i}{2} + 1$; i is even, $1 \le i \le j$.

Case 2: m is odd and $\frac{k}{2}$ is even.

$f(v_0) = 0$;
For the first block of mk vertices, if $m \ge 1$,

$f(v_i) = \frac{p_i+1}{2}$; $i \equiv p_i (mod\ k)$, p_i is odd,

$= \frac{k}{2} + \frac{p_i}{2}$; $i \equiv p_i (mod\ k)$, p_i is even,

$(t-1)k + 1 \le i \le tk$, $1 \le t \le m$, if t is odd.

$f(v_i) = \frac{k}{2} + \frac{p_i+1}{2}$; $i \equiv p_i (mod\ k)$, p_i is odd,

$= \frac{p_i}{2}$; $i \equiv p_i (mod\ k)$, p_i is even,
$(t-1)k + 1 \le i \le tk$, $1 \le t \le m$, if t is even.

For the second block of j vertices, where $1 \le j \le k-1$,

Sub-case I: $1 \le j \le \frac{k+2}{2}$.

$f(v_i') = \frac{k}{2} + \frac{i+1}{2}$; i is odd,

$= \frac{k}{4} + \frac{i}{2}$; i is even, $1 \le i \le j$.

Sub-case II: $\frac{k+4}{2} \le j \le k-2$.

$f(v_i') = \frac{k}{2} + \frac{i+1}{2}$; i is odd,

$= \frac{k-l}{4} + \frac{i}{2}$; i is even, $1 \le i \le j$, $\frac{k+l}{2} \le j \le \frac{k+l+2}{2}$,
where $l = 4, 8, ..., k-4$.

Sub-case III: $j = k - 1$.

$$f(v_i') = \frac{k}{2} + \frac{i+1}{2}; \quad i \text{ is odd}, 1 \leq i \leq j-2,$$
$$= 1; \quad i \text{ is odd}, i = j.$$

$$= \frac{i}{2} + 1; \quad i \text{ is even}, 1 \leq i \leq j.$$

Case 3: m is even and $\frac{k}{2}$ is odd.

$f(v_0) = 0;$

For the first block of mk vertices, if $m \geq 1$,

$$f(v_i) = \frac{k}{2} + \frac{p_i+1}{2}; \quad i \equiv p_i (mod\, k), p_i \text{ is odd},$$

$$= \frac{p_i}{2}; \quad i \equiv p_i (mod\, k), p_i \text{ is even},$$

$$(t-1)k + 1 \leq i \leq tk, 1 \leq t \leq m, \text{ if } t \text{ is odd.}$$

$$f(v_i) = \frac{p_i+1}{2}; \quad i \equiv p_i (mod\, k), p_i \text{ is odd},$$

$$= \frac{k}{2} + \frac{p_i}{2}; \quad i \equiv p_i (mod\, k), p_i \text{ is even},$$
$$(t-1)k + 1 \leq i \leq tk, 1 \leq t \leq m, \text{ if } t \text{ is even.}$$

For the second block of j vertices, where $1 \leq j \leq k-1$,

Sub-case I: $1 \leq j \leq \frac{k+4}{2}$.

$$f(v_i') = \frac{k}{2} + \frac{i+1}{2}; \quad i \text{ is odd},$$

$$= \frac{k-2}{4} + \frac{i}{2}; \quad i \text{ is even}, 1 \leq i \leq j.$$

Sub-case II: $\frac{k+6}{2} \leq j \leq k - 2$.

$$f(v_i') = \frac{k}{2} + \frac{i+1}{2}; \quad i \text{ is odd},$$

$$= \frac{k-l}{4} + \frac{i}{2}; \quad i \text{ is even}, 1 \leq i \leq j, \frac{k+l}{2} \leq j \leq \frac{k+l+2}{2},$$
$$\text{where } l = 6, 10, ..., k-4.$$

Sub-case III: $j = k - 1$.

