International Journal of Applied Mathematical Research, 1 (4) (2012) 643-658

www.sciencepubco.com/index.php/IJAMR

Fuzzy Linear Fractional Bi-Level Multi-Objective Programming Problems

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Abstract

The Kuhn-Tuker condition has become nowadays an important tool in the hands of investigation for checking the optimality in optimization literature. In the present paper with use of a Taylor series and Kuhn-Tucker conditions approach, we solve a fuzzy linear fractional bilevel multi-objective programming (FLFBL-MOP) problem. The Taylor series is an expansion of a series that represents a function. In the proposed approach, membership functions associated with each level(s) of the objective(s) of FLFBL-MOP problems are transformed and unified by using a Taylor series approach. By using the Kuhn-Tucker conditions, the problem is reduced to a single objective and finally, numerical example is given to illustrates the efficiency and superiority of the proposed approach.

Keywords: *Fuzzy programming, fractional programming, bi-level multi-objective programming, Kuhn-Tucker conditions, Taylor series.*

1 Introduction

Bi-level mathematical programming (BLMP) is identified as mathematical programming that solves decentralized planning problems with two decision makers (DMs) in a two level or hierarchical organization. The basic concept of the BLMP technique is that the upper level decision maker (ULDM) (the leader) sets his goals and/or decisions and then asks each subordinate level of the organization for their optima which are calculated in isolation; the lower level DM (LLDM) (the follower) decisions are then submitted and modified by the ULDM with consideration of the overall benefit for the organization; the process continued until a satisfactory solution is reached. In other words, although the ULDM independently optimizes its own benefits, the decision may be affected by the reaction of the LLDM. As a consequence, decision deadlock arises frequently and the problem of distribution of proper decision power is encountered in most of the practical decision situations. Most of the developments on BLMP problems focus on bi-level linear programming [1-4], and many others for bilevel nonlinear programming and bi-level multiobjective programming[3,5,6]. The use of the fuzzy set theory for decision problems with several conflicting objectives was first introduced by Zimmermann[7]. There after, various versions of fuzzy programming have been investigated and widely circulated in literature. In a hierarchical decision making context, it has been realized that each DM should have a motivation to cooperate with other, and a minimum level of satisfaction of the DM at a lower-level must be considered for overall benefit of the organization. The use of the concept of membership function of fuzzy set theory to BLMP problems for satisfactory decisions was first introduced by Lai[8]. The basic concept of the fuzzy programming approaches implies that the LLDM optimizes his/her objective function, taking a goal or preference of the ULDM into consideration. In the decision process, considering the membership functions of the fuzzy goals for the decision variables of the ULDM, the LLDM solves a FP problem with the set of constraints on an overall satisfactory degree of the ULDM. If the proposed solution is not satisfactory to the ULDM, the solution search is continued by redefining the elicited membership functions until a satisfactory solution is reached[9]. The main difficulty that arises with the FP approach of Shih et al [10], is that there is possibility of rejecting the solution again and again by the ULDM and reevaluation of the problem is repeatedly needed to reach the satisfactory decision, where the objectives of the DMs are overconflicting. Even inconsistency between the fuzzy goals of the objectives and the decision variables may arise. This makes the solution process a lengthy one [9]. Fuzzy programming approach to multi-level programming problems was studied by Sinha [11]. The Baky investigated the problem of Fuzzy goal programming algorithm for solving decentralized bi-level multi-objective programming problems[12]. In this paper, membership functions, which are associated with each objectives of each levels of FLFBL-MOP. are transformed to linear form by using first-order Taylor polynomial series. Here, the obtained Taylor series which has polynomial membership functions are equivalent to fractional membership functions which is associated to each objectives of each levels and Reduce the FLFBL-MOP into a single objective.

In other words, suitable transformation can be applied to formulate an equivalent fuzzy linear fractional bileve multi objective programming problem. The performance of the proposed method was experimentally validated by example. Results demonstrate that the proposed approach runs more effectively.

2 Problem Formulation

Assume that there are two levels in a hierarchy structure with upper level decision maker(ULDM)and lower level decision maker(LLDM). Let the vector of decision variables $x = (x_1, x_2)$ be partitioned between the two planners. The upper level decision maker has control over the vector $x_1 \in R^{n_1}$ and the lower level decision maker has control over the vector $x_2 \in R^{n_2}$, where $n = n_1 + n_2$. Furthermore, assume that

$$F_i(x_1, x_2) : R^{n_1} \times R^{n_2} \longrightarrow R^{m_i}, \quad i = 1, 2$$

are the upper level and lower level vector objective functions respectively. So the LFBL-MOP problem of maximization type may be formulated as follows[5,12]:

[1st level]

$$\underset{x_1}{\text{Max}} \; F_1(x_1, x_2) = \text{Max } (f_{11}(x_1, x_2), f_{12}(x_1, x_2), \ldots, f_{1m_1}(x_1, x_2))$$

where x_2 solves

[2nd level]

$$\text{Max} \quad F_2(x_1, x_2) = \text{Max } (f_{21}(x_1, x_2), f_{22}(x_1, x_2), \ldots, f_{2m_2}(x_1, x_2))$$

subject to

$$x \in G = \left\{ x \in R^n \mid A_1 x_1 + A_2 x_2 \leqslant b, x \geqslant 0, b \in R^m \right\} \neq \varnothing \tag{1}$$

