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Research Paper

Singular Values of One Parameter Family $\lambda\left(\frac{e^z-1}{z}\right)^m$

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Abstract

In the present paper, the singular values of one parameter family of entire functions $f_\lambda(z) = \lambda\left(\frac{e^z-1}{z}\right)^m$ and $f_\lambda(0) = \lambda$, $m \in \mathbb{N}\backslash\{0\}$, $\lambda \in \mathbb{R}\backslash\{0\}$, $z \in \mathbb{C}$ is investigated. It is shown that all the critical values of $f_\lambda(z)$ lie in the left half plane. It is also found that the function $f_\lambda(z)$ possesses infinitely many bounded singular values and lie inside the open disk centered at origin and having radius $|\lambda|$.

Keywords: *Critical values; Singular values.*

1. Introduction

It is known that if singular values exist in entire or transcendental functions, then the study of dynamical properties of such types of functions become more crucial. The dynamics of one parameter family λe^z that has only one singular value is studied in detail [1, 2]. This exponential family is simpler than other families which have more than one or infinitely many singular values. The dynamical properties of families of functions involving exponential maps are described in [3, 4, 5, 6]. The singular values of one parameter families of functions are discussed in [7, 8, 9]. The singular values of one parameter family of entire function $\frac{e^z-1}{z}$ are found in [10].

This work describes the singular values of one parameter family of function $\left(\frac{e^z-1}{z}\right)^m$ which is a generalization of $\frac{e^z-1}{z}$, where m is nonzero positive integer. The following one parameter family of transcendental entire functions which are neither even nor odd and not periodic, is considered:

$$\mathcal{T} = \left\{ f_\lambda(z) = \lambda\left(\frac{e^z-1}{z}\right)^m \text{ and } f_\lambda(0) = \lambda : m \in \mathbb{N}\backslash\{0\}, \lambda \in \mathbb{R}\backslash\{0\}, z \in \mathbb{C} \right\}$$

A point z^* is said to be a critical point of $f(z)$ if $f'(z^*) = 0$. The value $f(z^*)$ corresponding to a critical point z^* is called a critical value of $f(z)$. A point $w \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is said to be an asymptotic value for $f(z)$, if there exists a continuous curve $\gamma : [0, \infty) \to \hat{\mathbb{C}}$ satisfying $\lim_{t\to\infty} \gamma(t) = \infty$ and $\lim_{t\to\infty} f(\gamma(t)) = w$. A singular value of f is defined to be either a critical value or an asymptotic value of f. A function f is called critically bounded or it is said to be a function of bounded type if the set of all singular values of f is bounded, otherwise unbounded-type. The importance of singular values of transcendental entire or meromorphic functions can be seen in [11, 12, 13].

The organization of this paper is as follows: It is found that, in Theorem 2.1, the function $f_\lambda \in \mathcal{T}$ has no critical points in the right half plane and $f_\lambda(z)$ maps the left half plane inside the open disk. It is proved that the function $f_\lambda \in \mathcal{T}$ has infinitely many singular values in Theorem 2.2. In Theorem 2.3, it is shown that all the singular values of $f_\lambda \in \mathcal{T}$ are bounded and lie inside the open disk centered at origin and having radius $|\lambda|$.

2. Bounded singular values of $f_\lambda \in \mathcal{T}$

Suppose that $H^+ = \{z \in \hat{\mathbb{C}} : Re(z) > 0\}$ and $H^- = \{z \in \hat{\mathbb{C}} : Re(z) < 0\}$ are the right half and left half planes respectively. The following theorem describes that the function $f_\lambda \in \mathcal{T}$ has no critical points in the right half plane and $f_\lambda(z)$ maps the left half plane inside the open disk centered at origin and having radius $|\lambda|$:

Theorem 2.1 *Let $f_\lambda \in \mathcal{T}$. Then, $f'_\lambda(z)$ has no zeros in the right half plane H^+ and $f_\lambda(z)$ maps the left half plane H^- inside the open disk centered at origin and having radius $|\lambda|$.*

Proof. Since $f'_\lambda(z) = \lambda m(\frac{e^z-1}{z})^{m-1}\frac{(z-1)e^z+1}{z^2} = 0$, which implies $e^z = 1$ and $e^{-z} = 1-z$. Then, first equation gives $z = 2p\pi i$, where p is an integer; and for second equation, the real and imaginary parts are

$$e^{-x}\cos(y) = 1-x \tag{1}$$

$$e^{-x}\sin(y) = y \tag{2}$$

If $y = 0$, then, by Eq. (1), $e^x = \dfrac{1}{1-x}$. For $x > 1$, this is not valid since $e^x > 0$ and $\dfrac{1}{1-x} < 0$. When $x = 1$, it is trivially not hold . This is also false for $0 < x < 1$.

If $y \neq 0$, then, by Eq. (2), $\dfrac{\sin(y)}{y} = e^x$. This is not true for $y > 0$ because $\dfrac{\sin(y)}{y} < 1$ and $e^x > 1$ for $x > 0$. Since $\dfrac{\sin(y)}{y}$ is an even function, it is also not hold for $y < 0$.

Therefore, it shows that the function $f'_\lambda(z)$ has no zeros in H^+.

