International Journal of Applied Mathematical Research, 3 (4) (2014) 485-495
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www.sciencepubco.com/index.php/IJAMR
doi: 10.14419/ijamr.v3i4.3575
Research Paper

The M/M/1/K interdependent retrial queueing model with impatient customers and controllable arrival rates

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Abstract

In this paper an M/M/1/K interdependent retrial queueing model with controllable arrival rates and impatient customers is considered. The steady state solutions and the system characteristics are derived for this model. The analytical results are numerically illustrated and the effect of the nodal parameters on the system characteristics are studied and relevant conclusion is presented.

Keywords: *finite capacity, interdependent primary arrival and service processes, impatient customers, retrial queue, single server.*

1. Introduction

Queueing systems in which primary customers who find all the servers and waiting positions (if any) occupied may retry for service after a long period of time are called retrial queues. Between retrials a customer is said to be in orbit (in a sort of queue) and becomes a source of repeated calls. Due to impatience, a repeated call after the n^{th} unsuccessful retrials gives up further repetitions and abandons the system.

Detailed survey on retrial queues and bibliographical information have been obtained from Artalejo [2, 3, 4], Falin [5], Medhi [7], Falin and Templeton [6]. Artalejo [1] studied retrial queues with a finite number of sources. In most of the research work the authors have considered that the input flow of primary calls, intervals between repetitions, service times and decision not to retry for service are mutually independent. But the primary arrival and service processes are interdependent in practical situations. It is also assumed that whenever the queue size reaches a prescribed number R, the arrival reduces from λ_0 to λ_1 and it continues with that rate as long as the content in the queue is greater than some prescribed integer r $(0 \leq r < R)$. When the system size reaches r, the arrival rate changes back to λ_0 and the same process is repeated.

Much work has been reported in the literature regarding interdependent standard queueing model with controllable arrival rates. K.Srinivasa Rao, Shobha and P.Srinivasa Rao have discussed $M/M/1/\infty$ interdependent queueing model with controllable arrival rates in [8]. Srinivasan and Thiagarajan [9, 10, 11] have analysed $M/M/1/K$ interdependent queueing model with controllable arrival rates, $M/M/c//N$ loss and delay queueing system with interdependent queueing model with controllable arrival rates and no passing, $M/M/C/K/N$ interdependent queueing model with controllable arrival rates balking, reneging and spares. Recently Antline Nisha, Thiagarajan and Srinivasan [12] have studied $M/M/1/K$ interdependent retrial queueing model with controllable arrival rates. Although it is natural in the real world, there are only few works taking into consideration retrial phenomena involving the interdependent controllable arrival rates.

An attempt is made in this paper to obtain relevant results for the M/M/1/K interdependent retrial queueing model with impatient customers and controllable arrival rates. In section 2, the description of the model is given stating the relevant postulates. In section 3, the steady state equations are obtained. In section 4, the characteristics of the model are derived. In section 5, numerical results are calculated.

2. Description of the model

Consider a single server finite capacity retrial queueing system in which primary customers arrive according to the Poisson flow of rate λ_0 and λ_1, service times are exponentially distributed with rate μ. If a primary customer finds some server free, he instantly occupies it and leaves the system after service. Otherwise, if the server is busy, at the time of arrival of a primary call then with probability $H_1 \geq 0$ the arriving customer enters an orbit and repeats his demand after an exponential time with rate θ. Thus the Poisson flow of repeated call follow the retrial policy where the repetition times of each customer is assumed to be independent and exponentially distributed. If an incoming repeated call finds the line free, it is served and leaves the system after service, while the source which produced this repeated call disappears. Otherwise, if the server is occupied at the time of a repeated call arrival with probability H_2 retries for service again and with probability $(1 - H_2)$ the source leaves the system without service.

It is assumed that the primary arrival process $[X_1(t)]$ and the service process $[X_2(t)]$ of the systems are correlated and follow a bivariate Poisson process given by

$$P(X_1 = x_1, X_2 = x_2, ; t) = e^{-(\lambda i + \mu - \epsilon)t} \sum_{j=0}^{\min(x1,x2)} \frac{(\epsilon t)^j [(\lambda i - \epsilon)t]^{x1-j} [(\mu - \epsilon)t]^{x2-j}}{j!(x1-j)!(x2-j)!}$$

where x_1, x_2= 0,1,2........
$0 < \lambda_i, \mu$;
$0 < \epsilon <$min$(\lambda i, \mu)$,i=0,1
with parameters $\lambda_0, \lambda_1, \mu$ and ϵ as mean faster rate of primary arrivals, mean slower rate of primary arrivals, mean service rate and mean dependence rate (covariance between the primary arrival and service processes)respectively.

At time t, let $N(t)$ be the number of sources of repeated calls and $C(t)$ be the number of busy servers. The system state at time t can be described by means of a bivariate process C(t),N(t),t $\geq$ 0, where C(t)=1 or 0 according as the server is busy or idle, the process will be called CN process. If the service time is exponential, then $\{C(t), N(t)\}$ is Markovian. Let C and N be the numbers of customers in the service facility and in the orbit respectively in steady state. The processes $\{N(t), C(t)\}\,; t \geq 0$ is a markov process defined on the state space $(n, c)|n = \{0, 1, 2, ...r - 1, r, r + 1, ...k\}\,, c = \{0, 1\}$. The state probabilities at time t are defined as follows

From state $(0, n)$ transitions only to the following states are possible.

