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Research paper

An Inverse Problem for A Time-Fractional Heat Equation: Determination of A Time-Dependent Coefficient Via The Laplace Homotopy Analysis Method

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Abstract

This study addresses an inverse problem for a time-fractional heat equation involving the Caputo derivative, where the objective is to determine an unknown time-dependent thermal coefficient. The governing equation incorporates a fractional order time derivative, a spatially dependent diffusion term, and a known source term. To solve the inverse problem, we employ the Laplace Homotopy Analysis Method (LHAM)—a semi-analytical approach that combines the Laplace transform with the Homotopy Analysis Method. This technique allows for the construction of a convergent series solution for both the temperature distribution   and the unknown coefficient . By appropriately selecting the initial guess , the convergence-control parameter h becomes embedded in the solution process, reducing com-computational effort. The proposed method is validated through an illustrative example, demonstrating the effectiveness and accuracy of LHAM in solving time-fractional inverse problems with a source term.

Keywords: *Fractional Derivatives; Heat Equation; Homotopy Analysis Method; Inverse Problems; Laplace Transforms; Laplace Homotopy Analysis Method.*

1. Introduction

Classical integer-order differential equations often fall short in modeling the complex dynamics observed in real-world systems, particularly those exhibiting memory effects, hereditary behavior, or anomalous diffusion. To overcome these limitations, fractional calculus—which extends traditional differentiation and integration to non-integer orders—has emerged as a powerful tool for describing such phenomena with greater accuracy [1], [2].

Fractional differential equations (FDEs) are now widely employed across disciplines such as engineering, mathematical physics, biosciences, and epidemiology, due to their ability to account for long-term memory and spatial heterogeneity [3 - 5]. Time-fractional heat equations are extensively used to model anomalous diffusion and heat conduction in heterogeneous or porous media [6], [7].

Recent studies have shown that time-fractional differential equations and inverse problems naturally arise in various real-world systems. Examples include heat conduction in materials with memory, anomalous diffusion in groundwater flow, stress relaxation in viscoelastic solids, and dynamic response modeling in electrochemical batteries. In such problems, the Laplace Homotopy Analysis Method (LHAM) provides an effective semi-analytical framework for solving fractional inverse problems, particularly those involving unknown time-dependent coefficients.

Solving such equations—especially inverse problems where unknown parameters are estimated from observational data—poses significant analytical and computational challenges. A variety of solution techniques have been proposed. Classical methods such as Tikhonov regularization and optimization-based techniques transform the inverse problem into a stable minimization framework [8, 9]. Numerical approaches, including the finite difference method (FDM), finite element method (FEM), and spectral methods, have been extensively used to solve direct and inverse fractional models [10], [11].

To overcome the limitations of purely numerical schemes, various semi-analytical methods have been introduced. These include the Adomian Decomposition Method (ADM) [12], the Homotopy Perturbation Method (HPM) [13], and more recently, the Homotopy Analysis Method (HAM) [14], which provides a flexible framework for constructing series solutions with guaranteed convergence.

In this paper, we consider an inverse problem for a time-fractional heat equation involving the Caputo derivative, in which the objective is to determine an unknown time-dependent thermal coefficient. We adopt the Laplace Homotopy Analysis Method (LHAM)—a hybrid technique that combines the Laplace transform and HAM—to construct a convergent analytical series solution [15].

The structure of this paper is as follows: Section 2 introduces the governing time-fractional heat equation and the inverse problem. In Section 3, the LHAM-based solution methodology is developed. