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Research Paper

Some convergence and stability results for two new Kirk and Jungck-multi step type iterative processes

Mauricio Ramírez Juárez[1*], Arnoldo Bezanilla López[2]

[1,2] *Facultad de Ciencias Físico Matemáticas, Benemérita Universidad Autónoma de Puebla, Avenida San Claudio y 18 Sur, 72570 Puebla, México*
**Corresponding author E-mail: mausssrj@hotmail.com*

Abstract

In this work two new iterative processes called the "Jungck-Kirk generalized multi-step" and "Jungck-Kirk multi-step" are introduced and some convergence and stability results are proved for these iterative process. The results include results of almost stability and summable almost stability. Since these new iterative processes are more general than other ones extant in literature, some results of this work partially generalize results already proved in the existing literature.

Keywords: *Fixed point, Jungck type iterative process, Kirk-multistep iteration, Metric spaces, Stability of iterative processes*

1. Introduction and Preliminaries

There are many iterative processes related to fixed point theory, several of them have been obtained as generalizations of others ones. An iterative process is generally denoted by

$$x_0 \in X; \quad x_{n+1} = f(T, x_n), \quad \text{for all } n \in \mathbb{N},$$

where X is an ambient space, x_0 is an arbitrary initial point, $T : X \longrightarrow X$ is a mapping and f is some function. For example, if $f(T, x_n) = Tx_n$, then we obtain the well-known Picard iteration; $x_0 \in X$, $x_{n+1} = Tx_n$.

Recently, Gursoy, Karakaya and Rhoades [4] introduced a generalization of Kirk-Noor [3] iterative process and multistep iteration [22]. This iterative process is called Kirk-multistep iteration and is defined as

$$\begin{cases} x_0 \in X; \\ y_n^{k-1} = \sum_{i_k=0}^{s_k} \beta_{n,i_k}^{k-1} T^{i_k} x_n, & k \geq 2, \text{ for all } n \in \mathbb{N}; \\ y_n^p = \beta_{n,0}^p x_n + \sum_{i_{p+1}=1}^{s_{p+1}} \beta_{n,i_{p+1}}^p T^{i_{p+1}} y_n^{p+1}, & p = \overline{1, k-2}; \\ x_{n+1} = \alpha_{n,0} x_n + \sum_{i_1=1}^{s_1} \alpha_{n,i_1} T^{i_1} y_n^1, \end{cases} \tag{1}$$

where $\sum_{i_1=0}^{s_1} \alpha_{n,i_1} = 1$, $\sum_{i_{p+1}=0}^{s_{p+1}} \beta_{n,i_{p+1}}^p = 1$ for $p = \overline{1, k-1}$; α_{n,i_1}, $\beta_{n,i_{p+1}}^p \in [0,1]$ satisfying $\alpha_{n,i_1} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,i_{p+1}}^p \geq 0$, $\beta_{n,0}^p \neq 0$ for $p = \overline{1, k-1}$ and s_1, s_{p+1} for $p = \overline{1, k-1}$ are fixed integers with $s_1 \geq s_2 \geq \cdots \geq s_k$.

Other type of iterative processes are those known as Jungck type. They involve maps $S, T : Y \longrightarrow X$, where S is an injective map, X is a metric space and usually Y is an any set.

Here, we can consider Y as any set, $x_0 \in Y$, X endowed with a norm $|| \cdot ||$ and $T(Y) \subset S(Y)$.

Olaleru and Akewe [13] introduced an iterative process called *Jungck-multistep iterative process*. It is defined as,

$$\begin{cases} Sz_n^{k-1} = (1 - \beta_n^{k-1}) Sx_n + \beta_n^{k-1} T x_n & k \geq 2; \\ Sz_n^i = (1 - \beta_n^i) Sx_n + \beta_n^i T z_n^{i+1} & i = 1, 2, ..., k-2; \\ Sx_{n+1} = (1 - \alpha_n) Sx_n + \alpha_n T^i z_n^1, & n = 1, 2, ..., \end{cases} \tag{2}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^i\}_{n=0}^{\infty}$, $i = 1, 2, ..., k-1$, are real sequences in $[0, 1)$, such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ and k is a fixed integer.