(i) State $(1, n)$ with the probability that the arrival of a primary call during the interval $(t, t + dt)$ of time t, when the system is in faster rate of primary arrivals is $(\lambda_0 - \epsilon)dt + o(dt)$ and the system is in slower rate of primary arrivals is $(\lambda_1 - \epsilon)dt + o(dt)$.

(ii) State $(1, n-1)$ with the probability that the commencement of service of one of n sources in the orbit when the server is free during the interval $(t, t+dt)$ of time t, when the system is either in faster or slower rate of primary arrivals is $n\theta dt + o(dt)$.

Since the state $(0, n)$ means that the server is free, there is no transitions corresponding to the service completion.

Again state $(0, n)$ can be reached with transitions only from the following states.

(iii) State $(1, n)$ with the probability that the service completion of the call in service during the interval $(t, t+dt)$ of time t, when the system is either in faster or slower rate of primary arrivals is $(\mu - \epsilon)dt + o(dt)$.

From state $(1, n)$ transitions only to the following states are possible.

(iv) State $(1, n + 1)$ with the probability that the arrival of a primary call which is blocked and decides to try again during the interval $(t, t + dt)$ of time t, when the system is in faster rate of primary arrivals (form a source of repeated calls) is $H_1(\lambda_0 - \epsilon)dt + o(dt)$ and when the system is in slower rate of primary arrivals (form a source of repeated calls) is $H_1(\lambda_1-\epsilon)dt+o(dt)$.

(v) State $(0,n)$ with the probability that the service completion of the call in service during the interval $(t,t+dt)$ of time t, when the system is either in faster or slower rate of primary arrivals is $(\mu-\epsilon)dt+o(dt)$.

(vi) State $(1,n-1)$ with the probability that the arrival of a repeated call from a source which was blocked again and then decided to leave the system without service during the interval $(t,t+dt)$ of time t, when the system is either in faster or slower rate of primary arrivals is $n\theta(1-H_2)dt+o(dt)$.

Again state $(1,n)$ can be reached with transitions only from the following states.

(vii)State $(0,n)$ with the probability that the arrival of a primary call which is admitted to service as the server is free during the interval $(t,t+dt)$ of time t, when the system is in faster rate of primary arrival is $(\lambda_0-\epsilon)dt+o(dt)$ and the system is in slower rate of primary arrival is $(\lambda_1-\epsilon)dt+o(dt)$.

(viii) State $(0,n+1)$ with the probability that the commencement of service of one of the (n+1) source in the orbit as the server is free during the interval $(t,t+dt)$ of time t, when the system is either in faster and slower rate of primary arrivals is $(n+1)\theta dt+o(dt)$.

(ix) State $(1,n-1)$ with the probability that the arrival of a primary call which is not admitted to service as the server is busy during the interval $(t,t+dt)$ of time t, when the system is in faster rate of primary arrivals is $H_1(\lambda_0-\epsilon)dt+o(dt)$ and the system is in slower rate of primary arrivals is $H_1(\lambda_1-\epsilon)dt+o(dt)$.

(x) State $(1,n+1)$ with probability that the arrival of a repeated call from (n+1) sources which was blocked again as the server is busy and then decided to leave the system without service during the interval $(t,t+dt)$ of time t, when the system is either in faster or slower rate of primary arrivals is $(n+1)\theta(1-H_2)dt+o(dt)$

(xi) The probability that there is one primary arrival and one service completion during the interval $(t,t+dt)$ of time t, when the system is either in faster or slower rate of primary arrivals is $\epsilon dt+o(dt)$

3. Steady state equation

Let $P_{0,n,0}$ denote the steady state probability that there are n customers in the queue when the system is in the faster rate of primary arrivals and the server is idle.
Let $P_{0,n,1}$ denote the steady state probability that there are n customers in the queue when the system is in the slower rate of primary arrivals and the server is idle.
Let $P_{1,n,0}$ denote the steady state probability that there are n customers in the queue when the system is in the faster rate of primary arrivals and the server is busy.
Let $P_{1,n,1}$ denote the steady state probability that there are n customers in the queue when the system is in the slower rate of primary arrivals and the server is busy.
We observe that only $P_{0,n,0}$ and $P_{1,n,0}$ exists when $n=0,1,2,\ldots,r-1,r$;
$P_{0,n,0},P_{1,n,0},P_{0,n,1}$ and $P_{1,n,1}$ exist when $n=r+1,r+2,\ldots,R-2,R-1$;
$P_{0,n,1}$ and $P_{1,n,1}$ exists when $n=R,R+1,\ldots,K$.
Further $P_{0,n,0}=P_{1,n,0}=P_{0,n,1}=P_{1,n,1}=0$ if $n>K$.
The steady state equations are

$$-(\lambda_0-\epsilon)P_{0,0,0}+(\mu-\epsilon)P_{1,0,0}=0 \tag{1}$$

$$-\left[H_1(\lambda_0-\epsilon)+(\mu-\epsilon)\right]P_{1,0,0}+(\lambda_0-\epsilon)P_{0,0,0}+\theta P_{0,1,0}+\theta(1-H_2)P_{1,1,0}=0 \tag{2}$$

$$-(\lambda_0-\epsilon+n\theta)P_{0,n,0}+(\mu-\epsilon)P_{1,n,0}=0 \tag{3}$$

$$\begin{aligned}-\left[H_1(\lambda_0-\epsilon)+(\mu-\epsilon)+n\theta(1-H_2)\right]P_{1,n,0}+(\lambda_0-\epsilon)P_{0,n,0}+(n+1)\theta P_{0,n+1,0}+(n+1)\theta(1-H_2)P_{1,n+1,0}\\+H_1(\lambda_0-\epsilon)P_{1,n-1,0}=0,\qquad n=1,2,3,\ldots,r-1\end{aligned} \tag{4}$$

$$-(\lambda_0-\epsilon+r\theta)P_{0,r,0}+(\mu-\epsilon)P_{1,r,0}=0 \tag{5}$$

$$
\begin{aligned}
-\left[H_1(\lambda_0-\epsilon)+(\mu-\epsilon)+r\theta(1-H_2)\right]P_{1,r,0}+(\lambda_0-\epsilon)P_{0,r,0}+(r+1)\theta P_{0,r+1,0}+(r+1)\theta(1-H_2)P_{1,r+1,0}\\
+(r+1)\theta P_{0,r+1,1}+(r+1)\theta(1-H_2)P_{1,r+1,1}+H_1(\lambda_0-\epsilon)P_{1,r-1,0}=0
\end{aligned}
\tag{6}
$$

$$
-(\lambda_0-\epsilon+n\theta)P_{0,n,0}+(\mu-\epsilon)P_{1,n,0}=0 \tag{7}
$$

$$
\begin{aligned}
-\left[H_1(\lambda_0-\epsilon)+(\mu-\epsilon)+n\theta(1-H_2)\right]P_{1,n,0}+(\lambda_0-\epsilon)P_{0,n,0}+(n+1)\theta P_{0,n+1,0}+(n+1)\theta(1-H_2)P_{1,n+1,0}\\
+H_1(\lambda_0-\epsilon)P_{1,n-1,0}=0, \quad n=r+1,r+2\ldots,R-2
\end{aligned}
\tag{8}
$$

$$
-\big(\lambda_0-\epsilon+(R-1)\theta\big)P_{0,R-1,0}+(\mu-\epsilon)P_{1,R-1,0}=0 \tag{9}
$$

$$
-\left[H_1(\lambda_0-\epsilon)+(\mu-\epsilon)+(R-1)\theta(1-H_2)\right]P_{1,R-1,0}+(\lambda_0-\epsilon)P_{0,R-1,0}+H_1(\lambda_0-\epsilon)P_{1,R-2,0}=0 \tag{10}
$$

$$
-\big(\lambda_1-\epsilon+(r+1)\theta\big)P_{0,r+1,1}+(\mu-\epsilon)P_{1,r+1,1}=0 \tag{11}
$$

$$
\begin{aligned}
-\left[H_1(\lambda_1-\epsilon)+(\mu-\epsilon)+(r+1)\theta(1-H_2)\right]P_{1,r+1,1}+(\lambda_1-\epsilon)P_{0,r+1,1}+(r+2)\theta P_{0,r+2,1}\\
+(r+2)\theta(1-H_2)P_{1,r+2,1}=0
\end{aligned}
\tag{12}
$$

$$
-(\lambda_1-\epsilon+n\theta)P_{0,n,1}+(\mu-\epsilon)P_{1,n,1}=0 \tag{13}
$$

$$
\begin{aligned}
-\left[H_1(\lambda_1-\epsilon)+(\mu-\epsilon)+n\theta(1-H_2)\right]P_{1,n,1}+(\lambda_1-\epsilon)P_{0,n,1}+(n+1)\theta P_{0,n+1,1}+(n+1)\theta(1-H_2)P_{1,n+1,1}\\
+H_1(\lambda_1-\epsilon)P_{1,n-1,1}=0, \quad n=r+2,r+3,..,R-1
\end{aligned}
\tag{14}
$$

$$
-(\lambda_1-\epsilon+R\theta)P_{0,R,1}+(\mu-\epsilon)P_{1,R,1}=0 \tag{15}
$$

$$
\begin{aligned}
-\left[H_1(\lambda_1-\epsilon)+(\mu-\epsilon)+R\theta(1-H_2)\right]P_{1,R,1}+(\lambda_1-\epsilon)P_{0,R,1}+(R+1)\theta P_{0,R+1,1}+(R+1)\theta(1-H_2)P_{1,R+1,1}\\
+H_1(\lambda_0-\epsilon)P_{1,R-1,0}+H_1(\lambda_1-\epsilon)P_{1,R-1,1}=0
\end{aligned}
\tag{16}
$$

$$
-(\lambda_1-\epsilon+n\theta)P_{0,n,1}+(\mu-\epsilon)P_{1,n,1}=0 \tag{17}
$$

$$
\begin{aligned}
-\left[H_1(\lambda_1-\epsilon)+(\mu-\epsilon)+n\theta(1-H_2)\right]P_{1,n,1}+(\lambda_1-\epsilon)P_{0,n,1}+(n+1)\theta P_{0,n+1,1}+(n+1)\theta(1-H_2)P_{1,n+1,1}\\
+H_1(\lambda_1-\epsilon)P_{1,n-1,1}=0, \quad n=R+1,R+2,\ldots,K-1
\end{aligned}
\tag{18}
$$

$$
-\big(\lambda_1-\epsilon+K\theta\big)P_{0,K,1}+(\mu-\epsilon)P_{1,K,1}=0 \tag{19}
$$

$$
-(\mu-\epsilon)P_{1,K,1}+(\lambda_1-\epsilon)P_{0,k,1}+H_1(\lambda_1-\epsilon)P_{1,k-1,1}=0 \tag{20}
$$

