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Research Paper

Strong convergence for nonexpansive mappings by viscosity approximation methods in Hadamard manifolds

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Abstract

In 2010, Victoria Martin Marquez studied a nonexpansive mapping in Hadamard manifolds using Viscosity approximation method. Our goal in this paper is to study the strong convergence of the Viscosity approximation method in Hadamard manifolds. Our results improve and extend the recent research in the framework of Hadamard manifolds.

Keywords: *Hadamard Manifolds; Iteration Scheme; Nonexpansive Maps; Viscosity Method.*

1. Introduction

Recent developments in fixed point theory reflect that the iterative constructions of fixed points is vigorously proposed and analyzed for various classes of maps in different spaces. Viscosity approximation methods are very important because they are applied to convex optimization, linear programming, monotone inclusions and elliptic differential equations.

In 2000, the viscosity approximation method for selecting a particular fixed point of a given non-expansive mapping was introduced by Moudafi [9]. He established strong convergence of both implicit and explicit schemes in a Hilbert space. Further, in 2004, Xu[14] extended Moudafi's results[9] to the framework of uniformly smooth Banach spaces and proved the strong convergence of continuous scheme and iterative scheme. In Hilbert space, many authors studied the fixed point problems for the non-expansive mappings by the viscosity approximation methods and obtained a series of good results [1], [2], [7], [9], [12], [14].

In 2008, Qin et al. [17] introduced a modified Ishikawa iterative process for a pair of nonexpansive mappings and obtain a strong convergence theorem in the framework of uniformly smooth Banach spaces. They introduced the composite iteration process as follows:

$$\begin{aligned} x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n) y_n \\ y_n &= \beta_n x_n + (1-\beta_n) T_1 z_n \\ z_n &= \gamma_n x_n + (1-\gamma_n) T_2 x_n \end{aligned} \tag{1.1}$$

Where the sequence $\{\alpha_n\}$ in (0, 1) and $\{\beta_n\}, \{\gamma_n\}$ are sequences in [0, 1]. The sequence $\{x_n\}$ defined by (1.1) converges to a common fixed point of T_1 and T_2, which solves the variational inequality

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \le 0,$$

where f is a contraction and $p \in F(T_1) \cap F(T_2)$.

If $\{\gamma_n\}$ =1 in (1.1), this can be viewed as a modified Mann iteration process [6]:

$$x_{n+1}=\alpha_n f\left(x_n\right)+\left(1-\alpha_n\right)y_n$$

$$y_n=\beta_n x_n+\left(1-\beta_n\right)T_1 x_n \tag{1.2}$$

$\{\gamma_n\}$ =1 and $\{\beta_n\}$ =0 in (1.1), then it reduces to the algorithm considered by Xu [14].

In the recent years, some algorithms for solving variational inequalities and minimization problems have been extended from the Hilbert space framework to the more general setting of Riemannian manifolds [4], [5] and [10]. Motivated and inspired by the ongoing research in this direction, we establish the convergence of the viscosity method (1.1) for nonexpansive mappings in the setting of Hadamard manifolds, i.e., complete simply connected Riemannian manifolds of nonpositive sectional curvature.