Chugh and Kumar [3] defined the following iterative process called *Jungck-Kirk-Noor* :

$$\begin{cases} Sw_n = \gamma_{n,0} Sx_n + \sum_{l=1}^{t} \gamma_{n,l} T^l x_n, & \sum_{l=0}^{t} \gamma_{n,l} = 1; \\ Sz_n = \beta_{n,0} Sx_n + \sum_{j=1}^{s} \beta_{n,j} T^j w_n, & \sum_{j=0}^{s} \beta_{n,j} = 1; \\ Sx_{n+1} = \alpha_{n,0} Sx_n + \sum_{i=1}^{k} \alpha_{n,i} T^i z_n, & \sum_{i=0}^{k} \alpha_{n,i} = 1, \quad n = 0, 1, 2, ..., \end{cases} \tag{3}$$

where $k \geq s \geq t$, $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,j} \geq 0$, $\beta_{n,0} \neq 0$, $\gamma_{n,l} \geq 0$, $\gamma_{n,0} \neq 0$ $\alpha_{n,i}$, $\beta_{n,j}$, $\gamma_{n,l} \in [0, 1]$ where k, s and t are fixed integers.

Both the Jungck-multistep and the Jungck-Kirk-Noor iterative processes are generalizations of the Jungck-Noor Iterative process given in [14]. In the particular case when $Y = X$ and S is the identity map on X, then we obtain the multistep iteration scheme [22] and the Kirk-Noor [3] iterative process from (2) and (3) respectively. Furthermore, in this case, we can see (2) as a generalization of the Mann iterative process [11] and the Ishikawa iterative process [9], (3) is a generalization of the Mann iterative process [11], the Ishikawa iterative process [9], the Noor iterative process [10], the Kirk iterative process [10] and the Kirk-Ishikawa iterative process [17].

One of the issues in fixed point theory is the stability of iterative processes. A definition of stability was established by Harder and Hicks [6].

Ostrowski [21] seems to be the first to discuss the stability of iterative process. In 1964, he proved the stability of the process $x_{n+1} = Tx_n$ where T is a contraction.

In the same sense of Harder and Hicks's definition, Singh, Bhatnagar and Mishra [23] gave a definition of stability for Jungck-type iterative processes. Over time, other definitions of stability have appeared such as the *almost stability* (see Osilike [20]) and the *summable almost stability* (see Berinde [1]).

Below we put the definition of stability given by Singh, Bhatnagar and Mishra [23] and also we state the definitions of almost stability and summable almost stability, but in the context of the Jungck-type iterative processes.

Definition 1.1 *Let (X,d) be a metric space and Y an arbitrary set. Let $S,T:Y\longrightarrow X$ be two maps such that, S is an injective map and $T(Y)\subset S(Y)$. Let z be a coincidence point of T and S, i.e. $Tz=Sz=p$, for some $p\in X$. Let $x_0\in Y$ and $Sx_{n+1}=f(T,x_n)$ be an iterative process such that the sequence $\{Sx_n\}_{n=0}^{\infty}$ converges to p. Let $\{Sy_n\}_{n=0}^{\infty}$ be an arbitrary sequence in $S(Y)$ and $\epsilon_n=d(Sy_{n+1},f(T,y_n))$.*

- *If $\lim_{n\to\infty}\epsilon_n=0$ implies that $\lim_{n\to\infty}Sy_n=p$, then the iterative process $Sx_{n+1}=f(T,x_n)$ is called* (S,T)- stable *or* stable with respect to S and T.
- *If $\sum_{n=0}^{\infty}\epsilon_n<\infty$ implies that $\lim_{n\to\infty}Sy_n=p$, then the iterative process $Sx_{n+1}=f(T,x_n)$ will be called* almost (S,T)-stable *or* almost stable with respect to S and T.
- *If $\sum_{n=0}^{\infty}\epsilon_n<\infty$ implies that $\sum_{n=0}^{\infty}d(Sy_n,p)<\infty$, then the iterative process $Sx_{n+1}=f(T,x_n)$ will be called* summable almost (S,T)-stable *or* summable almost stable with respect to S and T.

