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Preliminary Concepts of Dynamical Systems

Nayana Mukherjee[1] **and Swarup Poria**[2]
[1] Department of Applied Mathematics, Indian School of Mines
Dhanbad, India
Email: nayanamukherjee810@gmail.com
[2] Department of Applied Mathematics, University of Calcutta
Kolkata, India
Email: *swarup_p*@yahoo.com

Abstract

Dynamical system is a young and vigorously growing area of research which promises enormous potential and opportunities. This paper aims to introduce some of the preliminary concepts of dynamical systems. Many application sides of the subject are noted to encourage the reader for future developments. Resources are supplied in the references for further reading.

Keywords: *Phase portrait, Limit Cycle , Lyapunov and asymptotic stability, Lyapunov function, Lyapunov exponent, Bifurcation, Chaos*

1 Introduction

Dynamics is the study of change, and a dynamical system is just a recipe for saying how a system of variables interacts and changes with time. A dynamical system may be thought of as an object of any nature, whose state evolves in time according to some dynamical law. The theory of dynamical system is a wide and independent field of scientific research.

Dynamical systems theory comprises a broad range of analytical, geometrical, topological, and numerical methods for analyzing differential equations and iterated mappings. The modern theory of dynamical systems derives from the work of H.J. Poincare (1854- 1912) on the three-body problem of celestial mechanics. Poincare's geometric methods were being extended to yield a much deeper understanding of classical mechanics, by Birkhoff and later Kolmogorov, Arnol'd, and Moser. On the theoretical side, nonlinear differential equations (nonlinear oscillators) stimulated the invention of new mathematical techniques by Van der Pol, Andronov, Littlewood, Cartwright, Levinson, and Smale. Lorenz's discovery in 1963 said that the solutions to his equations never settled down to equilibrium or to a periodic state instead they continued to oscillate in an irregular, aperiodic fashion. Moreover, if he started his simulations from two slightly different initial conditions, the resulting behaviors would soon become totally different. The implication was that the system was inherently unpredictable, tiny errors in measuring the current state of the atmosphere would be amplified rapidly, eventually leading to embarrassing forecasts. In 1971 Ruelle and Takens proposed a new theory for the onset of turbulence in fluids, based on abstract considerations about strange attractors. A few years later, R.M. May [1] found examples of chaos in iterated mappings arising in population biology, and stressed the pedagogical importance of studying simple nonlinear systems.

Dynamical Systems

System evolves with time in such a way that the states of the system at time t depend upon the states of the system at earlier times are called dynamical systems. In other words, a dynamical system consists of a set of possible states, together with a rule that determines the present states in terms of past states. According to the character of the state variables dynamical systems can be classified as a discrete dynamical system or as a continuous dynamical system.

2 Continuous Dynamical Systems

A continuous dynamical system is represented by a differential equation

$$\frac{dx}{dt} = f(x,t)\ ; x \in U \subseteq R^n, t \in R \tag{1}$$

possessing a unique solution $x(t, t_0) = x(t)$ satisfying the condition $x(t_0) = x_0$.
Example: Volterra Model

$$\begin{aligned} \frac{dH}{dt} &= rH - aHP \\ \frac{dP}{dt} &= bHP - mP \end{aligned}$$

where, H =density of prey at time t, P =density of predator at time t, r =intrinsic rate of prey population increase, a =predation rate coefficient, b =reproduction rate of predator per 1 prey eaten, and m =predator's mortality rate.

2.1 Autonomous and Non-autonomous Systems

A continuous dynamical system can be classified as autonomous system if right hand side of equation (1) does not depend explicitly on time t i.e. if $f(x,t) = f(x)$.
Example:

$$\dot{x} = ax(1-x)$$

This is an autonomous system because the right hand side $ax(1-x)$ does not depend on time t explicitly.
On the other hand if $f(x, y, t)$ explicitly depends on t then the dynamical system is called non-autonomous system.
Example: A periodically forced pendulum :

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -gsinx + Fcos\omega t \end{aligned}$$

is a non-autonomous dynamical system.

2.2 Phase Space and Phase Portrait

A set of phase variables of a system is a minimal set of variables which fully describes the state of the system. Phase space is the space generated by the phase variables i.e. phase space is the space generated by the generalized coordinates and generalized momenta of a physical system. A state of a system at any time is represented by a point in the system's phase space. Change of a system state over time is represented by a trajectory in the phase space. In other words, a trajectory is the path of an object in phase space as a function of time. A phase portrait is the collection of all possible trajectories of the system.
Example: The dynamical system

$$\begin{aligned} \frac{dH}{dt} &= rH - aHP \\ \frac{dP}{dt} &= bHP - mP \end{aligned}$$

has 2 dimensional phase space.