Write $s=\dfrac{\lambda_0-\epsilon}{\mu-\epsilon}, \quad \gamma=[H_1(\lambda_0-\epsilon)], \quad \delta=[H_2(\lambda_1-\epsilon)]$

From (1)-(4) we get,

$$
P_{0,n,0}=\frac{\gamma^n}{n!\,\theta^n}\frac{\prod_{i=0}^{n-1}(\lambda_0-\epsilon+i\theta)}{\prod_{i=1}^{n}(\mu-\epsilon)+(1-H_2)(\lambda_0-\epsilon+i\theta)}P_{0,0,0}, \qquad n\geq 1 \tag{21}
$$

$$
P_{1,n,0}=s\frac{\gamma^n}{n!\,\theta^n}\prod_{i=1}^{n}\frac{(\lambda_0-\epsilon+i\theta)}{(\mu-\epsilon)+(1-H_2)(\lambda_0-\epsilon+i\theta)}P_{0,0,0}, \qquad n\geq 0 \tag{22}
$$

From (5)-(8) we get,

$$
\begin{aligned}
P_{0,n,0} &= \frac{\gamma^n}{n!\,\theta^n}\frac{\prod_{i=0}^{n-1}(\lambda_0-\epsilon+i\theta)}{\prod_{i=1}^{n}(\mu-\epsilon)+(1-H_2)(\lambda_0-\epsilon+i\theta)}P_{0,0,0} \\
&\quad -\left\{\frac{A_1(r+1)!}{A_2\,n!\,\theta^{n-1-r}}\left[\left(\sum_{m=r}^{n-2}\frac{m!\,\theta^{m-r}}{r!}\gamma^{n-1-m}\prod_{i=m+1}^{n-1}\frac{(\lambda_0-\epsilon+i\theta)}{(\mu-\epsilon)+(1-H_2)(\lambda_0-\epsilon+i\theta)}\right)\right.\right. \\
&\quad \left.\left.+\frac{(n-1)!}{r!}\theta^{n-1-r}\right]\right\}P_{0,r+1,1} \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
P_{1,n,0} &= s\frac{\gamma^n}{n!\,\theta^n}\prod_{i=1}^{n}\frac{(\lambda_0-\epsilon+i\theta)}{(\mu-\epsilon)+(1-H_2)(\lambda_0-\epsilon+i\theta)}P_{0,0,0} \\
&\quad -\left[\frac{A_1(r+1)!}{A_2(\mu-\epsilon)n!\theta^{n-1-r}}\left(\sum_{m=r}^{n-1}\frac{m!\,\theta^{m-r}}{r!}\gamma^{n-1-m}\right.\right. \\
&\quad \left.\left.\prod_{i=m+1}^{n}\frac{(\lambda_0-\epsilon+i\theta)}{(\mu-\epsilon)+(1-H_2)(\lambda_0-\epsilon+i\theta)}\right)\right]P_{0,r+1,1} \\
&\qquad n=r+1,r+2,\cdots,R-1 \qquad (24)
\end{aligned}
$$

where

$$A_1=\Big[(\mu-\epsilon)+(1-H_2)(\lambda_1-\epsilon+(r+1)\theta)\Big], \qquad A_2=\Big[(\mu-\epsilon)+(1-H_2)(\lambda_0-\epsilon+n\theta)\Big]$$

From (9) and (10) we get,

$$P_{0,r+1,1}=\frac{A_3}{A_4}P_{0,0,0}$$

where

$$
\begin{aligned}
A_3 &= \left[\frac{\gamma^R}{(R-1)!\,\theta^{R-1}}\frac{\prod_{i=0}^{R-1}(\lambda_0-\epsilon+i\theta)}{\prod_{i=1}^{R-1}(\mu-\epsilon)+(1-H_2)(\lambda_0-\epsilon+i\theta)}\right] \\
A_4 &= \left[\left(\frac{A_1(r+1)!}{(R-1)!\,\theta^{R-2-r}}\sum_{m=r}^{R-2}\frac{m!\,\theta^{m-r}}{r!}\gamma^{R-1-m}\right.\right. \\
&\quad \left.\left.\prod_{i=m+1}^{R-1}\frac{(\lambda_0-\epsilon+i\theta)}{(\mu-\epsilon)+(1-H_2)(\lambda_0-\epsilon+(i)\theta)}\right)+A_1(r+1)\theta\right] \qquad (25)
\end{aligned}
$$

