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On Fixed and Coincidence Points Under Contractive Mappings in Non-Archimedean Fuzzy Metric Space

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Abstract

In this paper, we prove some coincidence and fixed point theorems for various contractive mappings in spherically complete non-archimedean fuzzy metric space. In this space, we also prove the metric locally constancy of a function f. Our results extend various known results in ultra metric space.

Keywords: *Spherically complete, M.l.c., Non-archimedean, Multivalued metric.*

Mathematics Subject Classification: 47H10, 54H25

1 Introduction

Non-Archimedean functional analysis has developed rapidly in recent years, as well as its application in mathematical physics. W.H.Schikhof [18] developed the theory of ultra metric calculus. Important contributors in ultra metric/non-archimedean spaces were Ljiljana Gajic, C.Petalas, Vidalis, Van Roovji, M. Zaharescu, K.P.R. Rao, G. N.V. Kishore and D. Mihet .
The theory of Fuzzy sets was introduced by Zadeh [11].Deng [20], Erceg [12] and Kaleva and Seikkala[14] have introduced the concept of fuzzy metric in different ways. George and Veeramani [1] gave a necessary and sufficient condition for a fuzzy metric space to be complete. Recently, Mihet [2,3] introduced the concept of non-archimedean fuzzy metric space and proved Banach Contraction theorem in this space.

The notion of metric locally constant function was introduced by M.Vajaitu and A. Zaharescu [13] in order to study certain groups of isometries on a given ultra metric space. Later on L. Gajic[10] obtained some results in spherically complete ultra metric space for generalized contractive mappings using the concept of metric locally constant. In spherically complete ultra metric space the continuity of maps are not necessary to obtained fixed point.
The aim of this paper to obtain some coincidence and fixed point theorems for various contractive mappings in spherically complete non-archimedean fuzzy metric space. Our results extend various known results in ultra metric space.

Definition 1.1 [5]: *A binary operation $*$: $[0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if it satisfies the following conditions:*

(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for all $a \in [0, 1]$,
(4) $a * b \le c * d$ whenever $a \le c$ and $b \le d$, for each $a, b, c, d \in [0, 1]$.

Example1.2: Two typical examples of continuous t-norm are $a * b = ab$ and $a * b = \min(a, b)$.

Definition 1.3: *The 3-tuple $(X, M, *)$ is called a non-Archimedean fuzzy metric space (shortly, N.A. FM-space) if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions:*

For all $x,y,z \in X$ and $s,t > 0$,
(FM-1)$M(x, y, 0) = 0$,
(FM-2)$M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$,
(FM-3)$M(x, y, t) = M(y, x, t)$,
(FM-4)$M(x, y, t) * M(y, z, s) \le M(x, z, \max\{t, s\})$
Or equivalently $M(x, y, t) * M(y, z, t) \le M(x, z, t)$
(FM-5)$M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous.
(FM-6) $\lim\limits_{n \to \infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$.

For $t \in (0,\infty)$, we define the closed ball $B[x, r, t]$ with centre $x \in X$ and radius $r \in (0,1)$ as

$$B[x, r, t] = \{y \in X ,M(x, y, t) > 1 - r\}.$$

Definition 1.4: *A N.A. FM-space $(X, M, *)$ is said to be spherically complete if every shrinking collection of balls in X has a non empty intersection.*

2 Main Results

Theorem 2.1: *Let $(X, M, *, \Diamond)$ be spherically complete non- Archimedean fuzzy metric space. If $T : X \to X$ is a mapping such that for every $x, y \in X$, $x \neq y$,*

$$M(Tx, Ty, t) \geq \min \{ M(x, Tx, t), M(x, y, t), M(y, Ty, t)\} \tag{2.1}$$

then T has a unique fixed point.

Proof: Let $B_a = B(a, 1\text{-}M(a, Ta, t), t)$ denote the closed spheres centered at a with the radii $1\text{-}M(a, Ta, t)$ and let A be the collection of these spheres for all $a \in X$.
The relation $B_a \leq B_b$ iff $B_b \subseteq B_a$ is a partial order on A.
Now, consider a totally ordered subfamily A1 of A. Since $(X, M, *, \Diamond)$ is spherically complete, we have that

$$\bigcap_{B_a \in A_1} B_a = B \neq \phi .$$

Let $b \in B$ and $B_a \in$ A1. Let $x \in B_b$. Then,

$$M(b, a, t) \geq (1\text{-}M(a,Ta, t)) = M(a, Ta, t). \tag{2.2}$$

If $a = b$ then $B_a = B_b$. Assume that $a \neq b$, let $x \in B_b$ .Then,

$$\begin{aligned} M(x, b, t) &\geq 1\text{-}(1\text{-}M(b, Tb, t) \\ &= M(b, Tb, t) \\ &\geq \min \{M(b, a,t), M(a, Ta, t), M(Ta,Tb, t)\} \\ &= \min \{ M(a, Ta,t), M(Ta, Tb, t)\}. \end{aligned}$$

For $M(Ta, Tb, t) > M(a, Ta, t)$ implies that

$$M(x, b, t) \geq M(a, Ta, t).$$

In opposite case, $M(Ta, Tb,t) \leq M(a, Ta, t)$.

$$\begin{aligned} M(x, b, t) &\geq M(b, Tb, t) \geq M(Ta, Tb, t) \\ &\geq \min\{M(a, Ta, t), M(a, b, t), M(b,Tb,t)\} \\ &= \min\{M(a, Ta, t), M(b, Tb, t)\}. \end{aligned}$$

Now, for $M(b, Tb, t) > M(a, Ta, t)$, we have

$$M(x, b, t) \leq M(a, Ta, t)$$

The inequality $M(b, Tb, t) \leq M(a, Ta, t)$ implies that $M(b, Tb, t) < M(b, Tb, t)$, which is a contradiction. So, we have proved that for $x \in B_b$

$$M(x, b, t) \geq M(a, Ta, t). \tag{2.3}$$

Now, we have that

$$M(x, a, t) \geq M(a, Ta, t).$$

So $x \in B_a$ and $B_b \subseteq B_a$ for any $B_a \in A_1$. Thus B_b is the upper bound for the family A. By Zorn's lemma A has a maximal element, say B_z, $z \in X$. We are going to prove that $z = Tz$.

Let us suppose the contrary, i.e. that $z \neq Tz$. Inequality (2.1) implies that

$$M(Tz, T(Tz), t) > M(z, Tz, t).$$

Now if $y \in B_{Tz}$ then $M(y, Tz, t) \geq 1-(1-M(Tz, T(Tz), t))$
$= M(Tz, T(Tz), t) > M(z, Tz, t).$

So

$$M(y, z, t) \geq \min\{M(y, Tz, t), M(Tz, z, t)\} = M(z, Tz, t).$$

This means that $y \in Bz$ and that $B_{Tz} \subseteq B_z$.

On the other hand $z \notin B_{Tz}$ since

$$M(z, Tz, t) < M(Tz, T(Tz), t).$$

So $B_{Tz} \underset{\neq}{\subset} B_z$. This is a contradiction with the maximality of B_z. Hence, we have that $z = Tz$.

Let u be a different fixed point. For $u \neq z$ we have that

$$M(z, u, t) = M(Tz, Tu, t) \geq \min\{M(Tz, z, t), M(z, u, t), M(u, Tu, t)\} = M(z, u, t)$$

which is a contradiction. The proof is completed.

We denote by F_X, the set of maps $f: X \to [0,+\infty)$.

Definition 2.2 *A function $f \in F_X$ is said to be metric locally constant(shortly, m.l.c.) provided that for any $x \in X$ and any y in the open $B(((x, f(x))$ one has $f(x) = f(y)$.*

Proposition 2.3: *Let M be an fuzzy ultra metric on X.*

(1) If $a,b \in X$, $\lambda > 0$, and $b \in B(a, \lambda, t)$ then $B(a, \lambda, t) = B(b, \lambda, t)$.