Any summable almost stable iterative process is also almost stable but the reverse is not always true (see Berinde [1]). Any T-stable iterative process is also almost stable, but the reverse is generally not true as was shown by Osilike [20, Example 1].

In the literature there are several fixed point theorems on stability where the following contractive condition introduced in [7] is employed (see for example [16], [15], [17], [5] and [4]).

$$d(Tx,Ty)\leq\delta d(x,y)+\varphi(d(x,Tx)),\ \text{where}\ \ \delta\in[0,1),\ \text{for all}\ \ x,y\in X, \tag{4}$$

where $\varphi:\mathbb{R}\longrightarrow\mathbb{R}$ is a monotone increasing mapping with $\varphi(0)=0$ and (X,d) is a metric space.

Rhoades [2] considered maps $T:X\longrightarrow X$ having a fixed point p and satisfying the condition

$$d(p,Tx)\leq\delta d(p,x),\ \text{for some}\ \ 0\leq\delta<1\ \text{and for each}\ x\in X. \tag{5}$$

If we put $\varphi(t)=0$ in inequality (4), we can get the definition of contraction used in the Banach fixed point theorem. If we take $\varphi(t)=Lt$, $L\geq0$, the inequality (4) reduces to the contractive definition due to Osilike [19]. It is easy to verify that any Zamfirescu contraction (see [24]) satisfies (4) with a particular mapping φ.

However, the mappings satisfying (4) need not have fixed points (see example in [19]), so, in stability results it has been necessary to assume the existence of the fixed point.

To our opinion it is better to work with the contractive condition defined by (5) than with (4) because if we suppose that T has a fixed point, then (4) implies (5) and using (5), we avoid doing unnecessary calculations. Furthermore, others contractive conditions such as (II) from [8] and the condition used in Corollary 2.2 from [18] involve (5), too.

In this work, we introduce the new iterative processes Kirk and Jungck-multi step type such that these ones generalize to the ones defined by (2) y (3). Also we prove that the new iterative processes are stable, almost stable and summable almost stable with respect to functions that satisfy a contractive condition like inequality (5) but in the context of the Jungck-type iterative processes. Furthermore our theorems include results that show that the new iterative methods can be used to approximate fixed points.

The proof of main result is based on the following lemma.

Lemma 1.2 (Berinde [1, Lemma 1]). *Let $\{u_n\}_{n=0}^{\infty}$ and $\{\epsilon_n\}_{n=0}^{\infty}$ be sequences of nonnegative numbers and $\delta\in[0,1)$, such that,*

$$u_{n+1}\leq\delta u_n+\epsilon_n,\quad n=0,1,... \tag{6}$$

L_1) If $\lim_{n\to\infty}\epsilon_n=0$, then $\lim_{n\to\infty}u_n=0$.

L_2) If $\sum_{n=0}^{\infty}\epsilon_n<\infty$, then $\sum_{n=0}^{\infty}u_n<\infty$.

2. Main results

Motivated by the iterative processes defined in the last section we define the following two iterative processes.

Definition 2.1 *Let $(X, \|\cdot\|)$ be a normed space and Y a set such that $X \subseteq Y$. Let $m \in \mathbb{N} \setminus \{1,2\}$ be a fixed number and $k_1, k_2, ..., k_m \in \mathbb{N}$. We suppose $S, T_{j,i_j} : Y \longrightarrow X$, for $j = 1, 2, ..., m$ and $i_j = 1, ..., k_j$, are maps such that S is an injective map and for any finite collection $\{\lambda_{i_j}\}_{i_j=0}^{k_j} \subset [0,1]$ and for each $x, y \in Y$ we have that $\left(\lambda_0 Sx + \sum_{i_j=1}^{k_j} \lambda_{i_j} T_{j,i_j} y\right) \in S(Y)$, for $j = 1, 2, .., m$. We suppose $\alpha_{n,i_j}^{(j)} \in [0,1]$ for each $j \in \{1, ..., m\}$, $i_j \in \{0, ..., k_j\}$ and $n \in \mathbb{N}$ and $\sum_{i_j=0}^{k_j} \alpha_{n,i_j}^{(j)} = 1$ for each $j \in \{1, ..., m\}$ and $n \in \mathbb{N}$.*