2. Preliminary notes

First of all, we give some definitions and notations, which can be easily found in [3], [11].
Let p∈M, where M is connected m-dimensional Riemannian manifold. A Riemannian manifold is a Riemannian metric $\langle .,.\rangle$, with the corresponding norm denoted by $\|.\|$. We denote the tangent space of M at p by T_pM. We define the length of a piecewise smooth curve, c: [a, b] →M joining p to q (i.e. c (a) = p and c (b) = q), by using the metric as $L(c)=\int_a^b \|c'(t)\|\,dt$. Then the Riemannian distance d (p, q) is defined to be the minimal length over the set of all such curves joining p to q, which induces the original topology on M. Let c be a smooth curve and ∇ be the Levi- Civita connection associated to (M, $\langle,\rangle$). A smooth vector field X along c is said to be parallel if $\nabla_{c'}$ X = 0. If c' is parallel, then c is a geodesic and here $\|c'\|$ is a constant. A geodesic joining p to q in M is said to be minimal geodesic if its length equals d(p, q).A geodesic triangle $\Delta(p_1, p_2, p_3)$ of a Riemannian manifold is a set consisting of three points p_1, p_2, p_3 and three minimal geodesic γ_i joining p_i to p_{i+1} , with i = 1, 2, 3(mod 3).
A Riemannian manifold is complete if for any p ∈ M, all geodesics emanating from p are defined for all $-\infty<t<\infty$. By the Hopf - Rinow theorem we know that if M is complete then any pair of points in M can be joined by a minimizing geodesic. Thus (M, d) is a complete metric space, and bounded closed subsets are compact.
Now, the exponential map $\exp_p$: T_pM→ M at p ∈M is such that $\exp_p v=\gamma_v(1,p)$ for each $v\in T_pM$, where $\gamma(.)=\gamma_v(.,p)$ is the geodesic starting at p with velocity v . Then $\exp_p tv=\gamma_v(t,p)$, for each real number t.

Definition 2.1 [11] A complete simply connected Riemannian manifold of non-positive sectional curvature is called a Hadamard Manifold.
Now, we present some basic results. We assume that M is a m-dimensional Hadamard manifold.

Proposition 2.1 [11] Let p ∈ M. Then $\exp_p$: T_pM→ M is a diffeomorphism, and for any two points p, q ∈ M there exists a unique normalized geodesic joining p to q, which is in fact a minimal geodesic. This result shows that M has the topology and differential structure similar to $\mathbb{R}_m$. Thus, Hadamard manifolds and Euclidean spaces have some similar geometrical properties.

Proposition 2.2 [11](comparison theorem for triangles). Let Δ (p_1, p_2, p_3) be a geodesic triangle. For each i = 1, 2, 3(mod3), by γ_i : [0, l_i] → M the geodesic joining p_i to p_{i+1}, and set $l_i=L\left(\gamma_i\right)$, $\alpha_i=\angle(\gamma_i'(0)-\gamma_{i-1}'(l_{i-1}))$. Then

$$\alpha_1+\alpha_2+\alpha_3\leq\pi\ ,$$

$$l_i^2+l_{i+1}^2-2l_i l_{i+1}\cos\alpha_{i+1}\leq l_{i-1}^2\ . \tag{2.1}$$

In terms of the distance and the exponential map, the inequality (2.1) can be rewritten as

$$d^2(p_i,p_{i+1})+d^2(p_{i+1},p_{i+2})-2\left\langle \exp^{-1}_{p_{i+1}}p_i,\exp^{-1}_{p_{i+1}}p_{i+2}\right\rangle\leq d^2(p_{i-1},p_i)\ , \tag{2.2}$$

Since $\left\langle \exp^{-1}_{p_{i+1}}p_i,\exp^{-1}_{p_{i+1}}p_{i+2}\right\rangle=d(p_i,p_{i+1})d(p_{i+1},p_{i+2})\cos\alpha_{i+1}$.

Proposition 2.3 [11] A subset K ⊆ M is to be convex if for any two points p and q in K, the geodesic joining p to q is contained in K, i.e., if γ : [a, b] → M is a geodesic such that $p=\gamma(a)$ and $q=\gamma(b)$, then γ ((1 - t) a + t b) ∈ K for all t ∈ [0, 1]. From now K will denote a nonempty, closed and convex set in M.

A real valued function f defined on M is said to be convex if for any geodesic γ of M, the composition function $f \circ \gamma : \mathrm{R} \to R$ is convex, that is,
$(f \circ \gamma)(t\,a + (1-t)b) \le t(f \circ \gamma)(a) + (1-t)(f \circ \gamma)(b)$ for any $a, b \in R$ and $0 \le t \le 1$.

