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k-trees, k-ctrees and Line Splitting Graphs

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Abstract

Let $G = (V, E)$ be a graph. For each edge e_i of G, a new vertex $e_i^{'}$ is taken and the resulting set of vertices is denoted by $E_1(G)$. The line splitting graph $L_s(G)$ of a graph G is defined as the graph having vertex set $E(G) \bigcup E_1(G)$ with two vertices adjacent if they correspond to adjacent edges of G or one corresponds to an element $e_i^{'}$ of $E_1(G)$ and the other to an element e_j of $E(G)$ where e_j is in $N(e_i)$. In this paper we characterize graphs whose line splitting graphs are $k - trees$ and $k - ctrees$.

Keywords: *$k - trees$, $k - ctrees$, line splitting graph, splitting graph.*

1 Introduction

By a graph $G = (V, E)$, we mean a finite, undirected graph without loops or multiple edges. For graph theoretic terminology, we refer to [1].

A vertex v of a graph G is called a star-vertex if all its neighboring vertices are independent.

A graph G is said to be $n - degenerate$ if every subgraph of G has a vertex of degree atmost n [3].

The open neighborhood $N(u)$ of a vertex u in $V(G)$ is the set of vertices adjacent to u. For each vertex u_i of G, a new vertex $u_i^{'}$ is taken and the resulting set of vertices is denoted by $V_1(G)$.

The splitting graph $S(G)$ of a graph G is defined as the graph having vertex set $V(G)\bigcup V_1(G)$ with two vertices adjacent if they correspond to adjacent vertices of G or one corresponds to an element $u_i^{'}$ of $V_1(G)$ and the other to an element w_j of $V(G)$ where w_j is in $N(u_i)$ [7]

The open neighborhood $N(e_i)$ of an edge e_i in $E(G)$ is the set of edges adjacent to e_i. For each edge e_i of G, a new vertex $e_i^{'}$ is taken and the resulting set of vertices is denoted by $E_1(G)$.

The line splitting graph $L_s(G)$ of a graph G is defined as the graph having vertex set $E(G)\bigcup E_1(G)$ with two vertices adjacent if they correspond to adjacent edges of G or one corresponds to an element $e_i^{'}$ of $E_1(G)$ and the other to an element e_j of $E(G)$ where e_j is in $N(e_i)$ [2].

Remark 1.1. *If $G = L(H)$ for some graph H, then $G = S(L(H))$.*

The simplest way to define a $k-tree$ for $k \geq 1$ is by recursion. A $k-tree$ of order $k+1$ is a complete graph of order $k+1$. A $k-tree$ of order $p+1, p \geq k+1$, can be obtained by joining a new vertex to any k mutually adjacent vertices of a $k-tree$ of order p. Note that $1-trees$ are generally known as 'trees' [4].

Remark 1.2. *[6] Every $2-tree$ is planar.*

The class of $k-ctrees$ (for $k \geq 1$) is the set of all graphs that can be obtained by the following recursive construction rule.

1. A totally disconnected graph of order k (i.e., $\overline{K_k}$) is a $k-ctree$.
2. To a $k-ctree$ $Q^{'}$ of order $n-1$ (where $n > k$), insert a new n^{th} vertex and join it to any set of k independent vertices of $Q^{'}$.

Note that $1-ctrees$ are generally known as trees [5].

Theorem 1.3. *[2] The line splitting graph $L_s(G)$ of a graph G is planar if and only if G is planar and (i) or (ii) holds :*

1. *G is either $K_{1,4}$ or $C_{2n}, n \geq 2$.*
2. *$\triangle(G) \leq 3$ and G has no subgraph homeomorphic from the subdivision graph of $K_{1,3}$ and also every block of G is either a K_2 or a triangle such that each triangle has atmost one cut-vertex.*

Theorem 1.4. *[4] Let G be a graph of order p and let $k < p$. Then the following assertions are equivalent :*

1. *G is a $k-tree$.*

2. *G is $k-$connected, triangulated and $K_{k+2}-$free.*

3. *G is $k-$connected, triangulated of size $kp - \binom{k+1}{2}$*

Theorem 1.5. *[1] If G is a (p, q) graph whose vertices have degree d_i, then the line graph of G, $L(G)$ has q vertices and Q_L edges, where $q_L = -q + \frac{1}{2}\sum d_i^2$.*

Theorem 1.6. *[5] Let G be a graph of order $p \geq 2k$. Then G is $k - ctree$ if and only if G is a $k-$degenerate, triangle-free graph of size $k(p-k)$.*

Theorem 1.7. *[5] A graph G of order $\geq k+1$ is a $k - ctree$ if and only if G has a star-vertex v of degree k and $G - v$ is a $k - ctree$.*