Let $x_0 \in X$ be an arbitrary point and define the iteration that generates the sequence $\{Sx_n\}_{n=0}^{\infty}$, by

$$\begin{cases} Sz_n^{(m)} = \alpha_{n,0}^{(m)} Sx_n + \displaystyle\sum_{i_m=1}^{k_m} \alpha_{n,i_m}^{(m)} T_{m,i_m} x_n; \\ Sz_n^{(j)} = \alpha_{n,0}^{(j)} Sx_n + \displaystyle\sum_{i_j=1}^{k_j} \alpha_{n,i_j}^{(j)} T_{j,i_j} z_n^{j+1}, \text{ for } j = 2, ..., m-1; \\ x_{n+1} = \alpha_{n,0}^{(1)} Sx_n + \displaystyle\sum_{i_1=1}^{k_1} \alpha_{n,i_1}^{(1)} T_{1,i_1} z_n^{(2)}. \end{cases}$$

This iterative process will be called the "Jungck-Kirk generalized multi-step iteration" or the "Jungck-Kirk generalized multi-step iterative process".

Let T be a map. If in the above definition, $T_{j,i_j} = T^{i_j}$ (here T^{i_j}, denotes the composition of T with itself i_j times), for $j = 1, 2, ..., m$ and $i_j = 1, ..., k_j$, then we obtain the following special case:

Definition 2.2 *Let $(X, \|\cdot\|)$ be a normed space and Y a set such that $X \subseteq Y$. Let $m \in \mathbb{N} \setminus \{1,2\}$ be a fixed number and $k_1, k_2, ..., k_m \in \mathbb{N}$. We suppose $S, T : Y \longrightarrow X$ are two maps such that, S is an injective map and for any finite collection $\{\lambda_{i_j}\}_{i_j=0}^{k_j} \subset [0,1]$ and for each $x, y \in Y$ we have that $\left(\lambda_0 Sx + \sum_{i_j=1}^{k_j} \lambda_{i_j} T^{i_j} y\right) \in S(Y)$, for $j = 1, 2, .., m$. We suppose $\alpha_{n,i_j}^{(j)} \in [0,1]$ for each $j \in \{1, ..., m\}$, $i_j \in \{0, ..., k_j\}$ and $n \in \mathbb{N}$, and $\sum_{i_j=0}^{k_j} \alpha_{n,i_j}^{(j)} = 1$, for each $j \in \{1, ..., m\}$ and $n \in \mathbb{N}$. Let $x_0 \in X$ be an arbitrary point and define the iteration that generates the sequence $\{Sx_n\}_{n=0}^{\infty}$, by*

$$\begin{cases} Sz_n^{(m)} = \alpha_{n,0}^{(m)} Sx_n + \displaystyle\sum_{i_m=1}^{k_m} \alpha_{n,i_m}^{(m)} T^{i_m} x_n; \\ Sz_n^{(j)} = \alpha_{n,0}^{(j)} Sx_n + \displaystyle\sum_{i_j=1}^{k_j} \alpha_{n,i_j}^{(j)} T^{i_j} z_n^{j+1}, \text{ for } j = 2, ..., m-1; \\ x_{n+1} = \alpha_{n,0}^{(1)} Sx_n + \displaystyle\sum_{i_1=1}^{k_1} \alpha_{n,i_1}^{(1)} T^{i_1} z_n^{(2)}. \end{cases}$$

This iterative process will be called "the Jungck-Kirk multi-step iteration" or the "Jungck-Kirk multi-step iterative process".

Remarks: Let us consider the Jungck-Kirk multi-step iterative process:

2.1 If $k_1 = k_2 = \cdots = k_m = 1$, then we obtain the Jungck-multistep iterative process (Olaleru and Akewe [13]).

2.2 If $m = 3$, we obtain the Jungck-Kirk-Noor iterative process (Chugh and Kumar [3]).

2.3 If $m = 3$, $k_1 = k_2 = k_3 = 1$, then we obtain the Jungck-Noor iterative process(Olatinwo [14]).

2.4 If $Y = X$, $S = I$, (I the identity function on X) we obtain the Kirk multistep iteration (Gursoy, Karakaya and Rhoades [4]).