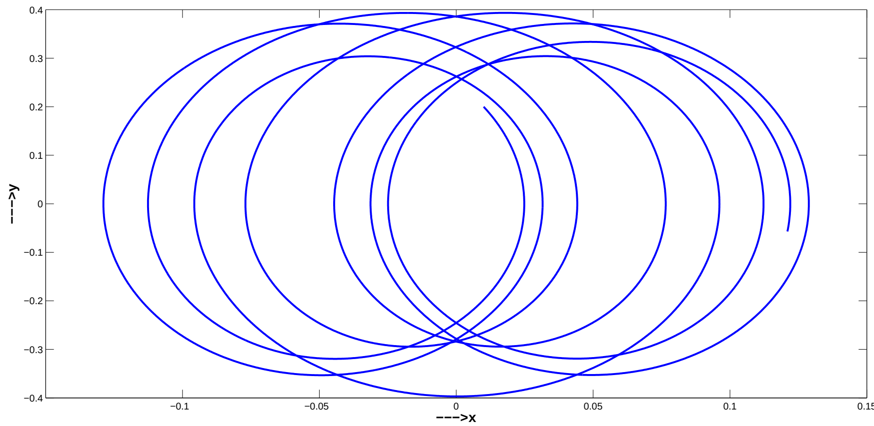

Figure 1: Phase portrait of periodically forced pendulam. The self intersection of the trajectory is observed.

2.3 Trajectories of Autonomous and Nonautonomous Systems

Any two trajectories of an autonomous system cannot intersect each other and not any single trajectory will intersect itself in the time evolution of the system. If two trajectories intersect, it indicates that at the point of intersection, the system can evolve in more than one possible directions. This violates the existence of unique solution of an autonomous dynamical system. On the other hand for non-autonomous system trajectories can have self intersection and two different trajctories can intersect in later time. Self intersection of the trajectories of the periodically forced pendulam is shown in figure 1.

2.4 The Vector Field

A vector field in the plane in the plane R^2 is determined by a vector function

$$\overrightarrow{F}(\overrightarrow{x}) = (F_1(x_1, x_2), F_2(x_1, x_2)).$$

At each point $\overrightarrow{x} \in R^2$ the vector $\overrightarrow{F}(\overrightarrow{x})$ is attached. One of the applications of vector fields is the visualization of solution of ordinary differential equations. Differential equations define a vector field at every point in the phase space. The solution of a differential equation with prescribed initial condition follows the flow of vectors. Vector field for Lotka-Volterra predator prey model

$$\frac{dH}{dt} = rH - aHP$$

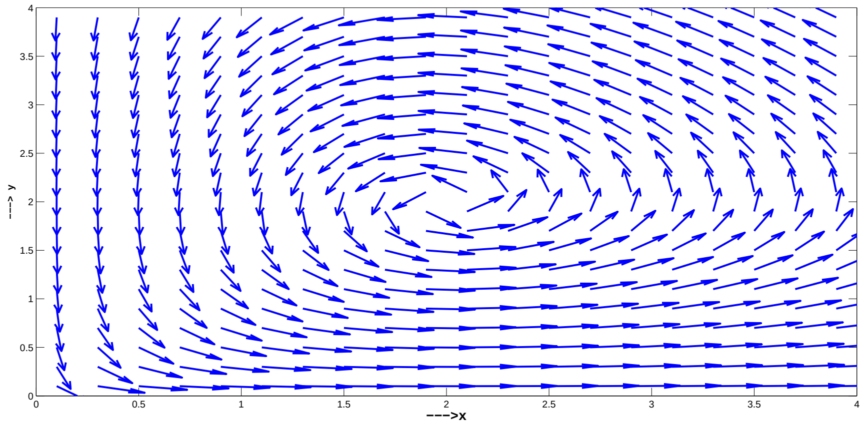

Figure 2: Vector field for Lotka Volterra Predator prey model with $r = m = 2$ and $a = b = 1$.

$$\frac{dP}{dt} = bHP - mP$$

is shown in figure 2.

2.5 Non-autonomous to Autonomous Conversion

Every non-autonomous dynamical system can be transformed into an autonomous system by increasing the phase space dimension by one.
Example :

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -gsinx + Fcos\omega t \end{aligned}$$

is non-autonomous with phase space of dimension 2. Assuming $t = z$ one can obtain the following system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -gsinx + Fcos\omega z \\ \dot{z} &= 1 \end{aligned}$$

This is an autonomous dynamical system with phase space dimension 3.

2.6 Equilibrium Point

An equilibrium point x^* of a continuous dynamical system is the point in the phase space where phase space velocity is zero. i.e, $[\frac{dx}{dt}]_{x=x^*} = 0\, i.e.\ f(x^*) = 0$.

Notice that the equilibrium points of a dynamical system are solutions of the ordinary differential equation. **Example**: $\frac{dx}{dt} = x(1-x)$ has equilibrium points 0 and 1.

2.7 Nonlinear Systems

A differential equation that can be written in the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + ... + a_n(x)y = F(x) \tag{2}$$

where $a_0, a_1, ..., a_n$ and F are functions of x only is called a linear differential equation of order n. A differential equation that does not satisfy this definition is called a nonlinear differential equation. The differential equation (2) is called an autonomous linear system if $a_0, a_1, ..., a_n$ are all constants and $F = 0$ otherwise it is called an nonautonomous system.
Appropriate definition of linear differential equation is given in the following paragraph. Let L be a differential operator then any ordinary differential equation can be written as

$$L(y) = f(x) \tag{3}$$

where $f(x)$ is any function of x. This differential equation is called homogeneous of $f(x) = 0$, otherwise it is called nonhomogeneous. Now consider

$$L = a_n D^n + a_{n-1}D^{n-1} + ... + a_1 D + a_0$$

where $D(y) = y',\ D^2(y) = y''$ etc and $a_0, a_1, ..., a_n$ are functions of x. One can easily check that the operator L satisfies the condition for linearlity i.e.

$$L(ay_1 + by_2) = aL(y_1) + bL(y_2).$$

Therefore an ordinary differential equation is said to be linear if and only if L is a linear operator otherwise it is called a nonlinear differential equation.