From (11)-(14), we recursively derive,

$$
\begin{aligned}
P_{0,n,1} &= \left\{\frac{A_1(r+1)!}{A_5\,n!\theta^{n-1-r}}\left[\left(\sum_{m=r}^{n-2}\frac{m!\,\theta^{m-r}}{r!}\delta^{n-1-m}\prod_{i=m+1}^{n-1}\frac{(\lambda_1-\epsilon+i\theta)}{(\mu-\epsilon)+(1-H_2)(\lambda_1-\epsilon+i\theta)}\right)\right.\right. \\
&\quad \left.\left.+\frac{(n-1)!}{r!}\theta^{n-1-r}\right]\right\}P_{0,r+1,1} \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
P_{1,n,1} &= \left[\frac{A_1(r+1)!}{(\mu-\epsilon)n!\theta^{n-1-r}}\left(\sum_{m=r}^{n-1}\frac{m!\,\theta^{m-r}}{r!}\delta^{n-1-m}\right.\right. \\
&\quad \left.\left.\prod_{i=m+1}^{n}\frac{(\lambda_1-\epsilon+i\theta)}{(\mu-\epsilon)+(1-H_2)(\lambda_1-\epsilon+i\theta)}\right)\right]P_{0,r+1,1} \\
&\qquad n=r+1,r+2,\cdots,R-1,R \qquad (27)
\end{aligned}
$$

where

$$A_5 = \left[(\mu-\epsilon)+(1-H_2)(\lambda_1-\epsilon+n\theta)\right]$$

A_1 is given by (23) and $P_{0,r+1,1}$ is given by (25)

From (15)-(20) we recursively derive,

$$P_{0,n,1} = \left[\frac{A_1(r+1)!}{A_5\, n!\theta^{n-1-r}}\left(\sum_{m=r}^{R-1}\frac{m!\,\theta^{m-r}}{r!}\delta^{n-1-m}\prod_{i=m+1}^{n-1}\frac{(\lambda_1-\epsilon+i\theta)}{(\mu-\epsilon)+(1-H_2)(\lambda_1-\epsilon+i\theta)}\right)\right]P_{0,r+1,1} \tag{28}$$

$$P_{1,n,1} = \left[\frac{A_1(r+1)!}{(\mu-\epsilon)n!\theta^{n-1-r}}\left(\sum_{m=r}^{R-1}\frac{m!\,\theta^{m-r}}{r!}\delta^{n-1-m}\prod_{i=m+1}^{n}\frac{(\lambda_1-\epsilon+i\theta)}{(\mu-\epsilon)+(1-H_2)(\lambda_1-\epsilon+i\theta)}\right)\right]P_{0,r+1,1} \tag{29}$$

where A_1 is given by (23),

A_5 is given by (26) and $P_{0,r+1,1}$ is given by (25).

Thus from (21)-(29), we find that all the steady state probabilities are expressed in terms of $P_{0,0,0}$.

4. Characteristics of the model

The following system characteristics are considered and their analytical results are derived in this system.

The probability $P(0)$ that the system is in faster rate of primary arrivals with the server idle and busy.

The probability $P(1)$ that the system is in slower rate of primary arrivals with the server idle and busy.

The probability $P_{0,0,0}$ that the system is empty.

The expected number of customers in the orbit Lq_0, when the system is in faster rate of primary arrivals with the server idle and busy.

The expected number of customers in the orbit Lq_1, when the system is in slower rate of primary arrivals with the server idle and busy.

The probability that the system is in faster rate of primary arrivals is

$$\begin{aligned} P(0) &= \sum_{n=0}^{K}P_{0,n,0}+\sum_{n=0}^{K}P_{1,n,0} \\ &= \left[\sum_{n=0}^{r}P_{0,n,0}+\sum_{n=r+1}^{R-1}P_{0,n,0}+\sum_{n=R}^{K}P_{0,n,0}\right]+\left[\sum_{n=0}^{r}P_{1,n,0}+\sum_{n=r+1}^{R-1}P_{1,n,0}+\sum_{n=R}^{K}P_{1,n,0}\right] \end{aligned}$$

Since $P_{0,n,0}$ and $P_{1,n,0}$ exist only when $n=0,1,2,\ldots,r-1,r,r+1,r+2,\ldots,R-2,R-1$, we get

$$P(0) = \left[\sum_{n=0}^{r}P_{0,n,0}+\sum_{n=r+1}^{R-1}P_{0,n,0}\right]+\left[\sum_{n=0}^{r}P_{1,n,0}+\sum_{n=r+1}^{R-1}P_{1,n,0}\right] \tag{30}$$

From (21) to (24) and (30), we get

$$\begin{aligned} P(0) &= \sum_{n=0}^{R-1} \frac{\gamma^n}{n!\,\theta^n} \frac{\prod_{i=0}^{n-1}(\lambda_0-\epsilon+i\theta)}{\prod_{i=1}^{n}(\mu-\epsilon)+(1-H_2)(\lambda_0-\epsilon+i\theta)} \left[\frac{\lambda_0+\mu-2\epsilon+n\theta}{\mu-\epsilon}\right] P_{0,0,0} \\ &\quad - \sum_{n=r+1}^{R-1} \left\{ \frac{A_1(r+1)!}{A_2\, n!\theta^{n-1-r}} \left[\left(\sum_{m=r}^{n-2} \frac{m!\,\theta^{m-r}}{r!} \gamma^{n-1-m} \right.\right.\right. \\ &\qquad \left. \prod_{i=m+1}^{n-1} \frac{(\lambda_0-\epsilon+i\theta)}{(\mu-\epsilon)+(1-H_2)(\lambda_0-\epsilon+i\theta)} \right) \\ &\qquad \left.\left. + \frac{(n-1)!}{r!}\theta^{n-1-r} \right] \left[\frac{\lambda_0+\mu-2\epsilon+n\theta}{\mu-\epsilon}\right] \right\} P_{0,r+1,1} \end{aligned} \tag{31}$$

where $A_1$and A_2 is given by (23), and $P_{0,r+1,1}$ is given by (25).

