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Research Paper

Fixed points in modular spaces with new type contractivity

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Abstract

In this present paper, we prove a common fixed point theorem for self maps in modular spaces. Also one corollary, which shows that our main theorem is generalized version of the main theorem of [A. Razani, E. Nabizadeh, M. Beyg Mohamadi and S. Homaei Pour, Abs. Appl. Anal. **2007**, Article ID 40575] is given.

Keywords: *Fixed point, contraction, modular, modular space.*

1. Introduction

The theory of modular spaces was introduced by Nakano [1] in 1950 and generalized by Musielak and Orlicz [2], Koshi and Shimogaki [3] and Yamamuro [4] and their collaborators. The monographic exposition of the theory of Orlicz spaces may be found in in the book of Krasnosel'skii and Rutickii[5]. We referred the reader for the theory of Orlicz spaces and modular spaces, to the books [6, 7]. Fixed point theorems in modular spaces, generalizing the classical Banach fixed point theorem in metric spaces, have been studied extensively by many mathematicians such as Arandelović, [8], Edelstein [9], Ćirić [10], Rakotch [11], Reich [12], Kirk [13]. In addition, Razani *et al.* [14] proved some fixed point theorems of non linear and asymptotic contractions in modular spaces. Also, quasi-contraction mappings in modular spaces without Δ_2–condition were investigated by Khamsi [15]. Kuaket and Kumam [16] proved the existence of fixed points of asymptotic pointwise contractions in modular spaces. Moreover Chen and Wang [17] proved the fixed points of asymptotic pointwise nonexpansive mappings in modular spaces.

In this paper we establish a fixed point theorem for self maps in modular spaces with new type contractivity.

Definition 1.1 *Let $\mathcal{X}$ be an arbitrary vector space over $\mathbb{F}(=\mathbb{R}$ or $\mathbb{C})$.*
A functional $\rho : \mathcal{X} \to [0, \infty]$ is called modular if for all $x, y \in \mathcal{X}$,
(i) $\rho(x) = 0$ if and only if $x = 0$,
(ii) $\rho(\alpha x) = \rho(x)$ for every $\alpha \in \mathbb{F}$ with $|\alpha| = 1$,
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.

Definition 1.2 *If (iii) in definition 1.1 is replaced by*

$$\rho(\alpha x + \beta y) \leq \alpha^s \rho(x) + \beta^s \rho(y),$$

for $\alpha, \beta \geq 0$, $\alpha + \beta = 1$ with an $s \in (0, 1]$, then we say that ρ is an s-convex modular, and if $s = 1$, ρ is called a convex modular.

A modular ρ defines a corresponding modular space, i.e., the vector space $\mathcal{X}_\rho$ given by *to*
$\mathcal{X}_\rho = \{x \in \mathcal{X} : \quad \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}$.*Let*ρ be a convex modular, the modular space $\mathcal{X}_\rho$ can be equipped with a norm called the Luxemburg norm, defined by

$$\|x\|_\rho = \inf\left\{\lambda > 0 \quad ; \quad \rho\left(\frac{x}{\lambda}\right) \leq 1\right\}.$$

Definition 1.3 *Let $\mathcal{X}_\rho$ be a modular space and let $\{x_n\}$ and x be in $\mathcal{X}_\rho$. Then*
(i) $\{x_n\}$ is said to be ρ–convergent to x and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.
(ii) $\{x_n\}$ is called ρ–Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.
(iii) A subset $\mathcal{S}$ of $\mathcal{X}_\rho$ is called ρ–complete complete if any ρ–Cauchy sequence is ρ–convergent to an element of $\mathcal{S}$.
(iv) A subset B of $\mathcal{X}_\rho$ is called ρ–closed if for any sequence $\{x_n\} \subseteq B$ with $x_n \xrightarrow{\rho} x$, we have $x \in B$.
(v) We say the modular ρ has the Fatou property if $\rho(x) \leq \liminf_{n\to\infty} \rho(x_n)$ whenever $x_n \xrightarrow{\rho} x$.
(vi) ρ is said to satisfy the Δ_2–condition if $\rho(2x_n) \to 0$ whenever $\rho(x_n) \to 0$ as $n \to \infty$.

Remark 1.4 *Note that, if $x \in \mathcal{X}_\rho$ then $\rho(ax)$ is an increasing function of $a > 0$. Suppose $0 < a < b$, then property (iii) of Definition 1.1 with $y = 0$ shows that $\rho(ax) = \rho\left(\frac{a}{b}bx\right) \leq \rho(bx)$ for all $x \in \mathcal{X}$. Moreover, if ρ is a convex modular on $\mathcal{X}$ and $|\alpha| \leq 1$, then $\rho(\alpha x) \leq \alpha\rho(x)$ and also $\rho(x) \leq \frac{1}{2}\rho(2x)$ for all $x \in \mathcal{X}$.*

Definition 1.5 *A function $T : \mathcal{X}_\rho \to \mathcal{X}_\rho$ is called ρ-continuous if*

$$\rho(x_n - x) \to 0, \text{ then } \rho\big(T(x_n) - T(x)\big) \to 0. \tag{2}$$

2. Main results

Throughout this paper, we assume that the modular ρ satisfies the Δ_2–condition. In this section, by using some ideas from [14, 18] we will prove a fixed point theorem for a new type of contractivity as follows.

Theorem 2.1 *Let $\mathcal{X}_\rho$ be a ρ-complete modular space, where ρ satisfies the Δ_2-condition. Suppose that $\varphi : \mathbb{R}^+ \to [0, \infty)$ is an increasing and upper semicontinuous function satisfying*

$$\varphi(t) < t, \quad (t > 0). \tag{3}$$

Let C be a ρ-closed subset of $\mathcal{X}_\rho$ and let $T, S : C \to C$ be mappings such that there exist $\alpha, \beta \in \mathbb{R}^+$ with $\alpha > \beta$, and

$$\rho\big(\alpha(Tx - Sy)\big) \leq \varphi\big(\rho(\beta(x - y))\big), \tag{4}$$

for all $x, y \in C$. Then T and S have a unique common fixed point in C.

First we prove that any fixed point of T is also a fixed point of S, and conversely. Suppose $Tx = x$, hence we have from (4)

$$0 \leq \rho\big(\alpha(x - Sx)\big) \leq \varphi\big(\rho(\beta(x - x))\big) = 0, \tag{5}$$

since $\alpha > 0$, so $Sx = x$. Similarly, if $Sx = x$, then $Tx = x$.

Now, we prove that if T and S have a common fixed point, then the fixed point is unique. Let $Tx = Sx = x$ and $Ty = Sy = y$. If $x \neq y$, then (4) implies that

$$\rho\big(\beta(x - y)\big) < \rho\big(\alpha(x - y)\big) = \rho\big(\alpha(Tx - Sy)\big) \leq \varphi\big(\rho(\beta(x - y))\big), \tag{6}$$

which is a contradiction. Therefore $x = y$.

Suppose $x_0 \in C$ and put $x_{2n+1} = Tx_{2n},\ x_{2n+2} = Sx_{2n+1}$ for all $n = 0, 1, 2, , \cdots$. We may suppose that for any n, $x_{n+1} \neq x_n$, otherwise T or S has a fixed point and the proof is complete. Now, we have

$$\begin{aligned} \rho\big(\alpha(x_{2n+1} - x_{2n})\big) &= \rho\big(\alpha(Tx_{2n} - Sx_{2n-1})\big) \leq \varphi\big(\rho\big(\beta(x_{2n} - x_{2n-1})\big)\big) \\ &< \rho\big(\beta(x_{2n} - x_{2n-1})\big), \end{aligned} \tag{7}$$

similarly

$$\rho\big(\alpha(x_{2n+2}-x_{2n+1})\big)=\rho\big(\alpha(Sx_{2n+1}-Tx_{2n})\big)\leq\varphi\big(\rho\big(\beta(x_{2n+1}-x_{2n})\big)\big) \\ <\rho\big(\beta(x_{2n+1}-x_{2n})\big). \tag{8}$$

Hence (7) and (8) imply that

$$\rho\big(\alpha(x_{n+1}-x_n)\big)\leq\varphi\big(\rho\big(\beta(x_n-x_{n-1})\big)\big)<\rho\big(\beta(x_n-x_{n-1})\big)\ (n\geq 1). \tag{9}$$

Consequently, $\{\rho\big(\alpha(x_{n+1}-x_n)\big)\}$ is decreasing and bounded from below. Hence $\{\rho\big(\alpha(x_{n+1}-x_n)\big)\}$ converges to z. Now, if $z\neq 0$,

$$z=\lim_{n\to\infty}\rho\big(\alpha(x_{n+1}-x_n)\big)\leq\lim_{n\to\infty}\varphi\big(\rho\big(\beta(x_n-x_{n-1})\big)\big) \\ <\lim_{n\to\infty}\varphi\big(\rho\big(\alpha(x_n-x_{n-1})\big)\big)=\varphi(z),$$

which is a contradiction, hence $z=0$.

Now, we show that $\{x_n\}$ is a ρ-cauchy sequence in $\mathcal{X}_\rho$. If $\{\beta x_n\}$ is not a ρ-cauchy sequence, then there exists $\varepsilon>0$ and sequences $\{m_k\},\{n_k\}$ of integers with $m_k>n_k\geq k$ and

$$\rho\big(\beta(x_{m_k}-x_{n_k})\big)\geq\varepsilon\quad(k\in\mathbb{N}). \tag{10}$$

Moreover, corresponding to odd numbers n_k, we can choose even numbers m_k in such a way that it is the smallest integer with $m_k>n_k$ such that

$$\rho\big(\beta(x_{m_{k-2}}-x_{n_k})\big)<\varepsilon. \tag{11}$$

In fact, let m_k be the smallest even number exceeding n_k for which (10) holds, and

$$N_k=\{m\in\mathbb{N}_e\mid\exists n_k\in\mathbb{N}_o;\rho\big(\beta(x_m-x_{n_k})\big)\geq\varepsilon,m>n_k\geq k\}.$$

It is clear that $N_k\neq\emptyset$ and by well ordering principle, the minimum element of N_k exists and is denoted by m_k, and clearly (11) holds.

Now, let $\alpha_0\in\mathbb{R}^+$ be such that $\frac{\beta}{\alpha}+\frac{1}{\alpha_0}=1$, then we have

$$\begin{aligned}\rho\big(\beta(x_{m_k}-x_{n_k})\big)&=\rho\big(\frac{\beta\alpha}{\alpha}\big(\alpha(x_{m_k}-x_{n_{k+2}})\big)+\frac{1}{\alpha_0}\big(\alpha_0\beta(x_{n_{k+2}}-x_{n_k})\big)\big)\\&\leq\rho\big(\alpha(x_{m_k}-x_{n_{k+2}})\big)+\rho\big(\alpha_0\beta(x_{n_{k+2}}-x_{n_k})\big)\\&\leq\varphi\big(\rho\big(\beta(x_{m_{k-1}}-x_{n_{k+1}})\big)\big)+\rho\big(\alpha_0\beta(x_{n_{k+2}}-x_{n_k})\big)\\&<\varepsilon+\rho\big(\alpha_0\beta(x_{n_{k+2}}-x_{n_k})\big).\end{aligned}$$

If $k\to\infty$, by Δ_2-condition, $\rho\big(\alpha_0\beta(x_{n_{k+2}}-x_{n_k})\big)\to 0$, hence $\lim_{k\to\infty}\rho\big(\beta(x_{m_k}-x_{n_k})\big)=\varepsilon$. Therefore,

$$\begin{aligned}\rho\big(\beta(x_{m_k}-x_{n_k})\big)&\leq\rho\big(\alpha(x_{m_{k+1}}-x_{n_{k+1}})\big)+\rho\big(2\alpha_0\beta(x_{m_k}-x_{m_{k+1}})\big)+\rho\big(2\alpha_0\beta(x_{n_{k+1}}-x_{n_k})\big)\\&\leq\varphi\big(\rho\big(\beta(x_{m_k}-x_{n_k})\big)\big)+\rho\big(2\alpha_0\beta(x_{m_k}-x_{m_{k+1}})\big)+\rho\big(2\alpha_0\beta(x_{n_{k+1}}-x_{n_k})\big).\end{aligned}$$