2.5 If $Y = X$, $S = I$, $m = 3$ and $\alpha_{n,0}^{(3)} = 1$, for each $n \in \mathbb{Z}^+$, gives us the Kirk-Ishikawa iterative process (Olatinwo [17]).

2.6 If $Y = X$, $S = I$, $m = 3$, $\alpha_{n,0}^{(2)} = 1$, $\alpha_{n,0}^{(3)} = 1$ and $\alpha_{n,i_1}^{(1)} = \alpha_{i_1}^{(1)}$, $i_1 = 0, 1, ..., k_1$, for each $n \in \mathbb{Z}^+$, then we obtain the Kirk iterative process (Kirk [10]).

The remarks 2.1 - 2.6 tell us how from the Jungck-Kirk multi-steps iteration, we get some iterative processes. Of course, from this iteration we can also obtain the Mann [11], Ishikawa[9], Noor [12] and multistep [13] iterative processes.

Now we give our main results.

Theorem 2.3 *Let $(X, ||\cdot||)$ be a normed space and Y a set such that $X \subseteq Y$. Let $m \in \mathbb{N} \setminus \{1, 2\}$ be a fixed number and $k_1, k_2, ..., k_m \in \mathbb{N}$. We suppose $S, T_{j,i_j} : Y \longrightarrow X$, for $j = 1, 2, ..., m$ and $i_j = 1, ..., k_j$ are maps such that S is an injective map and for any finite collection $\{\lambda_{i_j}\}_{i_j=0}^{k_j} \subset [0, 1]$ and for each $x, y \in Y$ we have that $\left(\lambda_0 Sx + \sum_{i_j=1}^{k_j} \lambda_{i_j} T_{j,i_j} y\right) \in S(Y)$, for $j = 1, 2, .., m$. We suppose also that there exists $q \in Y$, with $Sq = p$, such that*

a) $||T_{1,i_1}x - p|| \leq \delta ||Sx - p||$, *for some $\delta \in [0, 1)$, for each $x \in Y$ and $i_1 = 1, ..., k_1$;*

b) $||T_{j,i_j}x - p|| \leq ||Sx - p||$, *for each $x \in Y$, $i_j = 1, ..., k_j$ and $j = 1, 2, ..., m$.*

Let $\{Sx_n\}_{n=0}^{\infty}$ be the sequence generated by the Jungck-Kirk generalized multi-step iteration, with additional condition that there exists $\alpha < 1$, such that $0 < \alpha_{n,0}^{(1)} + \delta \leq \alpha$, for all $n \in \mathbb{N}$.

Let $\{Sy_n\}_{n=0}^{\infty}$ be an arbitrary sequence in $S(Y)$ and

$$Sw_n^{(m)} = \alpha_{n,0}^{(m)} Sy_n + \sum_{i_m=1}^{k_m} \alpha_{n,i_m}^{(m)} T_{m,i_m} y_n;$$

$$Sw_n^{(j)} = \alpha_{n,0}^{(j)} Sy_n + \sum_{i_j=1}^{k_j} \alpha_{n,i_j}^{(j)} T_{j,i_j} w_n^{j+1}, \text{ for } j = 2, ..., m-1;$$

$$\epsilon_n = \left|\left| Sy_{n+1} - \alpha_{n,0}^{(1)} Sy_n - \sum_{i_1=1}^{k_1} \alpha_{n,i_1}^{(1)} T_{1,i_1} w_n^{(2)} \right|\right|.$$

Then

i) q is a coincidence point of $T_{j,i_j} q = Sq = p$ for each $i_j \in \{1, ..., k_j\}$ and $j \in \{1, 2, ..., m\}$;

ii) the sequence $\{Sx_n\}_{n=0}^{\infty}$ converges to the point p;

iii) $\lim_{n\to\infty} \epsilon_n = 0$ if and only if $\lim_{n\to\infty} Sy_n = p$;

iv) if $\sum_{n=0}^{\infty} \epsilon_n < \infty$ then $\lim_{n\to\infty} Sy_n = p$;

v) if $\sum_{n=0}^{\infty} \epsilon_n < \infty$ then $\sum_{n=0}^{\infty} d(Sy_n, p) < \infty$.