The probability that the system is in slower rate of primary arrivals is

$$\begin{aligned} P(1) &= \sum_{n=0}^{K} P_{0,n,1} + \sum_{n=0}^{K} P_{1,n,1} \\ &= \left[\sum_{n=0}^{r} P_{0,n,1} + \sum_{n=r+1}^{R-1} P_{0,n,1} + \sum_{n=R}^{K} P_{0,n,1} \right] + \left[\sum_{n=0}^{r} P_{1,n,1} + \sum_{n=r+1}^{R-1} P_{1,n,1} + \sum_{n=R}^{K} P_{1,n,1} \right] \end{aligned}$$

Since $P_{0,n,1}$ and $P_{1,n,1}$ exists only when $n = r+1, r+2, \ldots, R-2, R-1, \ldots, K$, we get,

$$P(1) = \left[\sum_{n=r+1}^{R} P_{0,n,1} + \sum_{n=R+1}^{K} P_{0,n,1} \right] + \left[\sum_{n=r+1}^{R} P_{1,n,1} + \sum_{n=R+1}^{K} P_{1,n,1} \right] \tag{32}$$

From (26) to (29) and (32), we get

$$\begin{aligned} P(1) &= \sum_{n=r+1}^{R} \left\{ \frac{A_1(r+1)!}{A_5\, n!\theta^{n-1-m}} \left[\left(\sum_{m=r}^{n-2} \frac{m!\,\theta^{m-r}}{r!} \delta^{n-1-m} \prod_{i=m+1}^{n-1} \frac{(\lambda_1-\epsilon+i\theta)}{(\mu-\epsilon)+(1-H_2)(\lambda_1-\epsilon+i\theta)} \right) \right.\right. \\ &\qquad \left.\left. + \frac{(n-1)!}{r!}\theta^{n-1-r} \right] \left[\frac{\lambda_1+\mu-2\epsilon+n\theta}{\mu-\epsilon}\right] \right\} P_{0,r+1,1} \\ &\quad + \sum_{n=R+1}^{K} \left\{ \frac{A_1(r+1)!}{A_5\, n!\theta^{n-1-m}} \left(\sum_{m=r}^{R-1} \frac{m!\,\theta^{m-r}}{r!} \delta^{n-1-m} \prod_{i=m+1}^{n-1} \frac{(\lambda_1-\epsilon+i\theta)}{(\mu-\epsilon)+(1-H_2)(\lambda_1-\epsilon+i\theta)} \right) \right. \\ &\qquad \left. \left[\frac{\lambda_1+\mu-2\epsilon+n\theta}{\mu-\epsilon}\right] \right\} P_{0,r+1,1} \end{aligned} \tag{33}$$

where $P_{0,r+1,1}$ is given by (25).
The probability $P_{0,0,0}$ that the system is empty can be calculated form the normalizing condition.

$P(0) + P(1) = 1$

$$\begin{aligned}
{P_{0,0,0}}^{-1} &= \sum_{n=0}^{R-1} \frac{\gamma^n}{n!\,\theta^n} \frac{\prod_{i=0}^{n-1}(\lambda_0-\epsilon+i\theta)}{\prod_{i=1}^{n}(\mu-\epsilon)+(1-H_2)(\lambda_0-\epsilon+i\theta)} \left[\frac{\lambda_0+\mu-2\epsilon+n\theta}{\mu-\epsilon}\right] \\
&- \sum_{n=r+1}^{R-1} \left\{ \frac{A_1(r+1)!}{A_2\, n!\theta^{n-1-r}} \left[\left(\sum_{m=r}^{n-2} \frac{m!\,\theta^{m-r}}{r!} \gamma^{n-1-m} \prod_{i=m+1}^{n-1} \frac{(\lambda_0-\epsilon+i\theta)}{(\mu-\epsilon)+(1-H_2)(\lambda_0-\epsilon+i\theta)} \right) \right.\right. \\
&\qquad \left.\left. + \frac{(n-1)!}{r!}\theta^{n-1-r} \right] \left[\frac{\lambda_0+\mu-2\epsilon+n\theta}{\mu-\epsilon} \right] \right\} \frac{A_3}{A_4} \\
&+ \sum_{n=r+1}^{R} \left\{ \frac{A_1(r+1)!}{A_5\, n!\theta^{n-1-m}} \left[\left(\sum_{m=r}^{n-2} \frac{m!\,\theta^{m-r}}{r!} \delta^{n-1-m} \prod_{i=m+1}^{n-1} \frac{(\lambda_1-\epsilon+i\theta)}{(\mu-\epsilon)+(1-H_2)(\lambda_1-\epsilon+i\theta)} \right) \right.\right. \\
&\qquad \left.\left. + \frac{(n-1)!}{r!}\theta^{n-1-r} \right] \left[\frac{\lambda_1+\mu-2\epsilon+n\theta}{\mu-\epsilon} \right] \right\} \frac{A_3}{A_4} \\
&+ \sum_{n=R+1}^{K} \left\{ \frac{A_1(r+1)!}{A_5\, n!\theta^{n-1-m}} \left(\sum_{m=r}^{R-1} \frac{m!\,\theta^{m-r}}{r!} \delta^{n-1-m} \prod_{i=m+1}^{n-1} \frac{(\lambda_1-\epsilon+i\theta)}{(\mu-\epsilon)+(1-H_2)(\lambda_1-\epsilon+i\theta)} \right) \right. \\
&\qquad \left. \left[\frac{\lambda_1+\mu-2\epsilon+n\theta}{\mu-\epsilon} \right] \right\} \frac{A_3}{A_4} \qquad (34)
\end{aligned}$$