Proposition 2.4 [11] Let d: M × M→ R be a distance function. Then d is a convex function with respect to the product Riemannian metric, i.e., given any pair of geodesics γ_1: [0, 1] → M and γ_2 : [0, 1] → M the following inequality holds for all t ∈ [0, 1]:
$d(\gamma_1(t), \gamma_2(t)) \le (1-t)d(\gamma_1(0), \gamma_2(0)) + td(\gamma_1(1), \gamma_2(1))$
In particular, for each p ∈ M, the function d (·, p): M → R is a convex function.
Let P_K denote the projection onto K defined by

$$P_K(p) = \{p_0 \in K : d(p, p_0) \le d(p, q) \quad \text{for all } q \in K\}, \text{ for all } p \in M \quad .$$

Proposition 2.5 [13] For any point p ∈ M, P_K (p) is a singleton and the following inequality holds for all q ∈ K:

$$\left\langle \exp^{-1}_{P_K(p)} p, \exp^{-1}_{P_K(p)} q \right\rangle \le 0 .$$

Lemma 2.1: *[15], [16] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \le (1-\alpha_n)a_n + \alpha_n b_n \text{ , n} \ge 0,$$

Where $\{\alpha_n\}_{n=0}^{\infty} \subset (0,1)$ and $\{b_n\}_{n=0}^{\infty}$ such that

i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$

ii) Either $\limsup_{n\to\infty} b_n \le 0$ or $\sum_{n=0}^{\infty} |\alpha_n b_n| < \infty$.

Then the sequence $\{a_n\}_{n=0}^{\infty}$ converges to zero.

3. Main results

Let C be a closed convex subset of Hadamard manifold M, T_1, T_2: C→C be a pair of nonexpansive self-mappings and ψ : C→ C a contraction. Assume that the fixed point set F (T_1 T_2) = F (T_1) ∩ F (T_2) is nonempty. We next prove the convergence of an explicit algorithm to a fixed point of T which solves the variational inequality

$$\left\langle \exp^{-1}_{\bar{x}} \psi(\bar{x}), \exp^{-1}_{\bar{x}} x \right\rangle \le 0, \; \forall \; x \in Fix(T_1 T_2) \tag{3.1}$$

Let $x_0 \in M, \{\alpha_n\} \subset (0,1)$. Consider the iteration process

$$\begin{aligned} x_{n+1} &= \exp_{\psi(x_n)}\left((1-a_n)\exp^{-1}_{\psi(x_n)} y_n\right) \\ y_n &= \exp_{x_n}\left((1-b_n)\exp^{-1}_{x_n} T_1 z_n\right) \\ z_n &= \exp_{x_n}\left((1-c_n)\exp^{-1}_{x_n} T_2 x_n\right) \end{aligned} \tag{3.2}$$

which is equivalent to the following geodesic form of equation

$$\begin{aligned} x_{n+1} &= \gamma_{n_1}(1-a_n) \\ y_n &= \gamma_{n_2}(1-b_n) \\ z_n &= \gamma_{n_3}(1-c_n) \end{aligned}$$

where $\gamma_{n_k} : [0,1] \to M$ is the geodesic joining $\psi(x_n)$ to $T_2(x_n)$ for $n \ge 0$ and k = 1, 2, 3.

Theorem 3.1: *Let $C \subseteq M$ be a closed convex set and let T_1, T_2: $C \to C$ be a pair of nonexpansive mappings such that $F(T_1T_2) = F(T_1) \cap F(T_2) \neq \phi$. Let $x_0 \in M$ and $\psi : C \to C$ a ρ-contraction. Suppose that $\{a_n\} \in (0, 1)$ and $\{b_n\}, \{c_n\} \in [0,1]$ satisfies:*

i) $\sum_{n=0}^{\infty} a_n = \infty, a_n \to 0$

ii) $b_n \to 0, c_n \to 0$

iii) $\sum_{n=0}^{\infty} |a_{n+1} - a_n| < \infty$, $\sum_{n=0}^{\infty} |b_{n+1} - b_n| < \infty$ and $\sum_{n=0}^{\infty} |c_{n+1} - c_n| < \infty$

Then the sequence $\{x_n\}$ generated by the algorithm (3.2) converges strongly to $\bar{x} \in C$, the unique fixed point of the contraction $P_{F(T_1)\cap F(T_2)}\psi$. Moreover, the convergence point $\bar{x}$ is a solution of the variational inequality (3.1).

Proof: First we prove that $\{x_n\}$ is bounded.

We only prove the boundedness of $\{x_n\}$, since the boundedness of $\{\psi(x_n)\}$ is a direct consequence.

For this, take $x \in F(T_1) \cap F(T_2)$. Then by the convexity of the distance function and nonexpansivity of T_1 and T_2, we have

$$d(z_n, x) \leq d(\gamma_{n_3}(1-c_n), x)$$

$$\leq c_n d(x_n, x) + (1-c_n) d(T_2 x_n, x)$$

$$\leq c_n d(x_n, x) + (1-c_n) d(x_n, x)$$

$$\leq d(x_n, x) \tag{3.3}$$

Now (3.3) follows that

$$d(y_n, x) \leq d(\gamma_{n_2}(1-b_n), x)$$

$$\leq b_n d(x_n, x) + (1-b_n) d(T_1 z_n, x)$$

$$\leq b_n d(x_n, x) + (1-b_n) d(z_n, x)$$

$$\leq b_n d(x_n, x) + (1-b_n) d(x_n, x)$$

$$\leq d(x_n, x) \tag{3.4}$$

From (3.4), we have

$$d(x_{n+1}, x) \leq d(\gamma_{n_1}(1-a_n), x)$$

$$\leq a_n d(\psi(x_n), x) + (1-a_n) d(y_n, x)$$

$$\leq a_n (\rho d(x_n, x) + d(\psi(x), x)) + (1-a_n) d(x_n, x)$$

$$\leq \max\left\{ d(x_n, x), \frac{1}{1-\rho} d(\psi(x), x) \right\} \tag{3.5}$$

Now by mathematical induction, we have that

$$d(x_{n+1}, x) \leq \max\left\{ d(x_0, x), \frac{1}{1-\rho} d(\psi(x), x) \right\} \tag{3.6}$$

which implies, that $\{x_n\}$ is bounded, so $\{T_2x_n\},\{\psi(x_n)\},\{y_n\},\{z_n\}$ *and* $\{T_1z_n\}$ is also bounded.
Next, we claim that, d (x_{n+1}, x_n) $\to 0$ as n$\to\infty$.

$$d(x_{n+1},x_n)\le d\left(\gamma_{n_1}(1-a_n),\gamma_{n_1-1}(1-a_{n-1})\right)$$

$$\le d\left(\gamma_{n_1}(1-a_n),\gamma_{n_1-1}(1-a_n)\right)+d\left(\gamma_{n_1-1}(1-a_n),\gamma_{n_1-1}(1-a_{n-1})\right)$$

$$\le a_n d\left(\psi(x_n),\psi(x_{n-1})\right)+(1-a_n)d(y_n,y_{n-1})+|a_n-a_{n-1}|d\left(\psi(x_{n-1}),y_{n-1}\right)$$

$$\le a_n\rho d(x_n,x_{n-1})+(1-a_n)d(y_n,y_{n-1})+|a_n-a_{n-1}|d\left(\psi(x_{n-1}),y_{n-1}\right) \tag{3.7}$$

Similarly, we obtain

$$d(y_n,y_{n-1})\le d\left(\gamma_{n_2}(1-b_n),\gamma_{n_2-1}(1-b_{n-1})\right)$$

$$\le b_n d(x_n,x_{n-1})+(1-b_n)d(z_n,z_{n-1})+|b_n-b_{n-1}|d(T_1z_{n-1},x_{n-1}) \tag{3.8}$$

Further, we can obtain

$$d(z_n,z_{n-1})\le c_n d(x_n,x_{n-1})+(1-c_n)d(T_2x_n,T_2x_{n-1})+|c_n-c_{n-1}|d(T_2x_{n-1},x_{n-1})$$

$$\le d(x_n,x_{n-1})+|c_n-c_{n-1}|d(T_2x_{n-1},x_{n-1}) \tag{3.9}$$

Substituting (3.9) into (3.8), we get

$$d(y_n,y_{n-1})\le b_n d(x_n,x_{n-1})+(1-b_n)\left(d(x_n,x_{n-1})+|c_n-c_{n-1}|d(T_2x_{n-1},x_{n-1})\right)+|b_n-b_{n-1}|d(T_1z_{n-1},x_{n-1})$$

$$\le d(x_n,x_{n-1})+(1-b_n)|c_n-c_{n-1}|d(T_2x_{n-1},x_{n-1})+|b_n-b_{n-1}|d(T_1z_{n-1},x_{n-1}) \tag{3.10}$$

Putting the value from (3.10) into (3.7), we get

$$\begin{aligned}d(x_{n+1},x_n)\le{}& a_n\rho d(x_n,x_{n-1})+(1-a_n)\left(d(x_n,x_{n-1})+(1-b_n)|c_n-c_{n-1}|d(T_2x_{n-1},x_{n-1})+|b_n-b_{n-1}|d(T_1z_{n-1},x_{n-1})\right)\\&+|a_n-a_{n-1}|d\left(\psi(x_{n-1}),y_{n-1}\right)\end{aligned}$$

$$\le\left(1-(1-\rho)a_n\right)d(x_n,x_{n-1})+(1-a_n)(1-b_n)|c_n-c_{n-1}|d(T_2x_{n-1},x_{n-1})+(1-a_n)|b_n-b_{n-1}|d(T_1z_{n-1},x_{n-1})+|a_n-a_{n-1}|d\left(\psi(x_{n-1}),y_{n-1}\right)$$

$$\le\left(1-(1-\rho)a_n\right)d(x_n,x_{n-1})+|c_n-c_{n-1}|d(T_2x_{n-1},x_{n-1})++|b_n-b_{n-1}|d(T_1z_{n-1},x_{n-1})+|a_n-a_{n-1}|d\left(\psi(x_{n-1}),y_{n-1}\right)$$

$$\le\left(1-(1-\rho)a_n\right)d(x_n,x_{n-1})+L\left(|c_n-c_{n-1}|+|b_n-b_{n-1}|+|a_n-a_{n-1}|\right) \tag{3.11}$$

where $L\ge\max\{d(T_2x_{n-1},x_{n-1}),d(T_1z_{n-1},x_{n-1}),d(\psi(x_{n-1}),y_{n-1})\}$ for all n.

Now, by assumptions (i)-(iii), we have then
$\lim_{n\to\infty}a_n=0$, $\sum_{n=1}^{\infty}(1-\rho)a_n=\infty$ and $\sum_{n=1}^{\infty}\left(|c_n-c_{n-1}|+|b_n-b_{n-1}|+|a_n-a_{n-1}|\right)<\infty$

Hence by lemma 2.1, we obtain $\lim_{n\to\infty}d(x_{n+1},x_n)=0$
Also $d(T_1T_2x_n,x_n)\le d(x_n,x_{n+1})+d(x_{n+1},y_n)+d(y_n,T_1z_n)+d(T_1z_n-T_1T_2x_n)$

$$\le d(x_n,x_{n+1})+a_n d\left(\psi(x_n),y_n\right)+b_n d(x_n,T_1z_n)+c_n d(z_n-T_2x_n)$$

$$\leq \mathrm{d}(x_n, x_{n+1}) + a_n d(\psi(x_n), y_n) + b_n d(x_n, T_1 z_n) + c_n d(x_n - T_2 x_n)$$