$f(v_i') = \frac{k}{2} + \frac{i+1}{2}$; i is odd, $1 \leq i \leq j-2$,

$= 1$; i is odd, $i = j$.
$= \frac{i}{2} + 1$; i is even, $1 \leq i \leq j$.

Case 4: m is even and $\frac{k}{2}$ is even.
$f(v_0) = 0$;
For the first block of mk vertices, if $m \geq 1$,

$f(v_i) = \frac{k}{2} + \frac{p_i+1}{2}$; $i \equiv p_i (mod\, k)$, p_i is odd,

$= \frac{p_i}{2}$; $i \equiv p_i (mod\, k)$, p_i is even,
$(t-1)k + 1 \leq i \leq tk$, $1 \leq t \leq m$, if t is odd.

$f(v_i) = \frac{p_i+1}{2}$; $i \equiv p_i (mod\, k)$, p_i is odd,

$= \frac{k}{2} + \frac{p_i}{2}$; $i \equiv p_i (mod\, k)$, p_i is even,
$(t-1)k + 1 \leq i \leq tk$, $1 \leq t \leq m$, if t is even.

For the second block of j vertices, where $1 \leq j \leq k-1$,

Sub-case I: $1 \leq j \leq \frac{k+2}{2}$.

$f(v_i') = \frac{k}{2} + \frac{i+1}{2}$; i is odd,

$= \frac{k}{4} + \frac{i}{2}$; i is even, $1 \leq i \leq j$.

Sub-case II: $\frac{k+4}{2} \leq j \leq k-2$.

$f(v_i') = \frac{k}{2} + \frac{i+1}{2}$; i is odd,

$= \frac{k-l}{4} + \frac{i}{2}$; i is even, $1 \leq i \leq j$, $\frac{k+l}{2} \leq j \leq \frac{k+l+2}{2}$,
where $l = 4, 8, ..., k-4$.

Sub-case III: $j = k-1$.

$f(v_i^{'}) = \frac{k}{2} + \frac{i+1}{2}; \quad i \text{ is odd}, 1 \leq i \leq j-2,$

$\quad = 1; \quad i \text{ is odd}, i = j.$
$\quad = \frac{i}{2} + 1; \quad i \text{ is even}, 1 \leq i \leq j.$

The labeling pattern defined above covers all possible arrangement of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions for k-cordial labeling. Hence fans are k-cordial for all even k.
Illustration 2.4 The fan graph f_{28} and its 12- cordial labeling is shown in *Fig 2*.

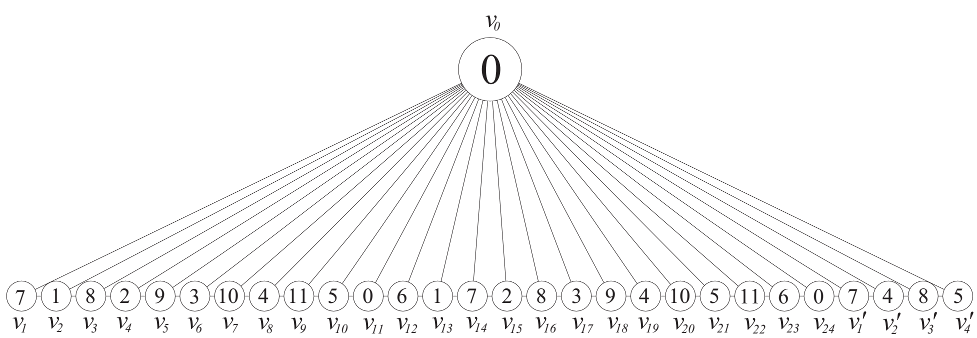

Fig 2 12-Cordial labeling of fan graph f_{28}.

Theorem 2.5 All the double fans Df_n are k-cordial for all k and $n = \frac{k+1}{2}$.

Proof: Let Df_n be the double fan. Let $v_1, v_2, ..., v_n$ be the path vertices and v_0 and $v_0^{'}$ be the two apex vertices of double fan Df_n. We note that $|V(G)| = n + 2$ and $|E(G)| = 3n - 1$.