Now we calculate the expected number of customers in the orbit. Let Lq denote the average number of customers in the orbit, then we have

$$Lq = Lq_0 + Lq_1$$

where

$$\begin{aligned}
Lq_0 &= \sum_{n=0}^{K} n\,P_{0,n,0} + \sum_{n=0}^{K} n\,P_{1,n,0} \\
&= \left[\sum_{n=0}^{r} n\,P_{0,n,0} + \sum_{n=r+1}^{R-1} n\,P_{0,n,0} + \sum_{n=R}^{K} n\,P_{0,n,0} \right] + \left[\sum_{n=0}^{r} n\,P_{1,n,0} + \sum_{n=r+1}^{R-1} n\,P_{1,n,0} + \sum_{n=R}^{K} n\,P_{1,n,0} \right]
\end{aligned}$$

When $n = 0, 1, 2, \ldots, r-1, r, r+1, r+2, \ldots, R-2, R-1$, we get

$$Lq_0 = \left[\sum_{n=0}^{r} n\,P_{0,n,0} + \sum_{n=r+1}^{R-1} n\,P_{0,n,0} \right] + \left[\sum_{n=0}^{r} n\,P_{1,n,0} + \sum_{n=r+1}^{R-1} n\,P_{1,n,0} \right] \qquad (35)$$

and

$$\begin{aligned}
Lq_1 &= \sum_{n=0}^{K} n\,P_{0,n,1} + \sum_{n=0}^{K} n\,P_{1,n,1} \\
&= \left[\sum_{n=0}^{r} n\,P_{0,n,1} + \sum_{n=r+1}^{R-1} n\,P_{0,n,1} + \sum_{n=R}^{K} n\,P_{0,n,1} \right] + \left[\sum_{n=0}^{r} n\,P_{1,n,1} + \sum_{n=r+1}^{R-1} n\,P_{1,n,1} + \sum_{n=R}^{K} n\,P_{1,n,1} \right]
\end{aligned}$$

When $n = r+1, r+2, \ldots, R-2, R-1, \ldots, K$, we get,

$$Lq_1 = \left[\sum_{n=r+1}^{R} n\,P_{0,n,1} + \sum_{n=R+1}^{K} n\,P_{0,n,1} \right] + \left[\sum_{n=r+1}^{R} n\,P_{1,n,1} + \sum_{n=R+1}^{K} n\,P_{1,n,1} \right] \qquad (36)$$

From (21)-(24),(35) and (36), we get

$$
\begin{aligned}
Lq \quad = \quad & \sum_{n=0}^{R-1} \frac{\gamma^n}{(n-1)!\,\theta^n} \frac{\prod_{i=0}^{n-1}(\lambda_0 - \epsilon + i\theta)}{\prod_{i=1}^{n}(\mu-\epsilon) + (1-H_2)(\lambda_0 - \epsilon + i\theta)} \left[\frac{\lambda_0 + \mu - 2\epsilon + n\theta}{\mu - \epsilon}\right] P_{0,0,0} \\
& - \sum_{n=r+1}^{R-1} \left\{ \frac{A_1 (r+1)!}{A_2\,(n-1)!\theta^{n-1-r}} \left[\left(\sum_{m=r}^{n-2} \frac{m!\,\theta^{m-r}}{r!} \gamma^{n-1-m} \right.\right.\right. \\
& \qquad \left. \prod_{i=m+1}^{n-1} \frac{(\lambda_0 - \epsilon + i\theta)}{(\mu-\epsilon) + (1-H_2)(\lambda_0 - \epsilon + i\theta)} \right) \\
& \qquad \left.\left. + \frac{(n-1)!}{r!} \theta^{n-1-r} \right] \left[\frac{\lambda_0 + \mu - 2\epsilon + n\theta}{\mu - \epsilon} \right] \right\} P_{0,r+1,1} \\
& + \sum_{n=r+1}^{R} \left\{ \frac{A_1 (r+1)!}{A_5\,(n-1)!\theta^{n-1-m}} \left[\left(\sum_{m=r}^{n-2} \frac{m!\,\theta^{m-r}}{r!} \delta^{n-1-m} \right.\right.\right. \\
& \qquad \left. \prod_{i=m+1}^{n-1} \frac{(\lambda_1 - \epsilon + i\theta)}{(\mu-\epsilon) + (1-H_2)(\lambda_1 - \epsilon + i\theta)} \right) \\
& \qquad \left.\left. + \frac{(n-1)!}{r!} \theta^{n-1-r} \right] \left[\frac{\lambda_1 + \mu - 2\epsilon + n\theta}{\mu - \epsilon} \right] \right\} P_{0,r+1,1} \\
& + \sum_{n=R+1}^{K} \left\{ \frac{A_1 (r+1)!}{A_5\,(n-1)!\theta^{n-1-m}} \left(\sum_{m=r}^{R-1} \frac{m!\,\theta^{m-r}}{r!} \delta^{n-1-m} \right.\right. \\
& \qquad \left. \prod_{i=m+1}^{n-1} \frac{(\lambda_1 - \epsilon + i\theta)}{(\mu-\epsilon) + (1-H_2)(\lambda_1 - \epsilon + i\theta)} \right) \\
& \qquad \left. \left[\frac{\lambda_1 + \mu - 2\epsilon + n\theta}{\mu - \epsilon} \right] \right\} P_{0,r+1,1}
\end{aligned} \tag{37}
$$