Where

$$f_{ij}(x_1, x_2) = \frac{c_{ij}x + \alpha_{ij}}{d_{ij}x + \beta_{ij}}$$

for i= 1, we have j = 1, 2, . . . , m_1, for ULDM objective functions,
for i=2, we have j = 1, 2, . . . , m_2, for LLDM objective functions,
and where

(1) $x_1 \in R^{n_1},\ x_2 \in R^{n_2}$,
(2)G is the bi-level convex constraints feasible choice set,
(3) m_1 is the number of first-level objective functions,
(4) m_2 is the number of second-level objective functions,
(5) m is the number of the constraints,
(6) A_i: $m \times n_i$ matrix, i = 1, 2,

(7) $c_{ij}, d_{ij} \in R^n, d_{ij}x + \beta_{ij} > 0$ for all $x \in G$,
(8) α_{ij}, β_{ij} are constants.

3 Fuzzy Linear Fractional Bilevel Multi-Objective Programming

If an imprecise aspiration level is introduced to each of the level of BFP, then we call it as a fuzzy level. The FLFBL-MOP can be written as follows:

$$
\begin{aligned}
& Find x_1 \\
& so as to satisfy \\
& f_{1j}(x_1,x_2) \lesssim g_{1j} \qquad (f_{1j}(x_1,x_2) \gtrsim g_{1j}),\ \text{j} = 1, 2, \ldots, m_1 \\
& where x_2 solves \\
& f_{2j}(x_1,x_2) \lesssim g_{2j} \qquad (f_{2j}(x_1,x_2) \gtrsim g_{2j}),\ j = 1, 2, ..., m_2 \\
& \qquad subject to \qquad\qquad (2) \\
& \qquad Ax_1 + Ax_2 \leqslant b \\
& \qquad x_1,\ x_2 \geqslant 0, x_1 \in R^{n_1},\ x_2 \in R^{n_2},\ A_1 \in R^{m \times n_1}, \\
& \qquad A_2 \in R^{m \times n_2},\ b \in R^m
\end{aligned}
$$

where g_{1j} is the aspiration level of the ULDM and g_{2j} is the aspiration level of the LLDM. $\lesssim$ and $\gtrsim$ indicate fuzziness of the aspiration levels, which are described as essentially less than and essentially more than , respectively. $\lesssim$ minimizes $f_{ij}(x_1,x_2)$ and $\gtrsim$ maximizes $f_{ij}(x_1,x_2)$.
A membership function must be described for each fuzzy level. A membership function can be ststed as follows for i=1, 2 and j=1,. . . , m_i:

if $f_{ij}(x_1,x_2) \gtrsim g_{ij}$

$$
\mu_{f_{ij}}(f_{ij}(x)) = \begin{cases} 1 & f_{ij}(x_1,x_2) \geqslant g_{ij} \\ \dfrac{f_{ij}(x_1,x_2) - \underline{t_{ij}}}{g_{ij} - \underline{t_{ij}}} & \underline{t_{ij}} \leqslant f_{ij}(x_1,x_2) \leqslant g_{ij} \\ 0 & f_{ij}(x_1,x_2) \leqslant \underline{t_{ij}} \end{cases}
$$

if $f_{ij}(x_1,x_2) \lesssim g_{ij}$

$$
\mu_{f_{ij}}(f_{ij}(x)) = \begin{cases} 1 & f_{ij}(x_1,x_2) \leqslant g_{ij} \\ \dfrac{\overline{t_{ij}} - f_{ij}(x_1,x_2)}{\overline{t_{ij}} - g_{ij}} & g_{ij} \leqslant f_{ij}(x_1,x_2) \leqslant \overline{t_{ij}} \\ 0 & f_{ij}(x_1,x_2) \geqslant \overline{t_{ij}} \end{cases} \qquad (3)
$$

where $\overline{t_{ij}}$ and $\underline{t_{ij}}$ are the upper and lower tolerance limits, respectively, for each fuzzy level.

4 The Taylor Series and Kuhn-Tucker Conditions for Solving Flfbl-Mop Problems

In the FLFBL-MOLP problems, membership functions associated to each of the objectives in each level are firstly transformed by using Taylor series and then a satisfactory value(s) for the variable(s) of the model is obtained by solving the fuzzy model, which has a single objective function. Here, the fractional linear membership functions from each objectives of each levels is converted to a linear polynomial on using Taylor series . Then, the FLFBL-MOLP on using Kuhn-Tucker conditions can be reduced to a single objective. The proposed approach can be explained in four steps.

Step 1. Determine $x_{ij}^* = (x_{ij}^{1*}, x_{ij}^{2*})$ (i = 1,2 and j = 1, 2,. . ., m_i) which is the value(s) that is used to maximize the each of the objectives in upper level and lower level membership Function $\mu_{f_{ij}}(x)$ associated to upper level and lower level $f_{ij}(x_1, x_2)$ (i = 1,2 and j = 1, 2, . . ., m_i) where n is the number of the variables.