Example: Linear system

$$y^{(2)} + x^2 y^{(1)} + sinx\ y = e^x$$

is a second order linear differential equation. This is a nonautonomous system.
Example: Nonlinear system

$$my^{(2)} + k_1 y + k_2 y^3 = 0$$

is a second order autonomous nonlinear differential equation.
If ϕ_1 and ϕ_2 are two solutions of a linear autonomous system then $c_1\phi_1 + c_2\phi_2$ is also a solution of the system i.e. the superposition principle holds for linear

autonomous systems. But for a nonlinear autonomous system the superposition principle does not hold. Generally, nonlinear problems are difficult to solve and are much less understandable than linear problems. Even if not exactly solvable, the outcome of a linear problem is rather predictable, while the outcome of a nonlinear problem is inherently not. Nonlinear problems are of interest to physicists and mathematicians because most physical systems are nonlinear in nature. Physical examples of linear systems are not very common.

2.8 Dissipative and Conservative Systems

Two kinds of DS are distinguished, namely, conservative and nonconservative. For a conservative system, the volume in phase space is conserved during time evolution. On the other hand for a nonconservative system, the volume is usually contracted. The contraction of phase volume in mechanical systems corresponds to loss of energy as result of dissipation. A growth of phase volume implies a supply of energy form outside to the system.

Let us consider a dynamical system $\dot{X} = f(X)$. Pick an arbitrary closed surface $S(t)$ of volume $V(t)$ in phase space. Taking points of S as initial conditions for trajectories let them evolve for infinitesimal time dt.Then S evolves into a new surface $S(t+dt)$ and $V(t)$ evolves to $V(t+dt)$. If $V(t) = V(t+dt)$ for all time t then the dynamical system is called conservative. Therefore for a conservative system, the volume in phase space is conserved during time evolution. On the other hand if $V(t) > V(t+dt)$ then dynamical system is called dissipative. In a dissipative system the volume in phase space is contracted. The contraction of phase volume in mechanical systems corresponds to loss of energy as result of dissipation. A growth of phase volume implies a supply of energy to the system which can be named negative dissipation. In general, a system in which the energy or phase volume varies are called dissipative systems. A system is conservative if $\nabla.f = 0$ and it is dissipative if $\nabla.f < 0$.
Example : Conservative system

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= x^3 - x\end{aligned}$$

is a conservative system since $\nabla.f = 0$.
Example : Dissipative system

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= sx - y - xz\\ \dot{z} &= xy - bz\end{aligned}$$

where $\sigma, r, b > 0$ are parameters. This is known as Lorenz system. For Lorenz system $\nabla . f = -\sigma - 1 - b < 0$. Therefore Lorenz system is a dissipative system.

2.9 Stability Concepts

Consider the first-order differential equation

$$\frac{dx}{dt} = f(x,t) \tag{4}$$

where $f(x,t)$ is defined and continuous for $t \in (a, \infty)$ and x from a certain domain D, possesses a bounded partial derivative $\frac{\partial f}{\partial x}$. We assume that the function $x = \phi(t)$ is a solution of the equation (4), which satisfies the initial condition $[x]_{t=t_0} = \phi(t_0)$, $t_0 > a$. We assume furthermore, that the function $x = x(t)$ is a solution of the same equation, which satisfies another initial condition $[x]_{t=t_0} = x(t_0)$. It is assumed that the solutions $\phi(t)$ and $x(t)$ are defined for all $t \geq t_0$.

Lyapunov Stability

The solution $x = \phi(t)$ of equation (4) is said to be stable in the sense of Lyapunov as $t \to \infty$ if , for every $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that for every solution $x = x(t)$ of that equation the following condition will be staisfied:

$$|x(t_0) - \phi(t_0)| < \delta \Rightarrow |x(t) - \phi(t)| < \epsilon$$

for all $t \geq t_0$ (we can always assume that $\delta \leq \epsilon$). We usually called Lyapunov stable as stable.

Example: The trivial solution $x = 0$ of the equation $\frac{dx}{dt} = 0$ is lyapunov stable.

Asymptotic Stability

The solution $x = \phi(t)$ of equation (4) is said to be asymptotically stable if
(a) the solution $x = \phi(t)$ is Lyapunov stable and
(b) there exist $\delta > 0$ such that for any solution $x = x(t)$ of (4), which satisfies the condition $|x(t_0) - \phi(t_0)| < \delta$, we have $\lim_{t \to \infty} |x(t) - \phi(t)| = 0$.

Example: The trivial solution $x = 0$ of the equation $\frac{dx}{dt} = 0$ is lyapunov stable but not asymptotically stable. The trivial solution $x = 0$ of the equation $\frac{dx}{dt} = -a^2 x, a =$constant is asymptotically stable.

Notice that stability of a nontrivial solution of a differential equation does not imply that the solution is bounded. Also the boundedness of solution of a differential equation does not imply that the solutions are stable. The concepts of boundedness and stability of solutions are mutually independent.