(2) If $a,b \in X$, $0 < \delta \leq \lambda$, then either $B(a, \lambda, t) \cap B(b, \lambda, t) = \phi$ or $B(b, \lambda, t) \subseteq B(a, \lambda, t)$. Hence, if a ball $B(a, \lambda, t)$ contains a ball $B(b, \lambda, t)$, then either the balls are the same or $\delta < \lambda$.

(3) Every ball is clopen(closed and open) in the topology defined by M.

Theorem 2.4: *Let $(X, M, *, \Diamond)$ be spherically complete non- Archimedean fuzzy metric space and $T : X \to X$ contractive mapping. Then there exist subset $B \subseteq X$ such that $T : B \to B$ and that the function $f(x) = M(x,Tx, t)$, $x \in B$, is m.l.c..*

Proof: Let $B_a = B(a, 1\text{-}M(a, Ta, t), t)$ denote the closed spheres centered at a with the radii 1-M(a, Ta, t) and let A be the collection of these spheres for all $a \in X$. The relation

$$B_a \le B_b \text{ iff } B_b \subseteq B_a$$

is a partial order on A.
Let A_1 be a totally ordered subfamily of A. Since $(X, M, *, \Diamond)$ is spherically complete,

$$\bigcap_{B_a \in A_1} B_a = B \neq \phi .$$

Let $b \in B$ and $Ba \in A_1$ then $b \in B_a$ so $M(b, a, t) \ge 1-(1-M(a,Ta, t)) = M(a,Ta, t)$.
If a = b then $B_a = B_b$. Assume that $a \neq b$, for any $x \in B_b$

$$M(x, a, t) \ge \min \{M(x, b, t), M(b,a, t)\} \ge M(a, Ta, t)$$

and

$$\begin{aligned} M(x, b, t) &\ge 1-(1-M(b, Tb, t) \\ &= M(b, Tb, t) \\ &\ge \min \{M(b, a,t), M(a, Ta, t), M(Ta,Tb, t)\} \\ &= \min \{ M(a, Ta,t), M(Ta, Tb, t)\} = M(a, Ta, t) \end{aligned}$$

So $B_b \subseteq B_a$ for any $B_a \in A_1$. Thus B_b is the upper bound for the family A_1. By Zorn's lemma there is a maximal element in A_1, say B_z.
For any $b \in B_z$

$$\begin{aligned} M(b, Tb, t) &\ge \min\{M(b, z, t), M(z, Tz, t), M(Tz, Tb, t)\} \\ &\ge \min\{M(b, z, t), M(z, Tz, t), M(z, b, t)\} = M(z, Tz, t) \end{aligned}$$

$B_b \cap B_z$ is nonempty (contains b) so by above Proposition,

$$B_b \subseteq B_z.$$

Since $Tb \in B_b$ we just prove that $T : Bz \to Bz$.
For z = Tz f(x) = 0 so theorem is proved.
For $z \neq Tz$ we are going to prove that f(b) = f(z) for every $b \in Bz$.
We know that $M(b, Tb, t) \ge M(z, Tz, t)$ for any $b \in Bz$. Let us suppose that for some $b \in B_z$

$$M(b, Tb, t) > M(z, Tz, t)$$
$$M(b, z, t) \ge M(z, Tz, t)$$

then

$$\begin{aligned} M(z, Tz, t) &\ge \min\{M(z, b, t), M(b, Tz, t)\} \\ &\ge \min\{M(z, b, t), M(b, Tb, t), M(Tb, Tz, t)\} \\ &\ge \min\{M(z, b, t), M(b, Tb, t), M(b, z, t)\} \\ &= \min\{M(z, b, t), M(b, Tb, t)\} \\ &= M(z, b, t) \end{aligned}$$

we obtain that $M(z, Tz, t) = M(b, z, t)$.
But

$$M(b, z, t) = M(z, Tz, t) \le M(b, Tb, t)$$

implies that $z \in Bz$ but $z \notin B_b$ and hence

$$B_b \underset{\neq}{\subset} B_z$$

which contradicts the maximality of Bz.

Thus we proved that f is m.l.c. on $B = B_z$.