Therefore, as $k\to\infty$, we get $\varepsilon\leq\varphi(\varepsilon)$, which is a contradiction. Hence $\{\beta x_n\}$ is a ρ-cauchy sequence, and by Δ_2-condition, $\{x_n\}$ is a ρ-cauchy sequence. Since $\mathcal{X}_\rho$ is complete, there is a $w\in C$ such that $\rho(x_n-w)\to 0$, as $n\to\infty$. Now, we show that w is the common fixed point of T and S. Put $x=x_{2n}$ and $y=w$ in (4), we have

$$\rho\big(\alpha(x_{2n+1}-Sw)\big)=\rho\big(\alpha(Tx_{2n}-Sw)\big)\leq\varphi\big(\rho\big(\beta(x_{2n}-w)\big)\big),$$

therefore $\rho\big(\alpha(w-Sw)\big)=\lim_{n\to\infty}\rho\big(\alpha(x_{2n+1}-Sw)\big)=0$, and so $w=Sw$. This completes the proof.

The following corollaries are immediate consequences of Theorem 2.1.

Corollary 2.2 *Let $\mathcal{X}_\rho$ be a ρ-complete modular space. Suppose that $\varphi:\mathbb{R}^+\to[0,\infty)$ is an increasing and upper semicontinuous function satisfying*

$$\varphi(t)<t,\quad(t>0). \tag{12}$$

Let C be a ρ-closed subset of $\mathcal{X}_\rho$ and let $T:C\to C$ be a mapping such that there exist $\alpha,\beta\in\mathbb{R}^+$ with $\alpha>\beta$, and

$$\rho\big(\alpha(Tx-Ty)\big)\leq\varphi\big(\rho\big(\beta(x-y)\big)\big), \tag{13}$$

for all $x,y\in C$. Then T has a unique fixed point in C.

Corollary 2.3 *Let $\mathcal{X}_\rho$ be a ρ-complete modular space. Let C be a ρ-closed subset of $\mathcal{X}_\rho$ and let $T, S : C \to C$ be mappings such that there exist $\alpha, \beta, \eta \in \mathbb{R}^+$ with $\alpha > \beta$ and $\eta \in (0, 1)$, and*

$$\rho\big(\alpha(Tx - Sy)\big) \leq \eta\big(\rho(\beta(x - y))\big), \tag{14}$$

for all $x, y \in C$. Then T and S have a unique common fixed point in C.

Corollary 2.4 *Let $\mathcal{X}_\rho$ be a ρ-complete modular space, where ρ is s-convex and satisfies the Δ_2-condition. Let C be a ρ-closed subset of $\mathcal{X}_\rho$ and let $T, S : C \to C$ be mappings such that there exist $\alpha, \beta, \eta \in \mathbb{R}^+$ with $\alpha > \max\{\beta, \eta\beta\}$ and*

$$\rho\big(\alpha(Tx - Sy)\big) \leq \eta^s\big(\rho(\beta(x - y))\big), \tag{15}$$

for all $x, y \in C$. Then T and S have a unique common fixed point in C.

Let β_0 be a constant such that $\alpha > \beta_0 > \max\{\beta, \eta\beta\}$. Then we have

$$\begin{aligned}\rho\big(\alpha(Tx - Sy)\big) &\leq \eta^s\big(\rho(\beta(x - y))\big) = \eta^s\big(\rho(\frac{\beta}{\beta_0}\beta_0(x - y))\big)\\ &\leq \big(\frac{\beta\eta}{\beta_0}\big)^s \rho(\beta_0(x - y)),\end{aligned}$$

where $\big(\frac{\beta\eta}{\beta_0}\big)^s < 1$. Hence by using Corollary 2.3, the result follows.

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