Step 2. Transform membership functions by using first-order Taylor polynomial series

$$\mu_{f_{ij}}(f_{ij}(x)) \cong \widehat{\mu_{f_{ij}}}(f_{ij}(x)) = \mu_{f_{ij}}(f_{ij}(x_{ij}^*)) + \Big((x_1 - x_{ij}^{1*})\frac{\partial}{\partial x_1} + (x_2 - x_{ij}^{2*})\frac{\partial}{\partial x_2}\Big)\mu_{f_{ij}}(f_{ij}(x_{ij}^*)) \tag{4}$$

$$\mu_{f_{ij}}(f_{ij}(x)) \cong \widehat{\mu_{f_{ij}}}(f_{ij}(x)) = \mu_{f_{ij}} f_{ij}(x_{ij}^*) + \sum_{k=1}^{2}(x_k - x_{ij}^{k*})\frac{\partial \mu_{f_{ij}} f_{ij}(x_{ij}^*))}{\partial x_k}$$

Step 3. Sum the Membership functions together for the upper level. Note that problem is solved by assuming that weights of the objectives in upper level are equal.

$$P(x) = \sum_{j=1}^{m_1}\Big(\mu_{f_{1j}}(f_{1j}(x_{1j}^*)) + \sum_{k=1}^{2}(x_k - x_{1j}^{k*})\frac{\partial \mu_{f_{1j}}(f_{1j}(x_{1j}^*))}{\partial x_k}\Big) \tag{5}$$

Step 4. After applying the Kuhn-Tucker conditions to the lower level of the objective problem, we find satisfactory $x^* = ({x^*}_1, \; {x^*}_2)$ by solving the reduced problem to a single objective.

FLFBP is converted into a new mathematical model. This model is represent as follows:

$$
\begin{aligned}
\text{Max} \quad & P(x) \\
s.t \quad & \\
& A_1 x_1 + A_2 x_2 + u = b \\
& wA_2 - \nu = \sum_{j=1}^{m_2} \frac{\partial \mu_{f_{2j}}(f_{2j}(x^*_{2j}))}{\partial x_2} \\
& wu = 0,\ x_2\nu = 0 \\
& x_1,\ x_2,\ w,\ u,\ \nu \geqslant 0
\end{aligned}
\tag{6}
$$

In this method, a zero-one variable, η and ξ, is added for each constraint $wu = 0$ and $x_2\nu = 0$, respectively. In addition, each of these constraints is replaced by two linear inequalities involving η and ξ and M, a large positive. The auxiliary formulation now becomes

$$
\begin{aligned}
Max \quad & P(x) \\
s.t \quad & \\
& A_1 x_1 + A_2 x_2 + u = b \\
& wA_2 - \nu = \sum_{j=1}^{m_2} \frac{\partial \mu_{f_{2j}}(f_{2j}(x^*_{2j}))}{\partial x_2} \\
& w \leqslant M\eta,\ u \leqslant M(1-\eta) \\
& x_2 \leqslant M\xi,\ \nu \leqslant M(1-\xi) \\
& \eta,\ \xi \in \{0,1\} \\
& x_1,\ x_2,\ w,\ u,\ \nu \geqslant 0
\end{aligned}
\tag{7}
$$

5 Numerical Example

An illusrative numerical example.

$$
\operatorname*{Max}_{x_1} \ (f_{11} = \frac{2x_1 + x_2 + 2}{2x_2 + 1}, f_{12} = \frac{x_1}{x_1 + x_2 + 1})
$$

where x_2 solve

$$
\text{Max} \ (f_{21} = \frac{3x_1 + 2x_2 - 1}{-x_1 + x_2 + 4}, f_{22} = \frac{-x_1 + x_2 + 3}{x_1 + 5}) \tag{8}
$$

subject to

$$
\begin{aligned}
& x_1 + x_2 \leqslant 4 \\
& x_1 - 2x_2 \leqslant 1 \\
& 3x_1 + x_2 \geqslant 3 \\
& x_1, x_2 \geqslant 0
\end{aligned}
$$

Let the fuzzy aspiration levels of the objectives in the bilevel to be (3, 0, 1, 1), respectively, so we have:

$Find x_1$

$so as to satisfy$

$$f_{11} = \frac{2x_1 + x_2 + 2}{2x_2 + 1} \gtrsim 3, \quad f_{12} = \frac{x_1}{x_1 + x_2 + 1} \gtrsim 0$$

$where x_2 solve$

$$f_{21} = \frac{3x_1 + 2x_2 - 1}{-x_1 + x_2 + 4} \gtrsim 1, \quad f_{22} = \frac{-x_1 + x_2 + 3}{x_1 + 5} \gtrsim 1 \tag{9}$$

$subject to$

$$\begin{aligned} &x_1 + x_2 \leqslant 4 \\ &x_1 - 2x_2 \leqslant 1 \\ &3x_1 + x_2 \geqslant 3 \\ &x_1, x_2 \geqslant 0 \end{aligned}$$

Let us assume that, the tolerance limits of the objectives in the bilevel are (1, 1, 1, 1) respectively. The membership functions of the bilevel are as follows:

First, membership functions are defined to be simple piecewise linear (see Fig.1). The membership functions of the levels are obtained as follows:

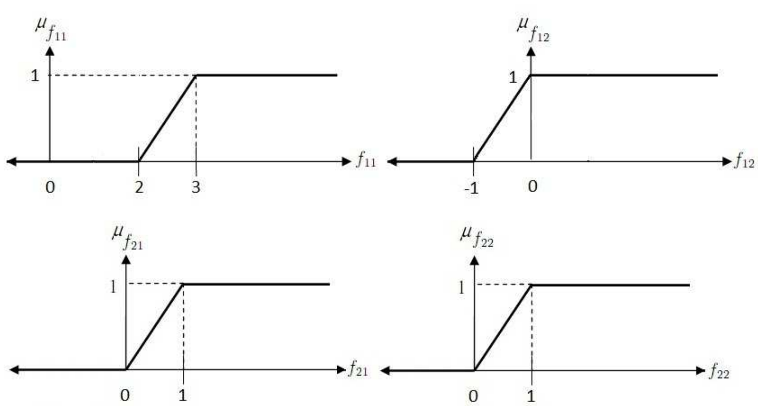

Figure 1: *simple piecewise linear membership functions.*

$$\mu_{f_{11}}(f_{11}(x)) = \begin{cases} 1 & \text{if } \ f_{11}(x_1, x_2) \geqslant g_{11} \\ \dfrac{f_{11}(x_1, x_2) - \underline{t_{11}}}{g_{11} - \underline{t_{11}}} & \text{if } \ \underline{t_{11}} \leqslant f_{11}(x_1, x_2) \leqslant g_{11} \\ 0 & \text{if } \ f_{11}(x_1, x_2) \leqslant \underline{t_{11}} \end{cases}$$

$$= \begin{cases} 1 & \text{if } \ f_{11}(x_1, x_2) \geqslant 3 \\ \dfrac{2x_1 - 3x_2}{2x_2 + 1} & \text{if } \ 2 \leqslant f_{11}(x_1, x_2) \leqslant 3 \\ 0 & \text{if } \ f_{11}(x_1, x_2) \leqslant 2 \end{cases} \tag{10}$$

In the same way, the other membership functions are formed as

$$\mu_{f_{12}}(f_{12}(x)) = \begin{cases} 1 & \text{if } \ f_{12}(x_1,x_2) \geqslant 0 \\ \dfrac{2x_1+x_2+1}{x_1+x_2+1} & \text{if } \ -1 \leqslant f_{12}(x_1,x_2) \leqslant 0 \\ 0 & \text{if } \ f_{12}(x_1,x_2) \leqslant -1 \end{cases} \tag{11}$$

$$\mu_{f_{21}}(f_{21}(x)) = \begin{cases} 1 & \text{if } \ f_{21}(x_1,x_2) \geqslant 1 \\ \dfrac{3x_1+2x_2-1}{-x_1+x_2+4} & \text{if } \ 0 \leqslant f_{21}(x_1,x_2) \leqslant 1 \\ 0 & \text{if } \ f_{21}(x_1,x_2) \leqslant 0 \end{cases} \tag{12}$$

$$\mu_{f_{22}}(f_{22}(x)) = \begin{cases} 1 & \text{if } \ f_{22}(x_1,x_2) \geqslant 1 \\ \dfrac{-x_1+x_2+3}{x_1+5} & \text{if } \ 0 \leqslant f_{22}(x_1,x_2) \leqslant 1 \\ 0 & \text{if } \ f_{22}(x_1,x_2) \leqslant 0 \end{cases} \tag{13}$$

If the problem is solved for each membership functions, one by one, then $\mu^*_{f_{11}}(f_{11}(1,0))$, $\mu^*_{f_{12}}(f_{12}(3{,}1))$, $\mu^*_{f_{21}}(f_{21}(3{,}1))$ and $\mu^*_{f_{22}}(f_{22}(0,\ 4))$ are obtained. Now the membership functions are transformed by using first-order Taylor polynomial series.

$$\begin{aligned} \mu_{f_{11}}(f_{11}(x)) \cong \widehat{\mu_{f_{11}}}(f_{11}(x)) = \mu_{f_{11}}(f_{11}(1,0)) + \Big((x_1-1)\times\frac{\partial}{\partial x_1} \\ +(x_2-0)\times\frac{\partial}{\partial x_2}\Big)\mu_{f_{11}}(f_{11}(1,0)) = 2x_1 - 5x_2 \end{aligned} \tag{14}$$

In the same manner, the other membership functions are transformed on using first-order Taylor polynomial series as follows:

$$\mu_{f_{12}}(f_{12}(x)) = 1.52 + .08x_1 - 0.12x_2 \tag{15}$$

$$\mu_{f_{21}}(f_{21}(x)) = 5.5 + 4x_1 - 1.5x_2 \tag{16}$$

$$\mu_{f_{22}}(f_{22}(x)) = 0.6 - 0.48x_1 + 0.2x_2 \tag{17}$$

The P(x) is obtained by adding (14) and (15) as follows:

$$P(x) = \mu_{f_{11}}(f_{11}(x)) + \mu_{f_{12}}(f_{12}(x)) = 1.52 + 2.08x_1 - 5.12x_2 \tag{18}$$

