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A Chebyshev wavelet operational method for solving stochastic Volterra-Fredholm integral equations

Fakhrodin Mohammadi

Department of Mathematics, Hormozgan University, P. O. Box 3995, Bandarabbas, Iran
E-mail: f.mohammadi62@hotmail.com

Abstract

In this paper, the stochastic operational matrix of Itô-integration for the Chebyshev wavelets is applied for solving stochastic Volterra-Fredholm integral equations. The main characteristic of the presented method is that it reduces stochastic Volterra-Fredholm integral equations into a linear system of equations. Convergence and error analysis of the Chebyshev wavelets basis is considered. The efficiency and accuracy of the proposed method was demonstrated by some non-trivial examples and comparison with the other existing methods.

Keywords: *Chebyshev wavelets, Itô integral, Brownian motion process, Stochastic Volterra-Fredholm integral equations, Stochastic operational matrix.*

1. Introduction

Stochastic analysis has been an interesting research area in mathematics, fluid mechanics, geophysics, biology, chemistry, epidemiology, microelectronics, theoretical physics, economics, and finance. The behavior of dynamical systems in these fields are often dependent on a noise source and a Gaussian white noise, governed by certain probability laws. This noise might be either due to thermal fluctuations, noise in somecontrol parameter, coarse-graining of a high-dimensional deterministic system with random initial conditions or the stochastic parameterization of small scales. The dynamical systems subject to noise can be modeled accurately using stochastic differential equations, stochastic integral equations, stochastic integro-differential equations or in more complicated cases stochastic partial differential equation[1, 2, 3, 4, 5, 6, 7, 8].

Since in many cases it is difficult to derive an explicit form of the solution of stochastic functional equations numerical approximation becomes a practical way to face this problem. Recently, many studies have been appeared which describe numerical solution of stochastic differential and integral equations [1, 4, 7, 8, 9, 3, 10, 11, 12, 13].

As a powerful tool, wavelets have been extensively used in signal processing, numerical analysis, and many other areas. Wavelets permit the accurate representation of a variety of functions and operators [14, 15, 17, 18, 19, 20, 21, 16]. In this paper, an stochastic operational matrix for the Chebyshev wavelets is derived. Then application of this stochastic operational matrix in solving stochastic Volterra-Fredholm integral equations is investigated. Some non-trivial examples are included to demonstrate the efficiency and accuracy of the proposed method. Also to verify the proposed method, numerical results are compared with the block pulse functions (BPFs) methed presented in [8].

This paper is organized as follows: In section 2 some basic definition and preliminaries about stochastic process

and Itô integral are presented. The Chebyshev wavelets and their properties are introduced in section 3. In section 4 stochastic operational matrix of the Chebyshev wavelets is derived. In section 5 application of this stochastic operational matrix in solving stochastic Voltera-Fredholm integral equations are described. In section 6 the efficiency of the proposed method is demonstrated by some non-trivial examples. Finally, a conclusion is given in section 7.

2. Preliminaries

In this section we review some basic definition of the stochastic calculus and the block pulse functions (BPFs).

2.1. Stochastic calculus

Definition 2.1. *(Brownian motion process) A real-valued stochastic process $B(t), t \in [0,T]$ is called Brownian motion, if it satisfies the following properties*
(i) The process has independent increments for $0 \leq t_0 \leq t_1 \leq ... \leq t_n \leq T$,
(ii) For all $t \geq 0$, $B(t+h) - B(t)$ has Normal distribution with mean 0 and variance h,
(iii) The function $t \rightarrow B(t)$ is continuous functions of t.