Theorem 1.8. *[5] Every $k - ctree$ is a $k-$degenerate, triangle-free graph.*

2 Main Results

2.1 k-trees and Line splitting graphs

Theorem 2.1. *There are only two graphs whose line splitting graphs are $2 - trees$. These graphs are $K_{1,3}$ and C_3.*

Proof. Suppose the line splitting graph $L_s(G)$ of a graph G is a $2-tree$. Clearly G is connected. By Remark 1.2, $L_s(G)$ is planar and hence by Theorem 1.3, G is planar and is either $K_{1,4}$ or $C_{2n}, n \geq 2$ or $\triangle(G) \leq 3$ and G has no subgraph homeomorphic from the subdivision graph of $K_{1,3}$ and also every block of G is either a K_2 or a triangle such that each triangle has atmost one cut-vertex. We consider the following cases depending on the magnitude of $\triangle(G)$.
Case 1. $\triangle(G) = 1$. Then $G = K_2$. Clearly, $L_s(G)$ is disconnected, a contradiction.
Case 2. $\triangle(G) = 2$. Then G is either a path or a cycle. Let G be a graph of order p and size q. We have the following subcases in this case.
Subcase 2.1. G is a path $P_p, p \geq 3$. For $p = 3$, $G = K_{1,2}$. But $L_s(K_{1,2})$ is not triangulated and hence by Theorem 1.4, $L_s(G)$ is not a $2 - tree$, a contradiction. For $p \geq 4$, $L_s(G)$ contains a cycle of length $n = 4$ without chords and therefore, $L_s(G)$ is not triangulated, a contradiction.
Subcase 2.2. G is a cycle C_p, $p \geq 3$. Then $L_s(G)$ has $2p$ vertices and $3p$ edges. Since a $2 - tree$ with $2p$ vertices contains $4p - 3$ edges, it follows that $3p < 4p - 3$ for all $p > 3$ and $3p = 4p - 3$ for $p = 3$. Hence $G = C_3$.
Case 3. $\triangle(G) = 3$. We consider the following subcases.

Subcase 3.1. G is not a tree. Then G has atleast one cycle. So G is a cycle together with a path of length ≥ 1, adjoined at some vertex. Then $L_s(G)$ contains a cycle of length $n = 4$ without chords and therefore, $L_s(G)$ is not triangulated, a contradiction.
Subcase 3.2. G is a tree other than $K_{1,3}$. Then $L_s(G)$ contains a cut-vertex and by Theorem 1.4, $L_s(G)$ is not a $2-tree$, a contradiction. Hence $G = K_{1,3}$.
Case 4. $\triangle(G) > 3$. Then $K_{1,4}$ is a subgraph of G. One can see that $L_s(K_{1,4})$ contains a subgraph isomorphic to K_4 and therefore $L_s(G)$ is not a $2-tree$, a contradiction.

From all the above cases, it follows that $G = K_{1,3}$ or C_3.

□

Theorem 2.2. *A line splitting graph $L_s(G)$ of order $2(k+1), k \geq 3$, is a $k-tree$ if and only if $G = K_{1,k+1}$.*

Proof. Let G be a (p,q) graph. It follows from Theorem 1.5, that the line graph $L(G)$ is $(q, -q + \frac{1}{2}\sum d_i^2)$ graph, where d_i is degree of each vertex $v_i \in V(G)$. Suppose that $L_s(G)$ is a $k-tree$, $k \geq 3$ of order $2(k+1)$. Then clearly $p_L = q = k+1$. Also, since $L_s(G) = S(L(G))$, $L_s(G)$ contains $3(-q + \frac{1}{2}\sum d_i^2)$ edges. Since $L_s(G)$ is a $k-tree$, we have,

$$\begin{aligned}|E(L_s(G))| &= 2(k+1)k - \frac{k(k+1)}{2}\\ &= \frac{3k(k+1)}{2}.\end{aligned}$$

This implies that

$$\begin{aligned}3(-q + \frac{1}{2}\sum d_i^2) &= \frac{3k(k+1)}{2}.\\ \text{So,}\quad q_L &= \frac{k(k+1)}{2}.\end{aligned}$$

Therefore, $L(G) = K_{k+1}$, $k \geq 3$ and hence G is $K_{1,k+1}$, $k \geq 3$.

Conversely, suppose that $K_{1,k+1}$, $k \geq 3$. Then $L(G) = K_{k+1}$. Hence $L_s(G) = S(L(G))$ is a $k-tree$ of order $2(k+1), k \geq 3$. □

2.2 k-ctrees and Line splitting graphs

Theorem 2.3. *There is only one graph whose the line splitting graph is $1-ctree$. That graph is P_3.*

Proof. Suppose the line splitting graph of a graph G is a $1-ctree$. Then $L_s(G)$ is a tree. Assume that G has a vertex u of $deg \geq 3$. Then any three edges incident with u form $K_{1,3}$. Consequently, $L_s(G)$ contains a triangle K_3. This is impossible since $L_s(G)$ is a tree. Hence $\triangle(G) \leq 2$. Then every component of G is either a cycle $C_n, n \geq 3$ or a path $P_n, n \geq 2$. If G is a cycle C_n, then $L_s(G)$ contains a subgraph C_n, which is impossible since $L_s(G)$ is a tree. So, G must be a path. Now, if the length of the path is 1, then $L_s(G)$ is $2K_1$, which is not $1-ctree$. Also, if the length of the path is ≥ 3 then $L_s(G)$ contains a cycle C_4, which is impossible. Hence $G = P_3$. □

Theorem 2.4. *There are only two graphs whose the line splitting graphs are $2-ctrees$. These graphs are K_2 and C_4.*

Proof. Suppose that the line splitting graph of a graph $G(p,q)$ is a $2-ctree$. Suppose $p \geq 5$. We consider the following cases.
Case 1. $\triangle(G) \geq 3$. Then $L_s(G)$ is not triangle-free. By Theorem 1.6, $L_s(G)$ is not a $2-ctree$, a contradiction.
Case 2.$\triangle(G) \leq 2$. Then G is either a cycle $C_p, p \geq 5$ or a path of length atleast 4. We consider the following subcases :
Subcase 2.1. $G = C_p, p \geq 5$, then $L_s(G)$ contains $2p$ vertices and $3p$ edges. But by Theorem 1.6, a $2-ctree$ on $2p$ vertices has $4p-4$ edges. Since $p \geq 5$ we have $3p < 4p-4$, a contradiction.
Subcase 2.2. $G = P_p, p \geq 5$,then $L_s(G)$ contains $2p-2$ vertices and $3p-6$ edges. But by Theorem 1.6, a $2-ctree$ on $2p-2$ vertices has $4p-8$ edges. Since $p \geq 5$ we have $3p-4 < 4p-8$, a contradiction.

In all the cases we arrive at a contradiction. Thus $p \leq 4$. In this case, $L_s(G)$ is isomorphic to one of the graphs, $\overline{K_2}$ and G_1, where G_1 is a graph shown in Figure 1. Consequently G is K_2 and C_4, respectively.

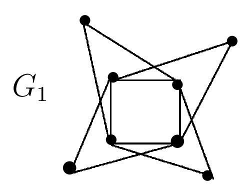

Figure 1.

□

Theorem 2.5. *$L_s(G)$ is a $k-ctree$, $k \geq 3$ if and only if $k = 2l, l \geq 2$ and $G = lK_2$.*

Proof. Suppose that $L_s(G)$ is a $k-ctree$, $k \geq 3$ of order p for some graph G. It follows from Theorem 1.8 that $L_s(G)$ is triangle-free graph. Assume that $p \geq k+1$. Then by Theorem 1.7, $L_s(G)$ contains a star-vertex u of degree k and $L_s(G) - u$ is a $k-ctree$. We consider the following cases:
Case 1. u corresponds to a newly introduced vertex e' for an edge e of G. Then e is adjacent to k edges in G. Since $k \geq 3$, $L_s(G)$ contains a triangle, a contradiction.
Case 2. u corresponds to an edge e of G. By construction of $L_s(G)$, e is

adjacent to $\frac{k}{2}$ edges in G. Since number of edges is an integer, k must be even. Also since $k \geq 3$, we have $k = 2l, l \geq 2$. So, e is adjacent to atleast two edges in G. We consider the following subcases :

Subcase 2.1. e is adjacent to more than two edges in G, then $L_s(G)$ contains a triangle, a contradiction.

Subcase 2.2. e is adjacent to exactly two edges at its same end vertices in G, then also $L_s(G)$ contains a triangle, a contradiction.

Subcase 2.3. e is adjacent to exactly two edges at its different end vertices in G, then $L_s(G) - e$ contains either a pendant vertex or an isolated vertex. Since $k \geq 3$, it follows that $L_s(G) - e$ is not a $k - ctree$,a contradiction.

In all the cases we arrive at a contradiction. So $p = k$. Hence $L_s(G)$ is a $k - ctree$ of order k. It follows that $L_s(G) = \overline{K}_k$. Now, if $k = 2l+1$, $l \geq 1$, then $L_s(G)$ has odd number of vertices, which is impossible. Hence, $k = 2l, l \geq 2$ and $G = lK_2$.

Conversely, suppose $G = lK_2, l \geq 2$. Then $L(G) = lK_1$ and hence $L_s(G)$ is $2lK_1$. i.e. kK_1, which is a $k - ctree$ of order $k = 2l, l \geq 2$. i.e. $k \geq 3$. □

Corollary 2.6. *There are no graphs whose line splitting graphs are $k - ctrees$ where $k = 2l + 1, l \geq 1$.*

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