Proof. By hypothesis b), we can see directly that q is a point of coincidence of all maps that are being considered. We omit the proof of part (ii), since it is similar to the test (iii).

To prove (iii) we must show two implications.

(*Necessity*). We suppose that $\lim_{n\to\infty} \epsilon_n = 0$.

$$\begin{aligned}
||Sy_{n+1}-p|| &\le \left|\left|Sy_{n+1}-\alpha_{n,0}^{(1)}Sy_n-\sum_{i_1=1}^{k_1}\alpha_{n,i_1}^{(1)}T_{1,i_1}w_n^{(2)}\right|\right| \\
&\quad+\left|\left|\alpha_{n,0}^{(1)}Sy_n+\sum_{i_1=1}^{k_1}\alpha_{n,i_1}^{(1)}T_{1,i_1}w_n^{(2)}-p\right|\right| \\
&=\epsilon_n+\left|\left|\alpha_{n,0}^{(1)}Sy_n+\sum_{i_1=1}^{k_1}\alpha_{n,i_1}^{(1)}T_{1,i_1}w_n^{(2)}-\alpha_{n,0}^{(1)}p-\sum_{i_1=1}^{k_1}\alpha_{n,i_1}^{(1)}p\right|\right| \\
&\le \epsilon_n+\alpha_{n,0}^{(1)}||Sy_n-p||+\sum_{i_1=1}^{k_1}\alpha_{n,i_1}^{(1)}||T_{1,i_1}w_n^{(2)}-p||
\end{aligned}$$

by hypothesis $a)$ we have,

$$||Sy_{n+1}-p|| \le \epsilon_n+\alpha_{n,0}^{(1)}||Sy_n-p||+\left(\sum_{i_1=1}^{k_1}\alpha_{n,i_1}^{(1)}\right)\delta||Sw_n^{(2)}-p||. \tag{7}$$

Let us now consider $||Sw_n^{(j)}-p||$, for $j=2,...,m-1$,

$$\begin{aligned}
||Sw_n^{(j)}-p|| &= \left|\left|\alpha_{n,0}^{(j)}Sy_n+\sum_{i_j=1}^{k_j}\alpha_{n,i_j}^{(j)}T_{j,i_j}w_n^{(j+1)}-\sum_{i_j=0}^{k_j}\alpha_{n,i_j}^{(j)}p\right|\right| \\
&\le \alpha_{n,0}^{(j)}||Sy_n-p||+\sum_{i_j=1}^{k_j}\alpha_{n,i_j}^{(j)}||T_{j,i_j}w_n^{(j+1)}-p||
\end{aligned}$$

by hypothesis $b)$ we have,

$$||Sw_n^{(j)}-p|| \le \alpha_{n,0}^{(j)}||Sy_n-p||+\left(\sum_{i_j=1}^{k_j}\alpha_{n,i_j}^{(j)}\right)||Sw_n^{(j+1)}-p||. \tag{8}$$

Similarly with $||Sw_n^{(m)}-p||$, we get

$$||Sw_n^{(m)}-p|| \le \alpha_{n,0}^{(m)}||Sy_n-p||+\left(\sum_{i_m=1}^{k_m}\alpha_{n,i_m}^{(m)}\right)||Sy_n-p|| = ||Sy_n-p||. \tag{9}$$

By (8) and (9), $||Sw_n^{(m-l)}-p|| \le ||Sy_n-p||$, for $l=0,1,...,m-2$, so

$$||Sw_n^{(2)}-p|| \le ||Sy_n-p||, \tag{10}$$

and by (7)

$$\begin{aligned}
||Sy_{n+1}-p|| &\le \epsilon_n+\alpha_{n,0}^{(1)}||Sy_n-p||+\left(\sum_{i_1=1}^{k_1}\alpha_{n,i_1}^{(1)}\right)\delta||Sy_n-p|| \\
&\le \epsilon_n+\left(\alpha_{n,0}^{(1)}+\delta\right)||Sy_n-p|| \\
&\le \epsilon_n+\alpha||Sy_n-p||.
\end{aligned}$$