Definition 2.2. *Let $\{\mathcal{N}_t\}_{t \geq 0}$ be an increasing family of σ-algebras of subsets of Ω. A process $g(t, \omega) : [0, \infty) \times \Omega \rightarrow \mathbb{R}^n$ is called $\mathcal{N}_t$-adapted if for each $t \geq 0$ the function $\omega \rightarrow g(t, \omega)$ is $\mathcal{N}_t$-measurable.*

Definition 2.3. *Let $\mathcal{V} = \mathcal{V}(S,T)$ be the class of functions $f(t, \omega) : [0, \infty) \rightarrow \Omega \times R$ such that*
(i) The function $(t, \omega) \rightarrow f(t, \omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel algebra on $[0, \infty)$ and $\mathcal{F}$ is the σ-algebra on Ω.
(ii) f is adapted to $\mathcal{F}_t$, where $\mathcal{F}_t$ is the σ -algebra generated by the random variables $B(s), s \leq t$.
(iii)$E\left(\int_S^T f^2(t, \omega) dt \right) < \infty$.

Definition 2.4. *(The Itô integral) Let $f \in \mathcal{V}(S,T)$, then the Itô integral of f is defined by*

$$\int_S^T f(t, \omega) dB_t(\omega) = \lim_{n \to \infty} \int_S^T \varphi_n(t, \omega) dB_t(\omega), \quad (\lim \ in \ L^2(P))$$

where, φ_n is a sequence of elementary functions such that

$$E\left(\int_s^T (f(t, \omega) - \varphi_n(t, \omega))^2 \, dt \right) \to 0, \quad as \quad n \to \infty.$$

For more details about stochastic calculus and integration please see [2].

2.2. Block pulse functions

BPFs have been studied by many authors and applied for solving different problems. In this section we recall definition and some properties of the block pulse functions [7, 8, 24].

The m-set of BPFs are defined as

$$b_i(t) = \begin{cases} 1 & (i-1)h \leq t < ih \\ 0 & otherwise \end{cases} \tag{1}$$

in which $t \in [0, T)$, $i = 1, 2, ..., m$ and $h = \frac{T}{m}$. The set of BPFs are disjointed with each other in the interval $[0, T)$ and

$$b_i(t)b_j(t) = \delta_{ij} b_i(t), i, j = 1, 2, ..., m, \tag{2}$$

where δ_{ij} is the Kronecker delta. The set of BPFs defined in the interval $[0, T)$ are orthogonal with each other, that is

$$\int_0^T b_i(t)b_j(t)dt = h\delta_{ij}, \quad i, j = 1, 2, ..., m. \tag{3}$$

If $m \to \infty$ the set of BPFs is a complete basis for $L^2[0, T]$, so an arbitrary real bounded function $f(t)$, which is square integrable in the interval $[0, T)$, can be expanded into a block pulse series as

$$f(t) \simeq \sum_{i=1}^{m} f_i b_i(t), \tag{4}$$

where

$$f_i = \frac{1}{h} \int_0^T b_i(t) f(t) dt, \quad i = 1, 2, ..., m. \tag{5}$$

Rewritting Eq. (4) in the vector form we have

$$f(t) \simeq \sum_{i=1}^{m} f_i b_i(t) = F^T \Phi(t) = \Phi^T(t) F, \tag{6}$$

in which

$$\Phi(t) = [b_1(t), b_2(t), ..., b_m(t)]^T,$$
$$F = [f_1, f_2, ..., f_m]^T. \tag{7}$$

Morever, any two dimensional function $k(s, t) \in L^2([0, T_1] \times [0, T_2])$ can be expanded with respect to BPFs such as

$$k(s, t) = \Phi^T(t) \Pi \Phi(t), \tag{8}$$

where $\Phi(t)$ is the m-dimensional BPFs vectors respectively, and Π is the $m \times m$ BPFs coefficient matrix with (i, j)-th element

$$\Pi_{ij} = \frac{1}{h_2 h_2} \int_0^{T_1} \int_0^{T_2} k(s, t) b_i(t) b_j(s) dt ds, \quad i, j = 1, 2, ..., m, \tag{9}$$

and $h_1 = \frac{T_1}{m}$ and $h_2 = \frac{T_2}{m}$. Let $\Phi(t)$ be the BPFs vector, then we have

$$\Phi^T(t) \Phi(t) = 1, \tag{10}$$

and

$$\Phi(t) \Phi^T(t) = \begin{pmatrix} b_1(t) & 0 & \cdots & 0 \\ 0 & b_2(t) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & b_m(t) \end{pmatrix}_{m \times m}. \tag{11}$$

For an m-vector F we have

$$\Phi(t) \Phi^T(t) F = \tilde{F} \Phi(t), \tag{12}$$

where $\tilde{F}$ is an $m \times m$ matrix, and $\tilde{F} = diag(F)$. Also, it is easy to show that for an $m \times m$ matrix A

$$\Phi^T(t) A \Phi(t) = \hat{A}^T \Phi(t), \tag{13}$$

where $\hat{A} = diag(A)$ is an m-vector.

3. Chebyshev wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function ψ called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets

$$\psi_{a,b}(t) = a^{-\frac{1}{2}} \psi\left(\frac{t-b}{a}\right), \quad a, b \in \mathbb{R}, \, a \neq 0. \tag{14}$$

The Chebyshev wavelets $\psi_{nm}(x) = \psi(k, n, m, x)$ are defined on the interval $[0, 1)$ by

$$\psi_{nm}(t) = \begin{cases} 2^{\frac{k+1}{2}} \tilde{T}_m(2^k t - (2n + 1)), & \frac{n}{2^k} \leq x \leq \frac{n+1}{2^k} \\ 0, & otherwise \end{cases} , \tag{15}$$

where

$$\tilde{T}_m(t) = \begin{cases} \frac{1}{\sqrt{\pi}}, & m = 0 \\ \sqrt{\frac{2}{\pi}} T_m(t), & m > 0 \end{cases} ,$$

and $T_m(t)$ are the well-known Chebyshev polynomials of degree m. The Chebyshev wavelets $\{\psi_{nm}(x) | n = 0, 1, \ldots, 2^k - 1, m = 0, 1, 2, \ldots, M - 1\}$ forms an orthonormal basis for $L^2_{w_n}[0, 1]$ with respect to the weight function $w_n(t) = w(2^{k+1}t - (2n + 1))$, in which $w(t) = \frac{1}{\sqrt{1-t^2}}$ [19, 20].

By using the orthonormality of the Chebyshev wavelets, any function $f(t)$ over $[0, 1)$; square-integrable with respect to the measure $\mathbf{w}(t)dt$; with $\mathbf{w}(t) = w_{nk}(t)$; for $\frac{n}{2^k} \leq t \leq \frac{n+1}{2^k}$; and $w_{nk}(t) = w(2^{k+1}t - 2n + 1)$; being $w(t) = \frac{1}{\sqrt{1-t^2}}$ can be expanded in terms of the Chebyshev wavelets as

$$f(t) \simeq \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(t) = C^T \Psi(t), \tag{16}$$

where $c_{mn} = (f(t), \psi_{mn}(t))_{w_{nk}}$ and $(.,.)_{w_{nk}}$ denotes the inner product on $L^2_{w_{nk}}[0, 1]$. If the infinite series in (16) is truncated, then it can be written as

$$f(t) \simeq \sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{mn} \psi_{mn}(x) = C^T \Psi(t), \tag{17}$$

where C and $\Psi(t)$ are $\hat{m} = 2^k M$ column vectors given by

$$C = \left[c_{00}, \ldots, c_{0(M-1)} | c_{10}, \ldots, c_{1(M-1)} |, \ldots, | c_{(2^k-1)0}, \ldots, c_{(2^k-1)(M-1)} \right]^T,$$

$$\Psi(x) = \left[\psi_{00}(t), \ldots, \psi_{0(M-1)}(t) | \psi_{10}(t), \ldots, \psi_{1(M-1)}(t) |, \ldots, | \psi_{(2^k-1)0}(t), \ldots, \psi_{(2^k-1)(M-1)}(t) \right]^T.$$

By changing indices in the vectors $\Psi(t)$ and C the series (4) can be rewritten as

$$f(t) \simeq \sum_{i=1}^{\hat{m}} c_i \psi_i(t) = C^T \Psi(t), \tag{18}$$

where

$$C = [c_1, c_2, \ldots, c_{\hat{m}}], \quad \Psi(x) = [\psi_1(x), \psi_2(x), \ldots, \psi_{\hat{m}}(x)],$$

and

$$c_i = c_{nm}, \quad \psi_i(t) = \psi_{nm}(t), \quad i = (n - 1)M + m + 1.$$

Similarly, any two dimensional function $k(s, t) \in L^2_{\mathbf{w} \otimes \mathbf{w}}([0, 1] \times [0, 1])$ can be expanded into Chebyshev wavelets basis as

$$k(s, t) \approx \sum_{i=1}^{\hat{m}} \sum_{j=1}^{\hat{m}} k_{ij} \psi_i(s) \psi_j(t) = \Psi^T(s) K \Psi(t), \tag{19}$$

where $K = [k_{ij}]$ is an $\hat{m} \times \hat{m}$ matrix and $k_{ij} = \left(\psi_i(s), (k(s, t), \psi_j(t))_{w_{nk}} \right)_{w_{nk}}$.

3.1. Chebyshev wavelets and BPFs

In this section we will review the relation between the Chebyshev wavelets and BPFs. It is worth mention that here we set $T = 1$ in definition of BPFs.

Theorem 3.1. *Let $\Psi(t)$ and $\Phi(t)$ be the $\hat{m}$-dimensional Chebyshev wavelets and BPFs vector respectively, the vector $\Psi(t)$ can be expanded by BPFs vector $\Phi(t)$ as*

$$\Psi(t) \simeq Q\Phi(t), \tag{20}$$

where Q is an $\hat{m} \times \hat{m}$ block matrix and

$$Q_{ij} = \psi_i \left(\frac{2j-1}{2\hat{m}} \right), i, j = 1, 2, ..., \hat{m} \tag{21}$$

Proof. Let $\phi_i(t), i = 1, 2, ..., \hat{m}$ be the i-th element of Chebyshev wavelets vector. Expanding $\phi_i(t)$ into an $\hat{m}$-term vector of BPFs, we have

$$\psi_i(t) \simeq \sum_{i=1}^{\hat{m}} Q_{ij} b_j(t) = Q_i^T \Phi(t), \quad i = 1, 2, ..., \hat{m}, \tag{22}$$

where Q_i is the i-th row and Q_{ij} is the (i, j)-th element of matrix Q. By using the orthogonality of BPFs we have

$$Q_{ij} = \frac{1}{h} \int_0^1 \psi_i(t) b_j(t) dt = \frac{1}{h} \int_{\frac{j-1}{\hat{m}}}^{\frac{j}{\hat{m}}} \psi_i(t) dt = \hat{m} \int_{\frac{j-1}{\hat{m}}}^{\frac{j}{\hat{m}}} \psi_i(t) dt, \tag{23}$$

by using mean value theorem for integrals in the last equation we can write

$$Q_{ij} = \hat{m} \left(\frac{j}{\hat{m}} - \frac{j-1}{\hat{m}} \right) \psi_i(\eta_i) = \psi_i(\eta_j), \quad \eta_j \in \left(\frac{j-1}{\hat{m}}, \frac{j}{\hat{m}} \right), \tag{24}$$

now by choosing $\eta_j = \frac{2j-1}{2\hat{m}}$ we have

$$Q_{ij} = \psi_i \left(\frac{2j-1}{2\hat{m}} \right), i, j = 1, 2, ..., \hat{m}. \tag{25}$$

and this prove the desired result. $\qquad \square$

The following Remark is the consequence of relations (12), (13) and Theorem 3.1.

Remark 3.2. *For an $\hat{m}$-vector F we have*

$$\Psi(t) \Psi^T(t) F = \tilde{F} \Psi(t), \tag{26}$$

in which $\tilde{F}$ is an $\hat{m} \times \hat{m}$ matrix as

$$\tilde{F} = Q \bar{F} Q^{-1}, \tag{27}$$

where $\bar{F} = diag\left(Q^T F \right)$. Moreover, it can be easy to show that for an $\hat{m} \times \hat{m}$ matrix A

$$\Psi^T(t) A \Psi(t) = \hat{A}^T \Psi(t), \tag{28}$$

where $\hat{A}^T = U Q^{-1}$ and $U = diag(Q^T A Q)$ is a $\hat{m}$-vector.

3.2. Convergence analysis

Here we investigate the convergence and error analysis of the Chebyshev wavelets basis.

Theorem 3.3. *Suppose* $f(x) \in L^2_{w_n}[0,1]$ *with bounded second derivative* $|f''| \leq L$, *and let* $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn} \psi_{mn}(x)$ *be its infinite Chebyshev wavelets expansion, then*

$$|c_{mn}| \leq \frac{\sqrt{2\pi}L}{(2n)^{\frac{5}{2}}(m^2 - 1)}, \tag{29}$$

this means the Chebyshev wavelets series converges uniformly to $f(x)$ *and*

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(x), \tag{30}$$

Proof. See [23]. □

Theorem 3.4. *Let* $f(x)$ *be a continuous function defined on* $[0,1)$, *with second derivatives* $f''(x)$ *bounded by* L, *then we have the following accuracy estimation*

$$\sigma_{M,k} \leq \left(\frac{\pi L^2}{2^4} \sum_{n=0}^{\infty} \sum_{m=M}^{\infty} \frac{1}{n^5(m^2-1)^2} + \frac{\pi L^2}{2^4} \sum_{n=2^k}^{\infty} \sum_{m=0}^{M-1} \frac{1}{n^5(m^2-1)^2} \right)^{\frac{1}{2}}, \tag{31}$$

where

$$\sigma_{M,k} = \left(\int_0^1 \left(f(x) - \sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(x) \right)^2 dx \right)^{\frac{1}{2}}.$$

Proof. We have

$$\sigma_{M,k}^2 = \int_0^1 \left(f(x) - \sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(x) \right)^2 dx$$

$$= \int_0^1 \left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(x) - \sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(x) \right)^2 dx$$

$$= \sum_{n=0}^{\infty} \sum_{m=M}^{\infty} c_{nm}^2 \int_0^1 \psi_{nm}^2(x) dx + \sum_{n=2^k}^{\infty} \sum_{m=0}^{M-1} c_{nm}^2 \int_0^1 \psi_{nm}^2(x) dx = \sum_{n=0}^{\infty} \sum_{m=M}^{\infty} c_{nm}^2 + \sum_{n=2^k}^{\infty} \sum_{m=0}^{M-1} c_{nm}^2,$$

now by considering the relation (29) the desired result is achieved. □

4. Stochastic operational matrix of Chebyshev wavelets

In this section we derive an stochastic operational matrix for Chebyshev wavelets. For this purpose we first remind some useful results for BPFs[7, 8].

Lemma 4.1. *[7] Let* $\Phi(t)$ *be the* $\hat{m}$*-dimensional BPFs vector defined in (7), then integration of this vector can be derived as*

$$\int_0^t \Phi(s)ds \simeq P\Phi(t), \tag{32}$$

where P is called the operational matrix of integration for BPFs and is given by

$$P = \frac{h}{2} \begin{bmatrix} 1 & 2 & 2 & \ldots & 2 \\ 0 & 1 & 2 & \ldots & 2 \\ 0 & 0 & 1 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 2 \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}_{\hat{m} \times \hat{m}} . \tag{33}$$

Lemma 4.2. *[7] Let $\Phi(t)$ be the $\hat{m}$-dimensional BPFs vector defined in (7), the Itô integral of this vector can be derived as*

$$\int_0^t \Phi(s)dB(s) \simeq P_s\Phi(t), \tag{34}$$

where P_s is called the stochastic operational matrix of BPFs and is given by

$$P_s = \begin{bmatrix} B\left(\frac{h}{2}\right) & B(h) & B(h) & \ldots & B(h) \\ 0 & B\left(\frac{3h}{2}\right) - B(h) & B(2h) - B(h) & \ldots & B(2h) - B(h) \\ 0 & 0 & B\left(\frac{5h}{2}\right) - B(2h) & \ldots & B(3h) - B(2h) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & B\left(\frac{(2\hat{m}-1)h}{2}\right) - B((\hat{m}-1)h) \end{bmatrix}_{\hat{m} \times \hat{m}} . \tag{35}$$

Now we are ready to derive a new operational matrix of stochastic integration for the Chebyshev wavelets basis. For this end we use BPFs and the matrix Q introduced in (20).

Theorem 4.3. *Suppose $\Psi(t)$ be the $\hat{m}$-dimensional Chebyshev wavelets vector defined in (17), the integral of this vector can be derived as*

$$\int_0^t \Psi(s)ds \simeq QPQ^{-1}\Psi(t) = \Lambda\Psi(t), \tag{36}$$

where Q is introduced in (20) and P is the operational matrix of integration for BPFs derived in (33).

Proof. Let $\Psi(t)$ be the Chebyshev wavelets vector, by using Theorem 3.1 and Lemma 4.1 we have

$$\int_0^t \Psi(s)ds \simeq \int_0^t Q\Phi(s)ds = Q \int_0^t \Phi(s)ds = QP\Phi(t), \tag{37}$$

now Theorem 3.1 give

$$\int_0^t \Psi(s)ds \simeq QP\Phi(t) = QPQ^{-1}\Psi(t) = \Lambda\Psi(t), \tag{38}$$

and this complete the proof. □

Theorem 4.4. *Suppose $\Psi(t)$ be the $\hat{m}$-dimensional Chebyshev wavelets vector defined in (17), the Itô integral of this vector can be derived as*

$$\int_0^t \Psi(s)dB(s) \simeq QP_sQ^{-1}\Psi(t) = \Lambda_s\Psi(t), \tag{39}$$

where Λ_s is called stochastic operational matrix for Chebyshev wavelets, Q is introduced in (20) and P_s is the stochastic operational matrix of integration for BPFs derived in (35).

Proof. Let $\Psi(t)$ be the Chebyshev wavelets vector, by using Theorem 3.1 and Lemma 4.2 we have

$$\int_0^t \Psi(s)dB(s) \simeq \int_0^t Q\Phi(s)dB(s) = Q \int_0^t \Phi(s)dB(s) = QP_s\Phi(t), \tag{40}$$

now Theorem 3.1 result

$$\int_0^t \Psi(s)dB(s) = QP_s\Phi(t) = QP_sQ^{-1}\Psi(t) = \Lambda_s\Psi(t), \tag{41}$$

and this complete the proof. □

5.   Numerical solution of stochastic Voltera-Fredholm integral equation

In this section, we use the stochastic operational matrix of Chebyshev wavelets for solving stochastic Voltera-Fredholm integral equations. In this way, consider the following stochastic Voltera-Fredholm integral equation

$$X(t) = f(t) + \int_{\alpha}^{\beta} X(s)k_1(s,t)ds + \int_0^t X(s)k_2(s,t)ds + \int_0^t X(s)k_3(s,t)dB(s), \quad t \in [0,T), \tag{42}$$

where $X(t)$, $f(t)$ and $k_i(s,t), i = 1, 2, 3$ are the stochastic processes defined on the same probability space (Ω, F, P), and $X(t)$ is unknown. Also $B(t)$ is a Brownian motion process and $\int_0^t k_3(s,t)X(s)dB(s)$ are the Itô integral. For sake of simplicity and without loss of generality we set $(\alpha, \beta) = (0, 1)$. Now, we approximate $X(t)$, $f(t)$ and $k_i(s,t), i = 1, 2, 3$ in term of $\hat{m}$-dimensional Chebyshev wavelets as follows

$$f(t) = F^T \Psi(t) = \Psi^T(t)F, \tag{43}$$

$$X(t) = X^T \Psi(t) = \Psi^T(t)X, \tag{44}$$

$$k_i(s,t) = \Psi^T(s)K_i\Psi(t) = \Psi^T(t)K_i^T\Psi(s), i = 1, 2, 3, \tag{45}$$

where X and F are Chebyshev wavelets coefficients vector, and $K_i, i = 1, 2, 3$ are Chebyshev wavelets coefficient matrices defined in Eq. (17) and Eq. (19). Substituting above approximations in Eq. (42), we have

$$X^T \Psi(t) = F^T \Psi(t) + X^T \left(\int_0^1 \Psi(s)\Psi^T(s) \right) K_1 \Psi(t)$$

$$+\Psi^T(t)K_2^T \left(\int_0^t \Psi(s)\Psi^T(s)X ds \right) + \Psi^T(t)K_3^T \left(\int_0^t \Psi(s)\Psi^T(s)X dB(s) \right),$$

using relation $\int_0^1 \Psi(s)\Psi^T(s)ds = I_{\hat{m} \times \hat{m}}$ and Remark 3.2 we get

$$X^T \Psi(t) = F^T \Psi(t) + X^T K_1 \Psi(t) + \Psi^T(t)K_2^T \left(\int_0^t \tilde{X}\Psi(s)ds \right) + \Psi^T(t)K_3^T \left(\int_0^t \tilde{X}\Psi(s)dB_i(s) \right),$$

where $\tilde{X}$ is an $\hat{m} \times \hat{m}$ matrix. Now applying the operational matrices Λ and Λ_s for Haar wavelets derived in Eqs. (36) and (39) we have

$$X^T \Psi(t) = F^T \Psi(t) + X^T K_1 \Psi(t) + \Psi^T(t)K_2^T \tilde{X}\Lambda \Psi(t) + \Psi^T(t)K_3^T \tilde{X}\Lambda_s \Psi(t), \tag{46}$$

by setting $Y_2 = K_2^T \tilde{X}\Lambda$, $Y_3 = K_3^T \tilde{X}\Lambda_s$ and using Remark 3.2 we derive

$$X^T \Psi(t) - X^T K_1 \Psi(t) - \hat{Y}_2^T \Psi(t) - \hat{Y}_3^T \Psi(t) = F^T \Psi(t), \tag{47}$$

in which $\hat{Y}_2$ and $\hat{Y}_3$ are $\hat{m} \times \hat{m}$ matrix and they are linear function of vector X. This equation is hold for all $t \in [0, 1)$, so we can write

$$X^T - X^T K_1 - \hat{Y}_2^T - \hat{Y}_3^T = F^T. \tag{48}$$

Since $\hat{Y}_2$ and $\hat{Y}_3$ are linear function of X, Eq. (48) is a linear system for unknown vector X. Solving this linear system and determining X, we can approximate solution of stochastic Voltera-Fredholm integral equation (42) by substituting obtained vector X in Eq. (44).

6.   Numerical examples

Here we demonstrate the efficiency and accuracy of the Chebyshev wavelets method (CWM) by some non-trivial examples. All algorithms are performed by Maple 17 with 20 digits precision.

Example 6.1. *Consider the following stochastic Volterra-Fredholm integral equation [8]*

$$X(t) = f(t) + \int_0^1 \cos(s+t)X(s)ds + \int_0^t (s+t)X(s)ds + \int_0^t e^{-3(s+t)}X(s)dB(s), \quad s, t \in [0, 1],$$

in which

$$f(t) = t^2 + \sin(1+t) - 2\cos(1+t) - 2\sin(t) - \frac{7t^4}{12} + \frac{1}{40}B(t),$$

and $X(t)$ is an unknown stochastic process defined on the probability space $(\Omega, \mathcal{F}, P)$ and $B(t)$ is a Brownian motion process. The proposed method in Section 5 are used for solving this stochastic Volterra-Fredholm integral equation. Fig. 1 presents the approximate solution computed by CWM for $\hat{m} = 64$. The numerical results derived by the CWM and BPFs method [8] are shown in Table 1.
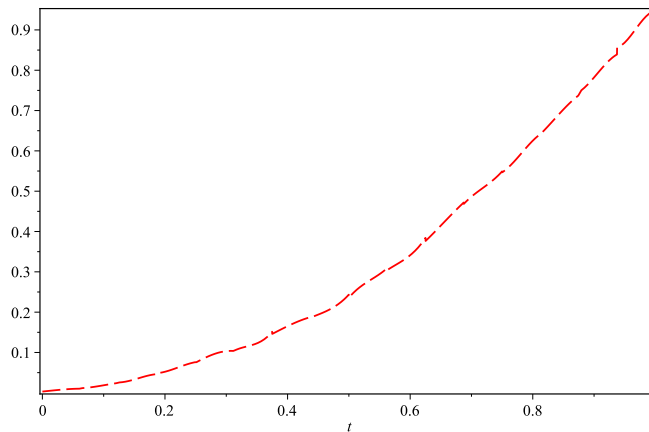

Figure 1: The approximate solution for $\hat{m} = 64$.