Now by assumptions (i) and (ii),

$$d(\mathrm{T_1 T_2 x}_n, \mathrm{x}_n) \to 0 \tag{3.12}$$

holds.
Put $T = T_1 T_2$. Since T_1 and T_2 are nonexpansive, we have T is also nonexpansive.
Using the fact that if $\bar{x} \in C$ is the unique fixed point of $P_{F(\mathrm{T_1}) \cap F(\mathrm{T_2})}\psi$, then by Proposition (2.5), we obtain

$$\left\langle \exp_{\bar{x}}^{-1} \psi(\bar{x}), \exp_{\bar{x}}^{-1} x_n \right\rangle = \left\langle \exp_{P_{F(\mathrm{T_1}) \cap F(\mathrm{T_2})}\psi(\bar{x})}^{-1} \psi(\bar{x}), \exp_{P_{F(\mathrm{T_1}) \cap F(\mathrm{T_2})}\psi(\bar{x})}^{-1} x \right\rangle \leq 0$$

Next we prove that

$$\limsup_{n\to\infty} \left\langle \exp_{\bar{x}}^{-1} \psi(\bar{x}), \exp_{\bar{x}}^{-1} x_n \right\rangle \leq 0, \tag{3.13}$$

where $\bar{x}$ is the unique fixed point of the contraction $P_{F(\mathrm{T_1}) \cap F(\mathrm{T_2})}\psi$.
Since we have proved that $\{x_n\}$ and $\{\psi(x_n)\}$ are bounded, $\left\{\left\langle \exp_{\bar{x}}^{-1} \psi(\bar{x}), \exp_{\bar{x}}^{-1} x_n \right\rangle\right\}$ is bounded; hence its upper limit exists. Thus we can find a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\limsup_{n\to\infty} \left\langle \exp_{\bar{x}}^{-1} \psi(\bar{x}), \exp_{\bar{x}}^{-1} x_n \right\rangle = \lim_{k\to\infty} \left\langle \exp_{\bar{x}}^{-1} \psi(\bar{x}), \exp_{\bar{x}}^{-1} x_{n_k} \right\rangle \tag{3.14}$$

Without loss of generality, we may assume that $x_{n_k} \to x^*$ for some $x^* \in M$, since $\{x_n\}$ is bounded. Using the convexity of distance function, we have

$$d\left(x_{n_k+1}, T\left(x_{n_k}\right)\right) \leq a_n d\left(\psi\left(x_{n_k}\right), T\left(x_{n_k}\right)\right).$$

Since $\left\{d\left(\psi\left(x_{n_k}\right), T\left(x_{n_k}\right)\right)\right\}$ is bounded as $\{x_n\}$ and $\{\psi(x_n)\}$ are bounded. By assumption (i) it follows that

$$\lim_{k\to\infty} d\left(x_{n_k+1}, T\left(x_{n_k}\right)\right) = 0 \text{ as } a_{n_k} \to 0$$

Now, using $d\left(x_{n_k}, T\left(x_{n_k}\right)\right) \leq d\left(x_{n_k+1}, x_{n_k}\right) + d\left(x_{n_k+1}, T\left(x_{n_k}\right)\right)$,
We obtain

$$\lim_{n\to\infty} d\left(x_{n_k}, T\left(x_{n_k}\right)\right) = 0.$$

Therefore

$$d\left(x^*, T\left(x^*\right)\right) \leq d\left(x^*, x_{n_k}\right) + d\left(x_{n_k}, T\left(x_{n_k}\right)\right) + d\left(T\left(x_{n_k}\right), T\left(x^*\right)\right) \to 0$$

Which shows that $x^* \in Fix(T)$. Then, since $\left\langle \exp_{\bar{x}}^{-1} \psi(\bar{x}), \exp_{\bar{x}}^{-1} x \right\rangle \leq 0$ for any $x \in Fix(T)$, we obtain that

$$\lim_{k\to\infty} \left\langle \exp_{\bar{x}}^{-1} \psi(\bar{x}), \exp_{\bar{x}}^{-1} x_{n_k} \right\rangle = \left\langle \exp_{\bar{x}}^{-1} \psi(\bar{x}), \exp_{\bar{x}}^{-1} x^* \right\rangle \leq 0 \tag{3.15}$$

Now combining (3.14) and (3.15), we obtain (3.13).

Finally, for the strong convergence, we show that $\lim_{n\to\infty} d(x_n, \bar{x}) = 0$

For this, consider the geodesic triangle $\Delta(k,l,m)$ and its comparison triangle $\Delta(k',l',m')\subset R^2$. Fix $n\geq 0$ and set $k=\psi(x_n)$, $l=y_n$, $m=\bar{x}$. So we can write $x_{n+1}=\exp_k\left((1-a_n)\exp_k^{-1}l\right)$. The comparison point of x_{n+1} in R^2 is

$$x'_{n+1}=a_n k'+(1-a_n)l'.$$

Then

$d\left(\psi(x_n),\bar{x}\right)=d(k,m)=\|k'-m'\|$ And $d(y_n,\bar{x})=d(l,m)=\|l'-m'\|$.

Let θ and θ' denote the angles at m and m', respectively. Therefore $\theta\leq\theta'$ by Lemma 3.5(1) [8, p.547] and then $\cos\theta'\leq\cos\theta$. Thus by Lemma 3.5(2) [8, p.547], we have

$$d^2(x_{n+1},\bar{x})\leq\|x'_{n+1}-m'\|^2$$

$$=\|a_n(k'-m')+(1-a_n)(l'-m')\|^2$$

$$=a_n^2\|k'-m'\|^2+(1-a_n)^2\|l'-m'\|^2+2a_n(1-a_n)\|k'-m'\|\|l'-m'\|\cos\theta'$$

$$\leq a_n^2 d^2\left(\psi(x_n),\bar{x}\right)+(1-a_n)^2 d^2(y_n,\bar{x})+2a_n(1-a_n)d\left(\psi(x_n),\bar{x}\right)d(y_n,\bar{x})\cos\theta$$

$$\leq a_n^2 d^2\left(\psi(x_n),\bar{x}\right)+(1-a_n)^2 d^2(x_n,\bar{x})+2a_n(1-a_n)d\left(\psi(x_n),\bar{x}\right)d(x_n,\bar{x})\cos\theta$$

$$\leq a_n^2 d^2\left(\psi(x_n),\bar{x}\right)+(1-a_n)^2 d^2(x_n,\bar{x})+2a_n(1-a_n)\Big(d\left(\psi(x_n),\psi(\bar{x})\right)+d\left(\left(\psi(\bar{x}),\bar{x}\right)\right)\Big)d(x_n,\bar{x})\cos\theta$$

$$\leq a_n^2 d^2\left(\psi(x_n),\bar{x}\right)+(1-a_n)^2 d^2(x_n,\bar{x})+2a_n(1-a_n)\left(\left\langle \exp_{\bar{x}}^{-1}\psi(\bar{x}),\exp_{\bar{x}}^{-1}x_n\right\rangle+\rho d^2(x_n,\bar{x})\right)$$

$$=a_n^2 d^2\left(\psi(x_n),\bar{x}\right)+\left[(1-a_n)^2+2a_n(1-a_n)\rho\right]d^2(x_n,\bar{x})+2a_n(1-a_n)\left(\left\langle \exp_{\bar{x}}^{-1}\psi(\bar{x}),\exp_{\bar{x}}^{-1}x_n\right\rangle\right)$$

$$=\left(1-2a_n+a_n^2+2a_n(1-a_n)\rho\right)d^2(x_n,\bar{x})+a_n^2 d^2\left(\psi(x_n),\bar{x}\right)+2a_n(1-a_n)\left(\left\langle \exp_{\bar{x}}^{-1}\psi(\bar{x}),\exp_{\bar{x}}^{-1}x_n\right\rangle\right)$$

$$=(1-\lambda_n)d^2(x_n,\bar{x})+\lambda_n\sigma_n$$

Where $\sigma_n=\frac{1}{\lambda_n}\left(a_n^2 d^2\left(\psi(x_n),\bar{x}\right)+2a_n(1-a_n)\left\langle \exp_{\bar{x}}^{-1}\psi(\bar{x}),\exp_{\bar{x}}^{-1}x_n\right\rangle\right)$

And $\lambda_n=2a_n-a_n^2-2a_n(1-a_n)\rho$.
Now using given hypothesis (i) and (3.15), $\lim_{n\to\infty}\sigma_n\leq 0$ and $\lim_{n\to\infty}\lambda_n=0$.
Also, by hypothesis (ii), we obtain $\sum_{n=0}^{\infty}\lambda_n=\infty$. Thus applying Lemma 2.1, we get $\lim_{n\to\infty}d(x_n,\bar{x})=0$.
This completes the proof.

Corollary 3.2 Let C $\subseteq$ M be a closed convex set and let T_1: C $\to$ C be a nonexpansive mapping such that F (T_1) $\neq\phi$. Let x_0 ∈ C is chosen arbitrarily and ψ : C→ C a ρ-contraction. Suppose that $\{a_n\}$ ∈ (0, 1) and $\{b_n\}$ ∈ [0, 1] satisfies:

i) $\sum_{n=0}^{\infty}a_n=\infty, a_n\to 0$
ii) $b_n<t$, for some $t\in[0,1)$;and
iii) $\sum_{n=0}^{\infty}|a_{n+1}-a_n|<\infty$ and $\sum_{n=0}^{\infty}|b_{n+1}-b_n|<\infty$.

Then the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = \exp_{\psi(x_n)}\left((1-a_n)\exp^{-1}_{\psi(x_n)} y_n\right)$$
$$y_n = \exp_{x_n}\left((1-b_n)\exp^{-1}_{x_n} T_1 x_n\right)$$

Converges strongly to $\bar{x} \in C$, the unique fixed point of the contraction $P_{F(T_1)}\psi$. Moreover, the convergence point $\bar{x}$ is a solution of the variational inequality

$$\left\langle \exp^{-1}_{\bar{x}} \psi(\bar{x}), \exp^{-1}_{\bar{x}} x \right\rangle \le 0 , \ \forall \ x \in Fix(T_1) \ .$$

Proof. we can obtain the desired result by taking $\{c_n\} = 1$ in theorem 3.1.

Corollary 3.3 Let M be a Hadamard manifold, C be a closed convex subset of M. Let T: C $\rightarrow$ C be a nonexpansive mapping with F (T) $\neq \phi$. Let $x_0 \in$ M is chosen arbitrarily and ψ : C$\rightarrow$ C a ρ-contraction. Suppose that $\{a_n\} \in (0, 1)$ satisfies:

i) $\lim_{n\to\infty} a_n = 0$

ii) $\sum_{n=0}^{\infty} a_n = \infty$

iii) $\sum_{n=0}^{\infty} |a_{n+1} - a_n| < \infty$.

Then the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = \exp_{\psi(x_n)}\left((1-a_n)\exp^{-1}_{\psi(x_n)} T(x_n)\right)$$

Converges strongly to $\bar{x} \in C$, the unique fixed point of the contraction $P_{F(T)}\psi$. Moreover, the convergence point $\bar{x}$ is a solution of the variational inequality.

$$\left\langle \exp^{-1}_{\bar{x}} \psi(\bar{x}), \exp^{-1}_{\bar{x}} x \right\rangle \le 0 , \ \forall \ x \in Fix(T) \ .$$

Proof . We can obtain the desired result by taking $\{b_n\} = 0$ and $\{c_n\} = 1$ in theorem 3.1.

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