Since $\left(\alpha_{n,0}^{(1)}+\delta\right)\le\alpha$

$$||Sy_{n+1}-p|| \le \epsilon_n+\alpha||Sy_n-p||. \tag{11}$$

By Lemma 1.2, $\lim_{n\to\infty} Sy_n = p$.

(*Sufficiency*). Now, we suppose that $\lim_{n\to\infty} ||Sy_n - p|| = 0$.

$$\begin{aligned}
\epsilon_n &= \left|\left| Sy_{n+1} - \alpha_{n,0}^{(1)} Sy_n - \sum_{i_1=1}^{k_1} \alpha_{n,i_1}^{(1)} T_{1,i_1} w_n^{(2)} \right|\right| \\
&\le ||Sy_{n+1} - p|| + \left|\left| \sum_{i_1=0}^{k_1} \alpha_{n,i_1}^{(1)} p - \alpha_{n,0}^{(1)} Sy_n - \sum_{i_1=1}^{k_1} \alpha_{n,i_1}^{(1)} T_{1,i_1} w_n^{(2)} \right|\right| \\
&\le ||Sy_{n+1} - p|| + \alpha_{n,0}^{(1)} ||Sy_n - p|| + \sum_{i_1=1}^{k_1} \alpha_{n,i_1}^{(1)} ||T_{1,i_1} w_n^{(2)} - p|| \\
&\le ||Sy_{n+1} - p|| + \alpha_{n,0}^{(1)} ||Sy_n - p|| + \left(\sum_{i_1=1}^{k_1} \alpha_{n,i_1}^{(1)} \right) \delta ||Sw_n^{(2)} - p||
\end{aligned}$$

From this and inequality (10), we get that

$$\begin{aligned}
\epsilon_n &\le ||Sy_{n+1} - p|| + \alpha_{n,0}^{(1)} ||Sy_n - p|| + \left(\sum_{i_1=1}^{k_1} \alpha_{n,i_1}^{(1)} \right) \delta ||Sy_n - p|| \\
&\le ||Sy_{n+1} - p|| + ||Sy_n - p||.
\end{aligned}$$

then,

$$\lim_{n\to\infty} \epsilon_n \le \lim_{n\to\infty} (||Sy_{n+1} - p|| + ||Sy_n - p||) = 0.$$

The statement (v) is true also by inequality (11) and Lemma 1.2 and (iv) is a consequence of (iii) and (v).

□

Corollary 2.4 *Let $(X, ||\cdot||)$ be a normed space and Y a set such that $X \subseteq Y$. Let $S, T : Y \longrightarrow X$ be maps such that S is an injective map and for any finite collection $\{\lambda_i\}_{i=0}^{k_j} \subset [0,1]$ and for each $x, y \in Y$ we have $\left(\lambda_0 Sx + \sum_{i=1}^{k_j} \lambda_i T^i y\right) \in S(Y)$, for each $j \in \{1, ..., m\}$. Also, we suppose there exists $q \in Y$, with $Sq = p$ such that*

c) $||Tx - p|| \le \delta ||Sx - p||$, *for some $\delta \in [0,1)$ and for each $x \in Y$;*

d) $||S(T(x)) - p|| \le ||T(S(x)) - p||$, *for each $x \in Y$;*

e) $||S^2 x - p|| \le ||Sx - p||$, *for each $x \in Y$.*

Let $\{Sx_n\}_{n=0}^{\infty}$ be a sequence generated by the Jungck-Kirk multi-step iteration, with additional condition that there exists $\alpha < 1$, such that $0 < \alpha_{n,0}^{(1)} + \delta \le \alpha$, for all $n \in \mathbb{N}$.