Table 1: Numerical results for $\hat{m} = 32$ and $\hat{m} = 64$.

t	$\hat{m} = 32$		$\hat{m} = 64$	
	CWM	BPFs[8]	CWM	BPFs[8]
0.2	0.0384640678	0.0566018117	0.0160672577	0.0162899633
0.4	0.1547742035	0.1550820154	0.1291019803	0.1151902625
0.6	0.3335285787	0.3908514112	0.3050923299	0.3840300664
0.8	0.6243209968	0.6338163380	0.5891794668	0.6993271966
1.0	0.9459824713	0.9684881988	0.9108351381	1.0017286969

Example 6.2. *Consider the following stochastic Volterra-Fredholm integral equation[8]*

$$X(t) = f(t) + \int_0^1 (s+t)X(s)ds + \int_0^t (s-t)X(s)ds + \frac{1}{125}\int_0^t \sin(s+t)X(s)dB(s), \quad s, t \in [0, 1],$$

where

$$f(t) = 2 - \cos(1) - (1+t)\sin(1) + \frac{1}{250}\sin(B(t)),$$

and $X(t)$ is an unknown stochastic process defined on the probability space $(\Omega, \mathcal{F}, P)$ and $B(t)$ is a Brownian motion process. The stochastic operational matrix of Chebyshev wavelets is employed for deriving numerical solution of this Volterra-Fredholm integral equation. Fig. 2 presents the approximate solution computed by the CWM for $\hat{m} = 64$. Table 2 shows the numerical results given by the CWM and BPFs method [8]. The numerical results reveal the efficiency of the proposed method.
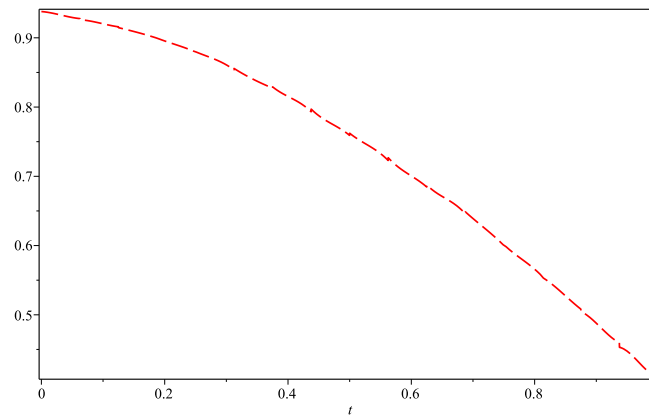
Figure 2: The approximate solution for $\hat{m} = 64$.

Table 2: Numerical results for $\hat{m} = 32$ and $\hat{m} = 64$.

t	$\hat{m} = 32$		$\hat{m} = 64$	
	CWM	BPFs[8]	CWM	BPFs[8]
0.2	0.8418226141	0.9860154776	0.8954496110	0.9833522815
0.4	0.7498444252	0.9432021950	0.8152919762	0.9157653040
0.6	0.6260729397	0.8554015473	0.7005129168	0.8042753408
0.8	0.4848493405	0.7250865831	0.5658723582	0.6954537702
1.0	0.3315123423	0.5459802735	0.4102839976	0.5713651151

7.   Conclusion

A computational method based on the Chebyshev wavelets and their Itô-integration operational matrix is proposed for solving stochastic Volterra-Fredholm integral equations. The main advantage of the proposed method is that it transforms stochastic Volterra-Fredholm integral equations into linear systems of algebraic equations which can be simply solved. Convergence and error analysis of the Chebyshev wavelets is investigated. The efficiency and accuracy of this method is shown by comparison with other existing methods on some non-trivial examples.

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