After applying the Kuhn-Tucker conditions to the lower level of the objectives problem, a new auxiliary problem is to be solved

$$
\begin{aligned}
\text{Max} \quad & P(x) = 1.52 + 2.08x_1 - 5.12x_2 \\
s.t & \\
& x_1 + x_2 + u_1 = 4 \\
& x_1 - 2x_2 + u_2 = 1 \\
& 3x_1 + x_2 - u_3 = 3 \\
& w_1 - 2w_2 - w_3 - \nu = -1.5 + 0.2 \\
& w_i \leqslant M\eta_i, \quad u_i \leqslant M(1-\eta_i) \\
& x_2 \leqslant M\xi, \quad \nu \leqslant M(1-\xi) \\
& \eta_i,\ \xi \in \{0,1\}, \\
& x_1,\ x_2,\ w_i,\ u_i,\ \nu \geqslant 0 \quad i = 1, \ldots, 3
\end{aligned} \tag{19}
$$

We solve the problem for M = 1000 and the solution is obtained as follows:

$$x_1^* = 1, \qquad x_2^* = 0,$$
$$f_{11}(x_1, x_2) = 4,\ f_{12}(x_1, x_2) = 0.5,\ f_{21}(x_1, x_2) = 0.67,\ f_{22}(x_1, x_2) = 0.33$$

and the membership values are

$$\mu_{f_{11}}(f_{11}) = 1, \quad \mu_{f_{12}}(f_{12}) = 1, \quad \mu_{f_{21}}(f_{21}) = 0.67, \quad \mu_{f_{22}}(f_{11}) = 0.33$$

The membership function values indicates that the levels $f_{11}\&f_{12}$ are satisfied 100% but f_{21} is satisfied 67% and f_{22} is satisfied 33% with the solutions of $x_1^* = 1$and $x_2^* = 0$.

Now further more we assume that the membership functions to be triangular (see Fig.2) which depends on three scalar parameters (a, b, c). Let f_{11} be depends on three scalar parameters (3, 4, 5) where f_{12} depends on (0, 0.5, 1), f_{21} on (0, 1, 3) and f_{22} be depends on (-1, 1, 3). As shown given by $f_{11}(x_1, x_2; 3, 4, 5)$, $f_{12}(x_1, x_2; (0, 0.5, 1)$, $f_{21}(x_1, x_2; (0, 1, 3)$ and $f_{22}(x_1, x_2; (-1, 1, 3)$. The membership functions of the level are obtained as follows:

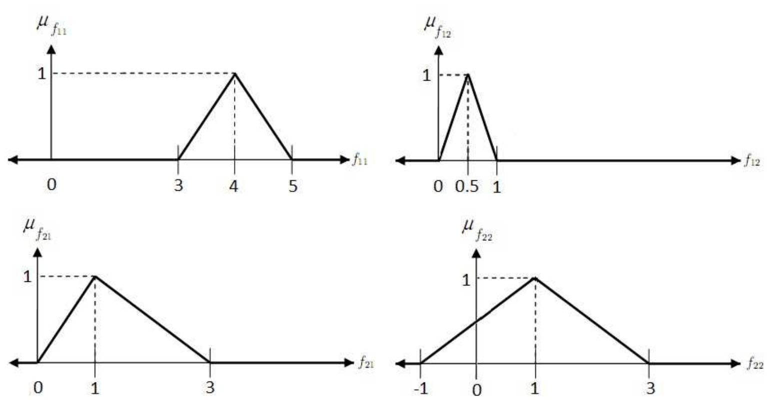

Figure 2: *Triangular membership functions.*

$$\mu_{f_{11}}(f_{11}(x)) = \begin{cases} 0 & \text{if } \ f_{11}(x_1,x_2) \geqslant c_{11} \\ \dfrac{c_{11} - f_{11}(x_1,x_2)}{c_{11} - b_{11}} & \text{if } \ b_{11} \leqslant f_{11}(x_1,x_2) \leqslant c_{11} \\ \dfrac{f_{11}(x_1,x_2) - a_{11}}{b_{11} - a_{11}} & \text{if } \ a_{11} \leqslant f_{11}(x_1,x_2) \leqslant b_{11} \\ 0 & \text{if } \ f_{11}(x_1,x_2) \leqslant a_{11} \end{cases}$$

$$= \begin{cases} 0 & \text{if } \ f_{11}(x_1,x_2) \geqslant 5 \\ \dfrac{-2x_1 + 9x_2 + 3}{2x_2 + 1} & \text{if } \ 4 \leqslant f_{11}(x_1,x_2) \leqslant 5 \\ \dfrac{2x_1 - 5x_2 - 1}{2x_2 + 1} & \text{if } \ 3 \leqslant f_{11}(x_1,x_2) \leqslant 4 \\ 0 & \text{if } \ f_{11}(x_1,x_2) \leqslant 3 \end{cases} \tag{20}$$

In the same way, the other membership functions are formed as

$$\mu_{f_{12}}(f_{12}(x)) = \begin{cases} 1 & \text{if } \ f_{12}(x_1,x_2) \geqslant 1 \\ \dfrac{x_2 + 1}{0.5x_1 + 0.5x_2 + 0.5} & \text{if } \ 0.5 \leqslant f_{12}(x_1,x_2) \leqslant 1 \\ \dfrac{x_1}{0.5x_1 + 0.5x_2 + 0.5} & \text{if } \ 0 \leqslant f_{12}(x_1,x_2) \leqslant 0.5 \\ 0 & \text{if } \ f_{12}(x_1,x_2) \leqslant 0 \end{cases} \tag{21}$$