Let $\{Sy_n\}_{n=0}^{\infty}$ be an arbitrary sequence in $S(Y)$ and

$$Sw_n^{(m)} = \alpha_{n,0}^{(m)} Sy_n + \sum_{i_m=1}^{k_m} \alpha_{n,i_m}^{(m)} T^{i_m} y_n;$$

$$Sw_n^{(j)} = \alpha_{n,0}^{(j)} Sy_n + \sum_{i_j=1}^{k_j} \alpha_{n,i_j}^{(j)} T^{i_j} w_n^{j+1}, \text{ for j=2,...,m-1};$$

$$\epsilon_n = \left|\left| Sy_{n+1} - \alpha_{n,0}^{(1)} Sy_n - \sum_{i_1=1}^{k_1} \alpha_{n,i_1}^{(1)} T^{i_1} w_n^{(2)} \right|\right|.$$

Then

i) *q is a coincidence point of S and T; even, $q = p$, i.e. p is a fixed point of T and S. Furthermore this fixed point is unique.*

ii) *the sequence $\{Sx_n\}_{n=0}^{\infty}$ converges to the point p;*

iii) $\lim_{n\to\infty} \epsilon_n = 0$ *if and only if* $\lim_{n\to\infty} Sy_n = p$;

iv) *if* $\sum_{n=0}^{\infty} \epsilon_n < \infty$ *then* $\lim_{n\to\infty} Sy_n = p$;

v) *if* $\sum_{n=0}^{\infty} \epsilon_n < \infty$ *then* $\sum_{n=0}^{\infty} d(Sy_n, p) < \infty$.

Proof. First, to prove (i) note that (e) implies that $Sp = p$. Since S is an injective map, then $q = p$. Now, by (c) we have $Tp = p$ and from (c) we obtain that if there exists another fixed point p', then $p' = p$.

The proofs of (ii), (iii), (iv) and (v) follow from Theorem 2.3. If we put $T_{j,i_j} = T^{i_j}$, for $i_j = 1, ..., k_j$ and $j = 1, ..., m$, in Theorem 2.3, we only need to observe that the assumptions $a)$ and $b)$ are satisfied. For this purpose let us fix $j \in \{1, ..., m\}$ and we consider $i_j \geq 2$,

$$||T^{i_j}x - p|| = ||TT^{i_j-1}x - p||.$$

By hypothesis $c)$, $||T^{i_j}x - p|| \leq \delta||ST^{i_j-1}x - p||$.
By hypothesis $d)$, $||T^{i_j}x - p|| \leq \delta||T^{i_j-1}Sx - p||$.

Repeating these steps i_j times, we get.

$$\begin{aligned} ||T^{i_j}x - p|| &\leq \delta^{i_j}||S^{i_j}x - p|| \\ &\leq \delta||S^2S^{i_j-2}x - p||. \end{aligned}$$

By hypothesis $e)$, $||T^{i_j}x - p|| \leq \delta||S^{i_j-1}x - p||$.

Repeating these steps i_j times, we get

$$||T^{i_j}x - p|| \leq \delta||Sx - p||.$$

So, $a)$ and $b)$ are satisfied.

□

In the particular case $Y = X$ and S the identity mapping in X in Corollary 2.4, we partially obtain the theorem 6 and 7 from [4]. But in this case we use the condition defined by the inequality (5) instead of the condition defined by (4).

Theorem 2.3 and Corollary 2.4 partially generalize theorems 3.1 and 3.2 of Chugh and Kumar [3]. This can be seen if we choose $m = 3$ in Corollary 2.4.

Unlike [4] and [3] in this work we include results of almost stability and summable almost stability in our theorems. But we say that our results *partially* generalize to others because we put the additional assumption $0 < \alpha^1_{n,0} + \delta \leq \alpha$, for some $\alpha < 1$.

We conclude with the following question: Does the Theorem 2.3 and the Corollary 2.4 remain true if we remove the restriction $0 < \alpha^1_{n,0} + \delta \leq \alpha < 1$?

3. Conclusion

In this work, we have introduced a general iterative method from which we can get others from the existing literature. Furthermore, we have given two results that partially generalize to the theorem 6 and 7 from [4] and the theorems 3.1 and 3.2 from [3].

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