Now, suppose that the line segment γ is defined by $\gamma(t) = tz$, $t \in [0,1]$. Further, let the function $g(z) = e^z$ for an arbitrary fixed $z \in \mathbb{C}$. Since $M \equiv \max_{t\in[0,1]} |g(\gamma(t))| = \max_{t\in[0,1]} |e^{tz}| < 1$ for $z \in H^-$, then

$$\int_\gamma g(z)dz = \int_0^1 g(\gamma(t))\gamma'(t)dt = z\int_0^1 e^{tz}dt = e^z - 1$$

$$|e^z - 1| = \Big|\int_\gamma g(z)dz\Big| \leq M|z| < |z|$$

$$\left|\left(\frac{e^z-1}{z}\right)^m\right| = \left|\frac{e^z-1}{z}\right|^m < 1 \text{ for all } z \in H^-$$

Consequently, we get

$$|f_\lambda(z)| = \left|\lambda\left(\frac{e^z-1}{z}\right)^m\right| < |\lambda| \text{ for all } z \in H^-.$$

Since $f_\lambda(2p\pi i) = 0$, this proves that the function $f_\lambda(z)$ maps H^- inside the open disk centered at origin and having radius $|\lambda|$. The proof of the theorem is completed.

The function $f_\lambda \in \mathcal{T}$ has infinitely many singular values which is shown in the following theorem:

Theorem 2.2 *Let $f_\lambda \in \mathcal{T}$. Then, the function $f_\lambda(z)$ has infinitely many singular values.*

Proof. For critical points, $f'_\lambda(z) = \lambda m(\frac{e^z-1}{z})^{m-1}\frac{(z-1)e^z+1}{z^2} = 0$. This gives the equations $e^z - 1 = 0$ and $(z-1)e^z + 1 = 0$. First equation gives $z = 2p\pi i$, then $f_\lambda(2p\pi i) = 0$. After simplifying, second equation gives

$$\frac{y}{\sin(y)} - e^{y\cot(y)-1} = 0 \tag{3}$$

$$x = 1 - y\cot(y) \tag{4}$$

From Fig. 1, it shows that Eq. (3) has infinitely many solutions. The theoretical solutions of Eq. (3) are given in [10]. Suppose that the solutions of Eq. (3) are $\{y_k\}_{k=-\infty,k\neq 0}^{k=\infty}$. From Eq. (4), we have $x_k = 1 - y_k\cot(y_k)$ for

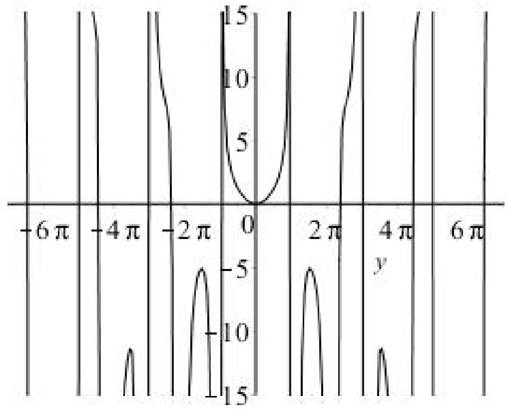

Figure 1: Graph of $\frac{y}{\sin(y)} - e^{y\cot(y)-1}$

$k = \pm1, \pm2, \pm3, \ldots$. It follows that $z_k = x_k + iy_k$ for k nonzero integer, are critical points of $f_\lambda(z)$ and the critical values, $f_\lambda(z_k) = \lambda\left(\dfrac{e^{z_k}-1}{z_k}\right)^m$, are different for distinct k. It shows that $f_\lambda \in \mathcal{T}$ has infinitely many critical values.

The finite asymptotic value of $f_\lambda \in \mathcal{T}$ is 0 since $f_\lambda(z)$ tends to 0 as z tends to infinity along negative real axis. Therefore, $f_\lambda \in \mathcal{T}$ possesses infinitely many singular values. This completes the proof.

In the following theorem, it is proved that the function $f_\lambda \in \mathcal{T}$ has bounded singular values:

Theorem 2.3 *Let $f_\lambda \in \mathcal{T}$. Then, all the singular values of $f_\lambda(z)$ are bounded and lie inside the open disk centered at origin and having radius $|\lambda|$.*

Proof. By Theorem 2.1, the function $f'_\lambda(z)$ has no zeros in H^+. Hence, all the critical points lie in H^-. Moreover, the function $f_\lambda(z)$ maps H^- inside the open disk centered at origin and having radius $|\lambda|$. It gives that all the critical values of $f_\lambda \in \mathcal{T}$ are lying inside the open disk centered at origin and having radius $|\lambda|$.

Since $f_\lambda \in \mathcal{T}$ has only one asymptotic value 0, consequently all the singular values of $f_\lambda \in \mathcal{T}$ are bounded and lie inside the open disk centered at origin and having radius $|\lambda|$. This proves the theorem.

It is seen that, for $m < 0$, the family of entire functions $\mathcal{T}$ will become family of meromorphic functions with infinitely many poles. It is another work for author.

3. Conclusion

In this work, the singular values of one parameter family of entire functions are described which is a generalization of investigated one parameter family of functions [10]. It was concluded that this family of functions has infinitely many bounded singular values and all these lie in the left half plane inside the open disk centered at origin.

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