Theorem 2.5: *Let $(X, M, *, \Diamond)$ be spherically complete non- Archimedean fuzzy metric space. If f and T are two self maps satisfying*

$$T(X) \subseteq f(X) \tag{2.4}$$

and

$$M(Tx, Ty, t) \ge \min\{ M(fx, fy, t), M(fx, Tx, t), M(fy, Ty, t)\} \tag{2.5}$$

then there exists $z \in X$ such that $fz = Tz$.
Further if f and T are coincidentally commuting at z then z is the unique common fixed point of f and T.

Proof: Let $B_a = (fa, 1\text{-}M(fa, Ta,t))$ denote the closed sphere centered at fa with the radius $1\text{-}M(fa, Ta,t)$ and let A be the collection of these spheres for all $a \in X$. Then the relation $B_a \le B_b$ iff $B_b \subseteq B_a$ is a partial order on A. Let A_1 be a totally ordered sub family of A.
Since $(X, M, *, \Diamond)$ is spherically complete , we have $\bigcap_{B_a \in A_1} B_a = B \neq \phi$.
Let $fb \in B$ and $B_a \in A_1$. Then $fb \in B_a$. Hence

$$M(fb, fa, t) \ge M(fa, Ta,t) \tag{i}$$

If $a = b$ then $B_a = B_b$. Assume that $a \neq b$.
Let $x \in B_b$. Then

$$\begin{aligned} M(x, fb,t) &\ge M(fb, Tb, t) \\ &\ge \min\{ M(fb, fa, t), M(fa, Ta, t), M(Ta, Tb, t)\} \\ &= \min\{M(fa, Ta, t), M(Ta, Tb, t)\} \qquad \text{from(i)} \\ &> \min\{M(fa, fb, t), M(fa, Ta, t), M(fb, Tb, t)\} \end{aligned}$$

from(2.5)

$$\begin{aligned} &= M(fa, Ta, t) \\ &= N(fa, Ta, t) \end{aligned} \tag{ii}$$

Now, $M(x, fa, t) \ge \min\{M(x, fb, t), M(fb, fa, t)\} \ge M(fa, Ta, t)$ from(i) and (ii).

Thus $x \in B_a$. Hence$B_b \subseteq B_a$ for any $B_a \in A_1$. Thus B_b is an upper bound in A for the family A_1 and hence by Zorn's Lemma, A has a maximal element, say B_z, $z \in X$.

Suppose $fz \neq Tz$. Since $Tz \in T(X) \subseteq f(X)$, there exists $w \in X$ such that $Tz = fw$. Clearly $z \neq w$. Now from (2.5) we have

$$M(fw, Tw, t) = M(Tz, Tw,t)$$
$$\geq \min \{M(fz, fw, t), M(fz, Tz, t), M(fw, Tw,t)\}$$
from(2.5)
$$= M(fz, fw,t)$$

Thus $fz \notin B_w$. Hence $B_z \not\subset B_w$. It is a contradiction to the maximality of B_z.
Hence $fz = Tz$.
Further assume that f and T are coincidentally commuting at z.
Then $f^2z = f(fz) = fTz = Tfz = T(Tz) = T^2z$.
Suppose $fz \neq z$. Now from (2.2), we have

$$M(Tfz, Tz, t) \geq \min\{M(f^2z, fz, t), M(f^zz, Tfz, t), M(fz, Tz, t)\}$$
$$= M(Tfz, Tz, t).$$

Hence $fz = z$. Thus $z = fz = Tz$. Uniqueness of common fixed point of f and T follows easily from(2.5).

Theorem 2.6: Let $(X,M,*,\Diamond)$ be spherically complete non- Archimedean fuzzy metric space. Let $f : X \to X$ and $T : X \to C(X)$ be satisfying

$$T(X) \subseteq f(X) \quad \forall \ x \in X, \tag{2.6}$$
$$H(Tx, Ty, t) \geq \min \{ M(fx, fy, t), M(fx, Tx, t), M(fy, Ty, t)\} \quad \forall x, y \in X , x \neq y \tag{2.7}$$

then there exists $z \in X$ such that $fz \in Tz$.
Further assume that

$$M(fx, fu, t) \geq H(Tfy, Tu, t) \quad \forall x, y , u \in X , fx \in Ty \tag{2.8}$$

and

f and T are coincidentally commuting at z. (2.9)
Then fz is the unique common fixed point of f and T.