Example: All solutions of $\frac{dx}{dt} = sin^2 x$ are bounded but $x(t) = 0$ solution of this differential equation is unstable. The solution $x(t) = t$ is a stable solution of the differential equation $\frac{dx}{dt} = 1$ but it is not bounded.

Stability Analysis of Equilibrium Points

Linear Stability Analysis (Local Stability Analysis)
We consider system

$$\begin{aligned}\dot{x} &= f(x,y)\\ \dot{y} &= g(x,y)\end{aligned}$$

Let (x^*, y^*) be fixed point of the system. Then $f(x^*, y^*) = 0$, $g(x^*, y^*) = 0$. Taylor series expansion in the neighborhood of (x^*, y^*) is

$$\begin{aligned} f(x,y) &= f(x^*,y^*) + (x-x^*)[\frac{\partial f}{\partial x}]_{(x^*,y^*)} + (y-y^*)[\frac{\partial f}{\partial y}]_{(x^*,y^*)} + (x-x^*)^2[\frac{\partial^2 f}{\partial x^2}]_{(x^*,y^*)}\\ &+ (y-y^*)^2[\frac{\partial^2 f}{\partial y^2}]_{(x^*,y^*)} + 2(x-x^*)(y-y^*)[\frac{\partial^2 f}{\partial x \partial y}]_{(x^*,y^*)} + \text{higher order terms.}\\ &\simeq (x-x^*)[\frac{\partial f}{\partial x}]_{(x^*,y^*)} + (y-y^*)[\frac{\partial f}{\partial y}]_{(x^*,y^*)}, \text{after linearization}\end{aligned}$$

Similarly, $g(x,y) \simeq (x-x^*)[\frac{\partial g}{\partial x}]_{(x^*,y^*)} + (y-y^*)[\frac{\partial g}{\partial y}]_{(x^*,y^*)}$, after linearization.
Therefore, very small disturbances in the neighborhood of fixed points will follow the linear equation

$$\begin{aligned}\dot{x} &= ax + by, \quad where \;\; a = [\frac{\partial f}{\partial x}]_{(x^*,y^*)}, b = [\frac{\partial f}{\partial y}]_{(x^*,y^*)}\\ \dot{y} &= cx + dy, \quad where \;\; c = [\frac{\partial g}{\partial x}]_{(x^*,y^*)}, d = [\frac{\partial g}{\partial y}]_{(x^*,y^*)}\end{aligned}$$

i.e,

$$\dot{X} = AX, \quad where \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\; and \;\; X = \begin{pmatrix} x \\ y \end{pmatrix}$$

The fixed point (x^*, y^*) is said to be linearly stable if all eigenvalues of A have negative real parts. The fixed point (x^*, y^*) is unstable if at least one eigenvalue of A have positive real part. If A has any 0 eigen value then linear stability analysis fails and nonlinear stability analysis is necessary.

Local and Global Stability Analysis
Global stability analysis of an equilibrium point can be done by Lyapunov function (without solving differential equation). A function $V(x_1, x_2, ..., x_n)$ is called Lyapunov function for the system of differential equations

$$\frac{dx_i}{dt} = f_i(x_1, x_2, x_3, ..., x_n); i = 1, 2, 3, ..., n$$

if

$$(a) \quad V(0,0,...,0) = 0$$
$$(b) \quad V(x_1, x_2, ..., x_n) \ \text{ is positive definite}$$
$$(c) \quad \frac{dV}{dt} \ \text{ is negative definite}$$

Origin is a globally asymptotically stable equilibrium point if there exist a Lyapunov function for the system in R^n.

Example:

$$\begin{aligned} \dot{x} &= -x \\ \dot{y} &= -y \end{aligned}$$

Let, $L(x,y) = (x^2 + y^2)/2$. Then L is positive definite and Therefore,

$$\frac{dL}{dt} = x\dot{x} + y\dot{y} = -x^2 - y^2$$

is negative definite. Hence $(0,0)$ is a global asymptotically stable equilibrium point for the system.

If

$$(a) \quad V(0,0,...,0) = 0$$
$$(b) \quad V(x_1, x_2, ..., x_n) \ \text{ is positive definite}$$
$$(c) \quad \frac{dV}{dt} \ \text{ is negative semi-definite}$$

Origin is a globally stable equilibrium point if there exist a Lyapunov function for the system in R^n.

Notice that Lyapunov function in certain sense is a generalised distance form the origin. The existence of Lyapunov function in a neighbourhood of an equilibrium point but non existence in the whole space implies the local stability of the equilibrium point. Note that $V(0) = 0$ is required. Otherwise choosing $V(x) = 1/(1 + |x|)$ we can prove that $\dot{x}(t) = x$ is locally stable. But actually the system $\dot{x}(t) = x$ is unstable at 0.

2.10 Periodic Orbit

A solution of $\dot{x} = f(x,t), x \in R^n$ through the point x_0 is said to be periodic of period T if there exists $T > 0$ such that $x(t, t_0) = x(t + T, x_0)$ for all $t \in R$. Any periodic orbit in the phase space is a closed curve. The systems which can be written in the form $\dot{x} = -\nabla V$ for some continuously differentiable, single valued scalar function $V(x)$ is called a gradient system with potential function V. Closed orbit does not exist in gradient systems. If one can find a Lyapunov function for a system then also existence of closed orbit can be ruled out. There are many other criteriones for rule out existence of closed orbit in a DS.