$$\mu_{f_{21}}(f_{21}(x)) = \begin{cases} 0 & \text{if } \ f_{21}(x_1,x_2) \geqslant 3 \\ \dfrac{-6x_1 + x_2 + 13}{-2x_1 + 2x_2 + 8} & \text{if } \ 1 \leqslant f_{21}(x_1,x_2) \leqslant 3 \\ \dfrac{3x_1 + 2x_2 - 1}{-x_1 + x_2 + 4} & \text{if } \ 0 \leqslant f_{21}(x_1,x_2) \leqslant 1 \\ 0 & \text{if } \ f_{21}(x_1,x_2) \leqslant 0 \end{cases} \tag{22}$$

$$\mu_{f_{22}}(f_{22}(x)) = \begin{cases} 0 & \text{if } \ f_{22}(x_1,x_2) \geqslant 3 \\ \dfrac{4x_1 - x_2 + 12}{2x_1 + 10} & \text{if } \ 1 \leqslant f_{22}(x_1,x_2) \leqslant 3 \\ \dfrac{x_2 + 8}{2x_1 + 10} & \text{if } \ -1 \leqslant f_{22}(x_1,x_2) \leqslant 1 \\ 0 & \text{if } \ f_{22}(x_1,x_2) \leqslant -1 \end{cases} \tag{23}$$

If

$$\mu_{f_{11}}(f_{11}(x)) = max(min(\frac{-2x_1 + 9x_2 + 3}{2x_2 + 1}, \ \frac{2x_1 - 5x_2 - 1}{2x_2 + 1}), \ 0) \tag{24}$$

and

$$\mu_{f_{12}}(f_{12}(x)) = max(min(\frac{x_2 + 1}{0.5x_1 + 0.5x_2 + 0.5}, \ \frac{x_1}{0.5x_1 + 0.5x_2 + 0.5}), \ 0) \tag{25}$$

and

$$\mu_{f_{21}}(f_{21}(x)) = max(min(\frac{-6x_1 + x_2 + 13}{-2x_1 + 2x_2 + 8}, \frac{3x_1 + 2x_2 - 1}{-x_1 + x_2 + 4}), 0) \tag{26}$$

and

$$\mu_{f_{22}}(f_{22}(x)) = max(min(\frac{4x_1 - x_2 + 12}{2x_1 + 10}, \frac{x_2 + 8}{2x_1 + 10}), 0) \tag{27}$$

Then the following results are obtained $\mu^*_{f_{11}}(f_{11}(1,0)), \mu^*_{f_{12}}(f_{12}((0,4)), \mu^*_{f_{21}}(f_{21}((1,0))$ and $\mu^*_{f_{22}}(f_{22}((0,4))$. Now as before the membership functions are transformed by using first-order Taylor polynomial series.

$$\mu_{f_{11}}(f_{11}(x)) \cong \widehat{\mu_{f_{11}}}(f_{11}(x)) = \mu_{f_{11}}(f_{11}(1,0)) + \Big((x_1 - 1) \times \frac{\partial}{\partial x_1} +$$

$$(x_2 - 0) \times \frac{\partial}{\partial x_2}\Big)\mu_{f_{11}}(f_{11}(1,0)) = -1 + 2x_1 - 7x_2 \tag{28}$$

In the same way, the other membership functions are transformed on using first-order Taylor polynomial series as follows:

$$\mu_{f_{12}}(f_{12}(x)) = 0.48 + 0.32x_1 - 0.24x_2 \tag{29}$$

$$\mu_{f_{21}}(f_{21}(x)) = 1.78 - 0.61x_1 - 0.22x_2 \tag{30}$$

$$\mu_{f_{22}}(f_{22}(x)) = 0.8 - .24x_1 + 0.1x_2 \tag{31}$$

Adding (28) and (29), we get P(x),

$$P(x) = \mu_{f_{11}}(f_{11}(x)) + \mu_{f_{12}}(f_{12}(x)) = -0.52 + 2.32x_1 - 7.24x_2 \tag{32}$$

After applying the Kuhn-Tucker conditions to the lower level of the objectives problem, a new auxiliary problem as below is to be solved

$$\begin{aligned}
\text{Max} \quad & P(x) = -0.52 + 2.32x_1 - 7.24x_2 \\
s.t \quad & \\
& x_1 + x_2 + u_1 = 4 \\
& x_1 - 2x_2 + u_2 = 1 \\
& 3x_1 + x_2 - u_3 = 3 \\
& w_1 - 2w_2 - w_3 - \nu = -0.22 + 0.1 \\
& w_i \leqslant M\eta_i, \quad u_i \leqslant M(1 - \eta_i) \\
& x_2 \leqslant M\xi, \quad \nu \leqslant M(1 - \xi) \\
& \eta_i, \ \xi \in \{0, 1\}, \\
& x_1, \ x_2, \ w_i, \ u_i, \ \nu \geqslant 0 \quad i = 1, \ldots, 3
\end{aligned} \tag{33}$$

The problem with M=1000 is solved and the solution is obtained as follows:

$$x_1^* = 1, \qquad x_2^* = 0,$$

$$f_{11}(x_1,x_2)=4,\;\; f_{12}(x_1,x_2)=0.5,\;\; f_{21}(x_1,x_2)=0.67,\;\; f_{22}(x_1,x_2)=0.33$$

and the membership values are

$$\mu_{f_{11}}(f_{11})=1, \quad \mu_{f_{12}}(f_{12})=1, \quad \mu_{f_{21}}(f_{21})=0.67, \quad \mu_{f_{22}}(f_{22})=0.67$$

The membership function values indicate that the level f_{11} is satisfied 67%, while f_{12} is satisfied 100% accordingly f_{21}, f_{22} are satisfiing with 0.67% with the solution of $x_1^*=1$and $x_2^*=0$, The result indicates that the proposed solution method is very simple, efficient and robust. Finally in the same fashion the membership functions are considered to be trapezoidal (see Fig.3) which dependeds on four scalar parameters (a, b, c, d). Let f_{11} be depends on four scalar parameters (2, 3, 4, 5), f_{12} on (-1, 0, 1, 3), f_{21} depends on four scalar parameters (0, 1, 3, 5) and f_{22} be depends on (0, 1, 2, 4), as given by $f_{11}(x_1,x_2;2,3,4,5), f_{12}(x_1,x_2;-1,0,1,3), f_{21}(x_1,x_2;\,0,1,3,5)$ and $f_{22}(x_1,x_2;0,1,2,4)$. The membership functions of the levels are obtained as follows:

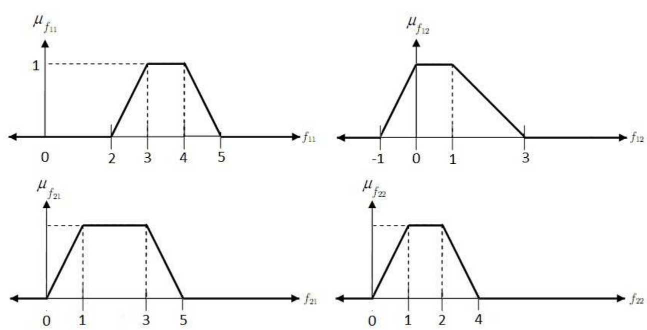

Figure 3: *Trapezoidal membership functions*

$$\mu_{f_{11}}(f_{11})=\begin{cases} 0 & \text{if } \; f_{11}(x_1,x_2)\geqslant d_{11} \\ \dfrac{d_{11}-f_{11}(x_1,x_2)}{d_{11}-c_{11}} & \text{if } \; c_{11}\leqslant f_{11}(x_1,x_2)\leqslant d_{11} \\ 1 & \text{if } \; b_{11}\leqslant f_{11}(x_1,x_2)\leqslant c_{11} \\ \dfrac{f_{11}(x_1,x_2)-a_{11}}{b_{11}-a_{11}} & \text{if } \; a_{11}\leqslant f_{11}(x_1,x_2)\leqslant b_{11} \\ 0 & \text{if } \; f_{11}(x_1,x_2)\leqslant a_{11} \end{cases}$$

$$\mu_{f_{11}}(f_{11}) = \begin{cases} 0 & \text{if } \ f_{11}(x_1, x_2) \geqslant 5 \\ \dfrac{-2x_1 + 9x_2 + 3}{2x_2 + 1} & \text{if } \ 4 \leqslant f_{11}(x_1, x_2) \leqslant 5 \\ 1 & \text{if } \ 3 \leqslant f_{11}(x_1, x_2) \leqslant 4 \\ \dfrac{2x_1 - 3x_2}{2x_2 + 1} & \text{if } \ 2 \leqslant f_{11}(x_1, x_2) \leqslant 3 \\ 0 & \text{if } \ f_{11}(x_1, x_2) \leqslant 2 \end{cases} \tag{34}$$

In the same way, the other membership functions are formed as

$$\mu_{f_{12}}(f_{12}) = \begin{cases} 0 & \text{if } \ f_{12}(x_1, x_2) \geqslant 3 \\ \dfrac{2x_1 + 3x_2 + 3}{2x_1 + 2x_2 + 2} & \text{if } \ 1 \leqslant f_{12}(x_1, x_2) \leqslant 3 \\ 1 & \text{if } \ 0 \leqslant f_{12}(x_1, x_2) \leqslant 1 \\ \dfrac{2x_1 + x_2 + 1}{x_1 + x_2 + 1} & \text{if } \ -1 \leqslant f_{12}(x_1, x_2) \leqslant 0 \\ 0 & \text{if } \ f_{12}(x_1, x_2) \leqslant -1 \end{cases} \tag{35}$$

$$\mu_{f_{21}}(f_{21}) = \begin{cases} 0 & \text{if } \ f_{21}(x_1, x_2) \geqslant 5 \\ \dfrac{-8x_1 + 3x_2 + 21}{-2x_1 + 2x_2 + 8} & \text{if } \ 3 \leqslant f_{21}(x_1, x_2) \leqslant 5 \\ 1 & \text{if } \ 1 \leqslant f_{21}(x_1, x_2) \leqslant 3 \\ \dfrac{3x_1 + 2x_2 - 1}{-x_1 + x_2 + 4} & \text{if } \ 0 \leqslant f_{21}(x_1, x_2) \leqslant 1 \\ 0 & \text{if } \ f_{21}(x_1, x_2) \leqslant 0 \end{cases} \tag{36}$$

$$\mu_{f_{22}}(f_{22}) = \begin{cases} 0 & \text{if } \ f_{22}(x_1, x_2) \geqslant 4 \\ \dfrac{5x_1 - x_2 + 17}{2x_1 + 10} & \text{if } \ 2 \leqslant f_{22}(x_1, x_2) \leqslant 4 \\ 1 & \text{if } \ 1 \leqslant f_{22}(x_1, x_2) \leqslant 2 \\ \dfrac{-x_1 + x_2 + 3}{x_1 + 5} & \text{if } \ 0 \leqslant f_{22}(x_1, x_2) \leqslant 1 \\ 0 & \text{if } \ f_{22}(x_1, x_2) \leqslant 0 \end{cases} \tag{37}$$

If

$$\mu_{f_{11}}(f_{11}(x)) = max(min(\frac{-2x_1 + 9x_2 + 3}{2x_2 + 1}, \ 1, \ \frac{2x_1 - 3x_2}{2x_2 + 1}), \ 0) \tag{38}$$

and

$$\mu_{f_{12}}(f_{12}(x)) = max(min(\frac{2x_1 + 3x_2 + 3}{2x_1 + 2x_2 + 2}, \ 1, \ \frac{2x_1 + x_2 + 1}{x_1 + x_2 + 1}), \ 0) \tag{39}$$

and

$$\mu_{f_{21}}(f_{21}(x)) = max(min(\frac{-8x_1 + 3x_2 + 21}{-2x_1 + 2x_2 + 8},\ 1,\ \frac{3x_1 + 2x_2 - 1}{-x_1 + x_2 + 4}),\ 0) \tag{40}$$

and

$$\mu_{f_{22}}(f_{22}(x)) = max(min(\frac{5x_1 - x_2 + 17}{2x_1 + 10},\ 1,\ \frac{-x_1 + x_2 + 3}{x_1 + 5}),\ 0) \tag{41}$$

then $\mu^*_{f_{11}}(f_{11}(1,0), \mu^*_{f_{12}}(f_{12}(3,1), \mu^*_{f_{21}}(f_{21}(1,0)$ and $\mu^*_{f_{22}}(f_{22}(0,4)$is obtained. Now, the membership functions are transformed by using first-order Taylor polynomial series.

In the same way, the other membership functions are transformed by using first-order Taylor polynomial series, in which we obtain.

$$\mu_{f_{12}}(f_{12}(x)) = 1.48 + 0.08x_1 - 0.12x_2 \tag{42}$$

$$\mu_{f_{21}}(f_{21}(x)) = 2.77 - 0.61x_1 - 0.22x_2 \tag{43}$$

$$\mu_{f_{22}}(f_{22}(x)) = 0.6 - 0.32x_1 + 0.2x_2 \tag{44}$$

The P(x) is obtained by adding (42) and (43) as follows:

$$P(x) = \mu_{f_{11}}(f_{11}(x)) + \mu_{f_{12}}(f_{12}(x)) = 1.48 + 2.08x_1 - 7.12x_2 \tag{45}$$

After applying the Kuhn-Tucker conditions to the lower level of the objectives problem, a new auxiliary problem is to be solved.

$$\begin{aligned}
&\text{Max} \quad P(x) = 1.48 + 2.08x_1 - 7.12x_2 \\
&\quad s.t \\
&\qquad x_1 + x_2 + u_1 = 4 \\
&\qquad x_1 - 2x_2 + u_2 = 1 \\
&\qquad 3x_1 + x_2 - u_3 = 3 \\
&\qquad w_1 - 2w_2 - w_3 - \nu = -0.22 + 0.2 \\
&\qquad w_i \leqslant M\eta_i, \quad u_i \leqslant M(1-\eta_i) \\
&\qquad x_2 \leqslant M\xi, \quad \nu \leqslant M(1-\xi) \\
&\qquad \eta_i,\ \xi \in \{0,1\}, \\
&\qquad x_1,\ x_2,\ w_i,\ u_i,\ \nu \geqslant 0 \quad i = 1,\dots,3
\end{aligned} \tag{46}$$

The problem with M=1000, is solved and the solution is obtained as follows:

$$x_1^* = 1, \qquad x_2^* = 0,$$

$$f_{11}(x_1,x_2) = 4,\ f_{12}(x_1,x_2) = 0.5,\ f_{21}(x_1,x_2) = 0.67,\ f_{22}(x_1,x_2) = 0.33$$

with the membership values as

$$\mu_{f_{11}}(f_{11}) = 1, \quad \mu_{f_{12}}(f_{12}) = 1, \quad \mu_{f_{21}}(f_{21}) = 0.67, \quad \mu_{f_{22}}(f_{22}) = 0.33$$

The membership function values indicate that the levels $f_{11}\&f_{12}$ are satisfied 100% while f_{21} is satisfied with 0.67% and f_{22} with 0.33% with the optimal solution of $x_1^* = 3$and $x_2^* = 1$.

6 Conclusion

In this paper, for solving FLFBL-MOP problems a powerful and robust method which is based on Taylor series and Kuhn-Tucker conditions is proposed. Membership functions associated with each level of each objective is transformed by using Taylor series. Actually, the FLFBL-MOP problem is reduced to an equivalent single objective linear programming problem by using the first-order Taylor polynomial series and Kuhn-Tucker conditions and then the proposed solution method is applied to numerical example to test the effect on the performance. It is obvious that the results show that the proposed method is more effective in reducing the complexity in computation for solving problems.

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