Using Little's formula, the expected waiting time of the customer in the orbit is calculated as

$$Wq = \frac{Lq}{\overline{\lambda}}$$

where $\overline{\lambda} = \lambda_0 P(0) + \lambda_1 P(1)$
Wq is calculated from (31),(33),(34) and (37)

This model includes the earlier models as particular cases. For example, when $H_1 = 1$ and $H_2 = 1$, we get the $M/M/1/K$ interdependent retrial queueing model with controllable arrival rates. When $H_1 = 1$, $H_2 = 1$ and $\theta \to \infty$, we get the standard $M/M/1/K$ interdependent queueing model with controllable arrival rates. When λ_0 tends to λ_1 and $\epsilon = 0$ this model reduces to $M/M/1/K$ retrial queueing model with impatient customers. When λ_0 tends to λ_1, $\epsilon = 0$, $H_1 = 1$ and $H_2 = 1$ this model reduces to $M/M/1/K$ retrial queueing model. When λ_0 tends to $\lambda_1, \epsilon = 0, H_1 = 1, H_2 = 1$ and $\theta \to \infty$, this model reduces to standard $M/M/1/K$ queueing model.

5. Numerical Illustrations

For various values of $\lambda_0, \lambda_1, \mu, \theta, H_1, H_2, \varepsilon, r, R, K$ the values of $P_{0,0,0}, P(0), P(1)$,
Lq and Wq are computed and tabulated in the tables 1 and 2

Table 1

r	R	K	λ_0	λ_1	μ	θ	H_1	H_2	ϵ	$P_{0,0,0}$	P(0)	P(1)
4	8	12	4	3	5	2	0.9	0.1	0.5	0.457578609	0.999841856	0.000158144
4	8	12	6	3	5	2	0.9	0.1	0.5	0.115219769	0.996998721	0.003001279
4	8	12	6	4	5	2	0.9	0.1	0.5	0.115188377	0.996727085	0.003272915
4	8	12	6	4	5	3	0.9	0.1	0.5	0.238055517	0.999580933	0.000419066
4	8	12	6	4	5	3	0.9	0.1	1	0.269761172	0.999740303	0.000259697
4	8	12	6	4	5	3	0.9	0.1	0	0.210846583	0.999222989	0.000777011
4	8	12	6	4	5	3	0.9	0.2	0.5	0.20718277	0.99926046	0.00073954
4	8	12	6	4	5	3	0.8	0.1	0.5	0.290201668	0.999798898	0.000201102
4	8	12	6	4	5	3	0.8	0.2	0.5	0.265412955	0.999626284	0.000373716
4	8	12	6	4	6	3	0.9	0.2	0.5	0.315127220	0.999643174	0.000356826

Table 2

r	R	K	λ_0	λ_1	μ	θ	H_1	H_2	ϵ	Lq	Wq
4	8	12	4	3	5	2	0.9	0.1	0.5	1.451269428	0.362831701
4	8	12	6	3	5	2	0.9	0.1	0.5	2.180027424	0.363883962
4	8	12	6	4	5	2	0.9	0.1	0.5	2.181468789	0.363975218
4	8	12	6	4	5	3	0.9	0.1	0.5	1.755364793	0.292601672
4	8	12	6	4	5	3	0.9	0.1	1	1.687937523	0.281347275
4	8	12	6	4	5	3	0.9	0.1	0	1.822547049	0.303836536
4	8	12	6	4	5	3	0.9	0.2	0.5	1.840693563	0.306857905
4	8	12	6	4	5	3	0.8	0.1	0.5	1.646901126	0.274501922
4	8	12	6	4	5	3	0.8	0.2	0.5	1.801099121	0.300220585
4	8	12	6	4	6	3	0.9	0.2	0.5	1.704395464	0.284099702

6. Conclusion

It is observed from the tables 1 and 2 that when λ_0 increases keeping the other parameters fixed, $P_{0,0,0}$ and P_0 decrease but P_1 and L_q increase. When λ_1 increases keeping the other parameters fixed, $P_{0,0,0}$ and P_0 decrease but P_1 and L_q increase. When θ increases keeping the other parameters fixed, $P_{0,0,0}$ and P_0 increase but P_1 and L_q decrease. when ϵ increases keeping the other parameters fixed, $P_{0,0,0}$ and P_0 increase but P_1 and L_q decrease. when μ increases keeping the other parameters fixed, $P_{0,0,0}$ and P_0 increase but P_1 and L_q decrease. When H_1 decreases keeping the other parameters fixed, $P_{0,0,0}$ and P_0 increase but P_1 and L_q decrease. When H_2 increases keeping the other parameters fixed, $P_{0,0,0}$ and P_0 decrease but P_1 and L_q increase.

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