2.11 Limit Cycle

Limit cycle is an isolated periodic orbit. Existence of limit cycle in a model implies that system exhibit self-sustained oscillations. For existence of limit cycle dynamical system must be nonlinear and its phase space should be at least two dimensional. A linear system can have closed orbits but they are not isolated. The famous Poincare Bendixson theorem can help us to show the existence of limit cycles in two dimensional phase space. An improved version of the Poincare Bendixson theorem states that if a trajectory is trapped in a compact region(subset of two dimensional phase space) then it must approach a fixed point, a limit cycle or something exotic called a cycle graph (an invariant set containing a finite number of fixed pints connected by a finite number of trajectories, all oriented either clockwise or counterclockwise). In the fundamental biochemical process called glycolysis, living cells obtain energy by breaking down sugar. A simple model of these glycolysis has been proposed by Selkov. The dimensionless form of the Selkov model is the following

$$\begin{aligned}\dot{x} &= -x + ay + x^2y \\ \dot{y} &= b - ay - x^2y\end{aligned}$$

where x and y are the concentrations of ADP (adenosine diphosphate) and F6P (fructose-6-phosphate), and $a, b > 0$ are kinetic parameters. Limit cycle oscillation of the system for $a = 0.08,\ b = 0.6$ is shown in figure 3.

2.12 Hyperbolic and Non-Hyperbolic Equlibruim Points

A hyperbolic equilibrium point of a continuous dynamical system are those equilibrium points at which the Jacobian matrix has no eigenvalue with zero real part. On the other hand if an equilibrium point is not hyperbolic then it is called non-hyperbolic. Hartman Grobman theorem states that the stability type of the hyperbolic equilibrium point is fully captured by the linearized system. A phase portrait is structurally stable at hyperbolic equlibrium point i.e. topology of the phase portrait cannot be changed by an arbitrary small perturbation to the vector field. However, at non-hyperbolic equilibrium point phase portrait is structurally unstable and qualitatively different phase portrait arrive for an arbitrarily small perturbation to the vector field.

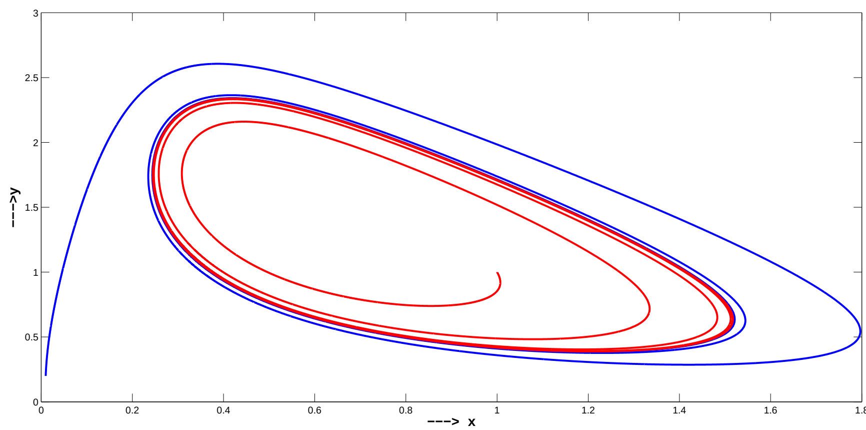

Figure 3: Limit cycle solution of glycolytic oscillator with initial condition (a) (1,1) with red line (b)((0.01,0.2) with blue line are shown

2.13 Bifurcation Point

The qualitative structure of flow can change as parameters are varied, fixed points can be created or destroyed, or their stability can change. These qualitative changes in the dynamics are called bifurcation and the parameter values at which they occur are called bifurcation point. More precisely, a point (x_0, α) is called bifurcation point of a dynamical system (with parameter α) if systems behavior changes qualitatively at (x_0, α). A necessary condition for (x_0, α) to be a bifurcation point is that it must be a nonhyperbolic equilibrium point at the critical parameter value. Hartman Grobman theorem implies that any qualitative change or bifurcation must be reflected in the linear dynamics. The qualitative structure of equilibrium point remain fixed unless the equlibrium point loses its hyperbolicity. The loss of hyperbolicity of equlibrium point occurs in one of the two following ways.
(a). Occurance of a simple real zero eigenvalue of the jacobian matrix at the critical parameter value. This type of bifurcation is called steady state bifurcation. Most typical steady state bifurcations are saddle-node bifurcation, transcritical bifurcation and pitchfork bifurcation.
(b). Occurance of a simple pair of purely imaginary eigenvalues of the jacobian matrix at the critical parameter value. This type of bifurcation is known as Hopf bifurcation.

2.14 Lyapunov Exponent

The Lyapunov exponent or Lyapunov characteristic exponent of a dynamical system is a quantity that characterizes the rate of separation of infinitesimally close trajectories with time. Quantitatively, two trajectories in phase space with initial separation δZ_0 changes with time in the following rule.

$$|\delta Z(t)| \approx e^{\lambda t}|\delta Z_0|$$

where λ is the Lyapunov exponent.