To define k-cordial labeling $f : V(G) \rightarrow Z_k$ we consider the following two cases.
Case 1: k is odd.

Sub-case I: $\frac{k+1}{2}$ is odd.

$f(v_0) = 0;$
$f(v_0^{'}) = (k-1);$
$f(v_i) = \frac{k+i}{2}; \quad i \text{ is odd},$

$= \frac{k-1+2i}{4}; \quad i$ is even, $1 \leq i \leq n$.

Sub-case II: $\frac{k+1}{2}$ is even.

$f(v_0) = 0;$
$f(v_0^{'}) = (k-1);$
$f(v_i) = \frac{k+i}{2}; \qquad i$ is odd,

$= \frac{k-3+2i}{4}; \quad i$ is even, $1 \leq i \leq n$.

Case 2: k is even.

Sub-case I: $\frac{k}{2}$ is odd.

$f(v_0) = 0;$
$f(v_0^{'}) = (k-1);$
$f(v_i) = \frac{k-1+i}{2}; \quad i$ is odd,

$= \frac{k-2+2i}{4}; \quad i$ is even, $1 \leq i \leq n$.

Sub-case II: $\frac{k}{2}$ is even.

$f(v_0) = 0;$
$f(v_0^{'}) = (k-1);$
$f(v_i) = \frac{k-1+i}{2}; \quad i$ is odd,

$= \frac{k+2i}{4} - 1;$ i is even, $1 \leq i \leq n$.

The labeling pattern defined above covers all possible arrangement of vertices. In each possibility the graph under consideration satisfies the vertex conditions and edge conditions for k-cordial labeling.Hence the graph of double fan admits k-cordial labeling.

Illustration 2.6 The double fan Df_{22} and its 43-cordial labeling is shown in *Fig 3*.

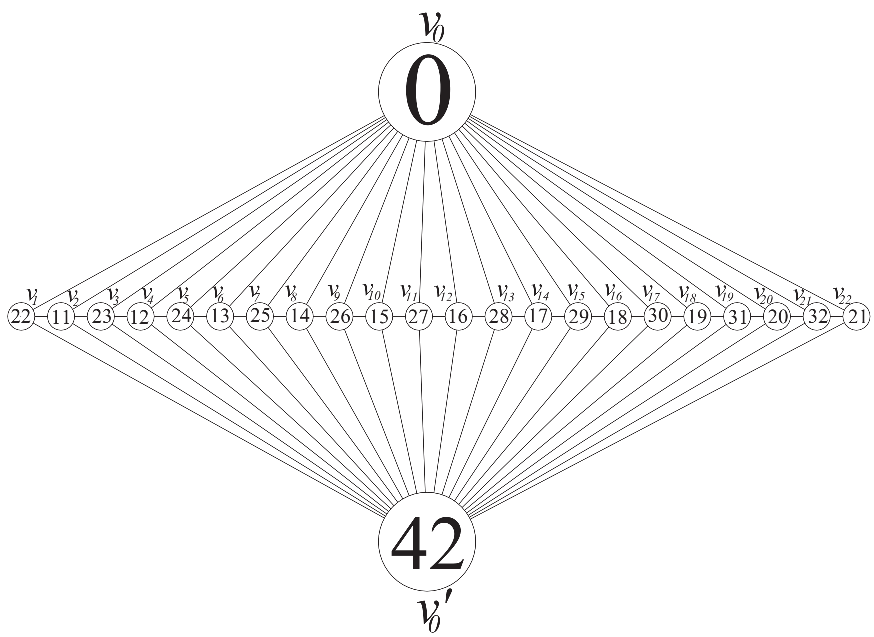

Fig 4 43-Cordial labeling of double fan Df_{22}.

Concluding Remarks

Here we have contributed general result for fan and a particular result for double fan to the theory of k-cordial labeling. To derive similar results for other graph families is an open problem.

References

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