Proof: Let $B_a = (fa, 1\text{-}M(fa, Ta,t))$ denote the closed sphere centered at fa with the radius $1\text{-}M(fa, Ta,t)$ and let A be the collection of these spheres for all $a \in X$. Then the relation $B_a \leq B_b$ iff $B_b \subseteq B_a$ is a partial order on A. Let A_1 be a totally ordered sub family of A.
Since $(X ,M,*,\Diamond)$ is spherically complete , we have $\bigcap_{B_a \in A_1} B_a = B \neq \phi$.

Let fb $\in$ B and $B_a \in A_1$. Then fb $\in B_a$. Hence

$$M(fb, fa, t) \geq M(fa, Ta,t) \quad \text{(i)}$$

If a = b then $B_a = B_b$. Assume that $a \neq b$.
Let $x \in B_b$. Then

$$M(x, fb,t) \geq M(fb, Tb, t).$$

Since Ta is compact, there exists $u \in$ Ta such that

$$M(fa, u, t) = M(fa, Ta, t) \quad \text{(ii)}$$

Consider

$$\begin{aligned} M(fb, Tb, t) &= \inf_{c \in Tb} M(fb, c, t) \\ &\geq \min \{M(fb, fa, t) ,M(fa, u, t), \inf_{c\in Tb} M(u,c,t) \} \\ &\geq \min \{M(fb, Ta, t) ,M(Ta, Tb, t)\} \quad \text{from (i) and (ii)} \\ &> \min \{ M(fb, Ta, t) ,M(Ta, fb, t)\} \quad \text{from (i) and (2.7)} \end{aligned}$$

Thus $$M(fb, Tb, t) > M(fa, Ta, t) \quad \text{(iii)}$$

Now,

$$\begin{aligned} M(x, fa,t) &\geq \min \{M(x,fa, t) ,M(fb, fa, t)\} \\ &\geq M(fa, Ta, t) \quad \text{from (i) and (iii).} \end{aligned}$$

Thus $x \in B_a$. Hence $B_b \subseteq B_a$ for any $B_a \in A_1$. Thus B_b is an upper bound in A for the family A_1 and hence by Zorn's Lemma, A has a maximal element, say B_z, $z \in X$.
Suppose fz $\notin$ Tz. Since Tz is compact, there exists $k \in$ Tz such that M(fz, Tz, t) = M(fz, k, t). From (2.6), there exists $w \in X$ such that k = fw. Thus

$$M(fz, Tz, t) = M(fz, fw, t) \quad \text{(iv)}$$

Clearly $z \neq w$. Now,

$$\begin{aligned} M(fw, Tw, t) &\geq H(Tz, Tw, t) \\ &\geq \min \{M(fz, fw, t), M(fz, Tz, t), M(fw, Tw, t)\} \\ &= M(fz, fw, t) \quad \text{from (iv)} \end{aligned}$$

Hence, fz $\notin B_w$. Thus $B_z \not\subset B_w$. It is a contradiction to the maximality of B_z.
Hence fz $\in$ Tz. Further assume (2.8) and (2.9). Write fz = p. Then, $p \in$ Tz. From (2.8), M(p, fp, t) = M(fz, fp, t) $\geq$ H(Tfz, Tp, t) = H(Tp, Tp, t) = 0. This implies that fp = p.
From (2.9), p = fp $\in$ fTz $\subseteq$ Tfz = Tp. Thus fz = p is a common fixed point of f and T.
Suppose $q \in X$, $q \neq p$ is such that q= fq $\in$ Tq. From (2.7) and (2.8) we have

$$\begin{aligned} M(p, q, t) = M(fp, fq, t) &\geq H(Tfp, Tq, t) = H(Tp, Tq, t) \\ &> \min\{M(fp,fq, t), M(fp, Tp, t), M(fq, Tq, t)\} \\ &= M(p, q, t). \end{aligned}$$

This implies that p = q. Thus p = fz is the unique common fixed point of f and T.

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