The rate of separation can be different for different orientations of initial separation vector. An n dimensional dynamical system have n Lyapunov exponents. For a periodic behaviour of a system largest Lyapunov exponent should be negative, for a quasiperiodic behaviour largest lyapunov exponent should be 0 and for chaotic behaviour the largest Lyapunov exponent must be positive. For a hypercahotic system at least two lyapunov exponents are positive.

3 Discrete Dynamical Systems

In a discrete dynamical system the time variable is discrete i.e, $t \in Z$ or N. Instead of a differential equation the evolution is determined by a difference equation. For example $x_{t+1} = f(x_t, x_{t-1}, .., x_{t-n})$; $t \in Z$ or N is the form of a discrete dynamical system. Logistic map, $x(t+1) = ax(t)(1 - x(t))$,where $a \in [0, 4]$ and $x \in [0, 1]$ is a discrete dynamical system.

3.1 Fixed Point

A fixed point x^* of a discrete dynamical system is the point where $x^* = f(x^*)$. Fixed points of the logistic map are $x = 0, (a - 1)/a$.

3.2 Orbit of a Point

Let f be a map $f : X \to X$ and x_0 be a point in X. Then orbit of x_0 under f is the set $\{x_0, f(x_0), f^2(x_0), ...\}$. Let us consider a map $f : N \to N$ given by $f(x) = ax(1 - x)$, then the orbit of x_0 under the map f is given by,$\{x_0, ax_0(1 - x_0), a[x_0(1 - x_0][1 - ax_0(1 - x_0)], ..\}$.

3.3 Stability Analysis of Fixed Points

Let f be a map on X and let p be a real number such that $f(p) = p$. If all points sufficiently close to p are attracted to p, then p is called a sink or an attracting fixed point. If all points sufficiently close to p are repelled from p, then p is called a source or a repelling fixed point. If $|f'(p)| < 1$ then p is a sink and if $|f'(p)| > 1$ then p is a source.

Example:

Stablity analysis of the fixed points of the map $x_{n+1} = ax_n(1 - x_n)$ defined in $[0, 1]$ and $0 \leq a \leq 4$.

Solution: Every fixed point x^* of the map satisfy the following equation

$$\begin{aligned} x^* &= ax^*(1 - x^*) \\ \text{i.e. } x^* &= 0, 1 - \frac{1}{a} \\ \text{Now } f'(x^*) &= a(1 - 2x^*) \end{aligned}$$

At the fixed point $x^* = 0$, $f'(x^*) = a$. Therefore if $|a| < 1$,then $|f'(x^*)| < 1$ at $x^* = 0$ Hence $x^* = 0$ is stable for $a < 1$. Now consider the fixed point $x^* = 1 - \frac{1}{a}$. At this x^*, $f'(x^*) = a[1 - 2 + \frac{2}{a}] = 2 - a$.Therefore if $|2 - a| < 1$ then $|f'(x^*)| < 1$ i.e if $1 < a < 3$ then x^* is stable.

Let $f = (f_1, f_2, ..., f_m)$ be a map on R^m The Jacobian matrix of f at the fixed point P, denoted by $Df(P)$. If the magnitude of each eigenvalue of $Df(P)$ is less than 1, then P is a sink or a stable fixed point and if the magnitude of at least one eigenvalue of $Df(P)$ is greater than 1, then P is a unstable fixed point.

3.4 Period-k Orbit and its Stability

Let $f : X \to X$ be a map then period-k points of f are those points of X for which $f^k(x) = x$ but $f^i(x) \neq x$ for $i = 1, 2, 3, ..., (k - 1)$. Let $\{x_1, x_2, x_3, ..., x_{k-1}, x_k\}$ are period-k orbit of f such that $x_2 = f(x_1), x_3 = f(x_2), ..., f(x_k) = x_1$. Condition for stability of period-k points of the map f is

$$|f'(x_1).f'(x_2).f'(x_3)....f'(x_{k-1}).f'(x_k)| < 1.$$

Notice that fixed points of a map can be defined as period-1 points of the map.

3.5 Hyperbolic and Non-hyperbolic Fixed Points

Let f be a map on R^m $(m \geq 1)$. Assume that $f(P) = P$.Then the fixed point P is called hyperbolic if none of the eigenvalues of $Df(P)$has magnitude 1. If P is hyperbolic and if at least one eigenvalue of $Df(P)$ has magnitude greater than 1 and at least one eigenvalue has magnitude less than 1 but no eigen value with magnitude equal to one then P is called a saddle. (For a periodic

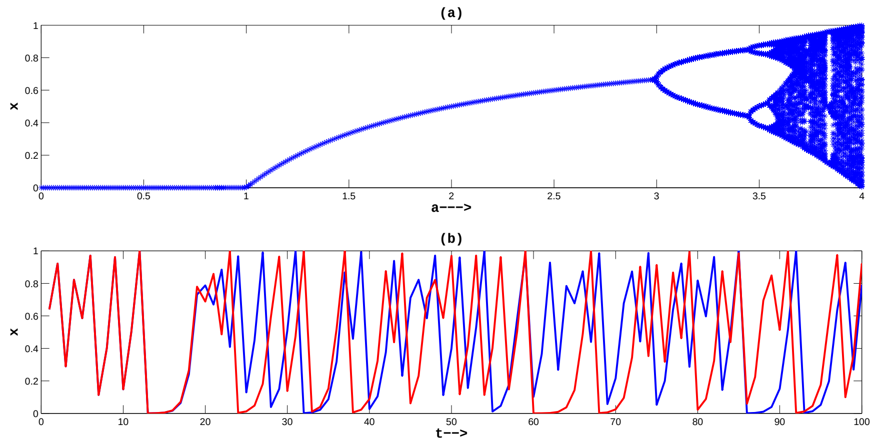

Figure 4: (a) Bifurcation diagram of logistic map. (b) Sensitive dependence on initial conditions is shown. Blue line denotes the time evolution of logistic map $4x(1-x)$ for initial condition $x_0 = 0.2$ and red line indicates the time evolution of the same map for $x_0 = 0.200001$. It is clear from the figure that after some time the two trajectories are totally different.

point of period k, replace f by f^k .) Saddles are unstable fixed points. If even one eigenvalue of $Df(P)$ has magnitude greater than 1, then P is unstable in the sense previously described. Almost all perturbation of the orbit will be magnified under iteration. In a small ϵ neighbourhood of P, f behaves very much like a linear map with an eigenvalue that has magnitude greater than 1, that is, the orbits of most points near P diverge from P.

3.6 Bifurcation of Maps

If an eigenvalue reaches the unit circle, then fixed point is no longer hyperbolic and a bifurcation can occur. The loss of hyperbolicity of a fixed point occurs in one of the three following ways.

(a) Jacobian matrix have a simple real eigenvalue 1. This type of bifurcation is known as steady state bifurcation for maps. The saddle-node, transcritical and pitchfork bifurcations are examples of steady state bifurcation.

(b) A simple conjugate pair of eigenvalues of the Jacobian matrix lying on the unit circle. We refer this case as Hopf bifurcation for maps.

(c) A simple real eigenvalue of Jacobian matrix is -1. In this case period doubling bifurcation occures. This bifurcation is also known as flip bifurcation or subharmonic bifurcation.

4 Chaos

A dynamical system is said to be chaotic, if it has the following properties:
(a) it must be sensitive to initial conditions.
(b) it must be topologically mixing.
(c) its periodic orbits must be dense.
the necessary requirements for a deterministic continuous dynamical system to be chaotic are that the system must be nonlinear and be at least three dimensional. At least one Lyapunov exponent must be positive for chaotic systems. For a hyper-chaotic system at least two Lyapunov exponents are positive.

Sensitivity to initial conditions means that each point in such a system is arbitrarily closely approximated by other points with significantly different future trajectories. Thus, an arbitrarily small perturbation of the current trajectory may lead to significantly different future behaviour.

Sensitivity to initial conditions is popularly known as the butterfly effect, so called because of the title of a paper given by Edward Lorenz in 1979 to the American Association for the Advancement of Science in Washington, D.C. entitled Predictability: Does the Flap of a Butterflys Wings in Brazil set off a Tornado in Texas? The flapping wing represents a small change in the initial condition of the system, which causes a chain of events leading to large-scale phenomena. Had the butterfly not flapped its wings, the trajectory of the system might have been vastly different.

Topologically mixing means that the system will evolve over time so that any given region or open set of its phase space will eventually overlap with any other given region. Here, 'mixing' is really meant to correspond to the standard intuition: the mixing of coloured dyes or fluids is an example of a chaotic system.

Linear systems are never chaotic; for a dynamical system to display chaotic behaviour it has to be nonlinear. Also, by the Poincare Bendixson theorem, a continuous dynamical system on the plane cannot be chaotic; among continuous systems only those whose phase space is non-planar (having dimension at least three) can exhibit chaotic behaviour. However, a discrete dynamical system (such as the logistic map) can exhibit chaotic behaviour in a one-dimensional or two-dimensional phase space.
The bifurcation diagram of logistic map is shown in figure 4(a) and sensitivity to initial condition of chaotic logistic map is shown in figure 4(b). Time evolution of x and y component of the chaotic Lorenz system are shown in figure 5(a) and figure 5(b) respectively. The phase diagram of chaotic Lorenz system in the xz phase plane is presented in figure 5(c) and sensitivity to initial condition of chaotic Lorenz system is plotted in figure 5(d).

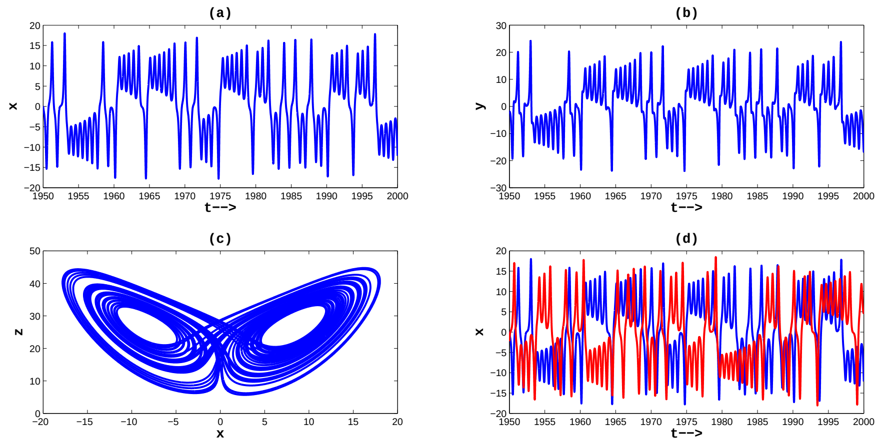

Figure 5: (a) Time evolution of the x-component of chaotic Lorenz system. (b) Time evolution of the y-component of chaotic Lorenz system. (c) Phase diagram of the Lorenz system in xz plane. (d) Sensitive dependence on initial conditions for Lorenz system is shown. Blue line denotes the time evolution of x-component of Lorenz system for initial conditions $x = 0.01, y = 0.0, z = 0.2$ and red line indicates the time evolution of the same system for $x = 0.01, y = 0.000001, z = 0.2$.

4.1 Distinguishing Random from Chaotic Data

It can be difficult to tell from data whether a physical or other observed process is random or chaotic, because in practice no time series consists of pure signal. There will always be some form of corrupting noise, is present as round-off or truncation error. Thus any real time series, even if mostly deterministic, will contain some randomness.

All methods for distinguishing deterministic and stochastic processes rely on the fact that a deterministic system always evolves in the same way from a given starting point. Thus, given a time series to test for determinism, one can (a) pick a test state,(b) search the time series for a similar or nearby state and (c) compare their respective time evolutions.

Define the error as the difference between the time evolution of the test state and the time evolution of the nearby state. A deterministic system will have an error that either remains small (stable, regular solution) or increases exponentially with time (chaos). A stochastic system will have a randomly distributed error. Random processes are fundamentally different than deterministic processes. Two successive realizations of a random process will give different sequences, even if the initial state is the same but for a chaotic system if initial conditions are same the generated data will be same.

5 Poincare Section

Poincase section gives us a geometric depiction of the trajectories in a lower dimensional space. Suppose we have a 3 dimensional (3D) flow. Instead of directly studying the flow in 3D, consider its intersection with a plane S. Let successive points of intersection of the trajectory with the S plane (from up to down direction) are the points $P_0, P_1, P_2, ..$ respectively. Then the points $P_0, P_1, P_2, P_3, ...$ form the 2D Poincare section. Here Poincare section is a continuous mapping T of the plane S into itself such that $P_{k+1} = T(P_k)$. Since the flow is deterministic P_0 determines P_1, P_1 determines P_2 etc.

The Poincare section reduces a continuous dynamical system to a discrete dynamical system. The constructed map is known as Poincare map. However the time interval of the map from point to point is not necessarily constant. With the help of Poincare map we shall be able to classify different types of flow. For limit cycle oscillation of a flow we shall get fixed point of the Poincare map. Poincare section of quasiperiodic flow looks like a continuous closed curve. For chaotic flow poincare section will be a dispersed set of points.

6 Applications

Applications of dynamical system theory in Physics, Biology, Chemistry and Engineering was discussed by Strogatz [2]. From the work of Helmholtz and Frank in the last century through to that of Hodgkin, Huxley, and many others in this century, physiologists have repeatedly used mathematical methods and models to help their understanding of physiological processes [3]. Modern areas of physiological research demands solid understanding of differential equations, including phase plane analysis and stability theory. Application of dynamical system theory in biological, medical, ecological, phychological and social sciences are going to play an increasingly important role in future major discoveries. Behavioral ecology is another important area of reaserch. How bird flocks, school of fish, and so on reach community discisions is another exciting relatively new area of research [4]. Dynamical system theory is successfully applied for marital inetraction and divorce prediction [5]. Numerical calculations of the dynamics of the Solar System and its constituents, integrations of stellar orbits in a galaxy and extensive simulations of the gravitational n-body problem (modelling star and galaxy clusters) have become a major part of mainstream geophysical and astrophysical research. Applications of the theory of nonlinear Hamiltonian dynamical systems to such problems are discussed extensively by Oded Regev [6].

Experimental observations have pointed out that, chaotic systems are common in nature. They can be found, for example, in Chemistry (Belouzov Zhabotinski reaction), in Nonlinear Optics (lasers), in Electronics (Chua Matsumoto circuit), in Fluid Dynamics (Rayleigh Benard convection), etc. Many natural phenomena can also be characterized as being chaotic. They can be found in meteorology, solar system, heart and brain of living organisms and so on. Although chaos theory was originally developed in the context of the physical sciences, Radzicki and Butler amongst others have noted that social, ecological, and economic systems tend to be characterized nonlinear relationships and complex interactions evolve dynamically over time [7]. This recognition has led to a surge of interest in applying chaos theory to number of fields incluidng ecology,medicine, international relations and economics. Although there is an enormous amount of research interest in the subject, there is only a small number of practical suggestions based on chaos theory which can be applied to everyday life in an attempt to retard ageing and optimise health. However, further research may confirm that by following these suggestions and recommendations it may be possible to stimulate the body and mind to work optimally and to postpone age-related disease and disability [8].

7 Summary

In this article, we introduce some of the preliminary concepts of dynamical system. Application areas of the dynamical system theory are discussed briefly. References are supplied for further reading. Many open areas for future research are discussed.

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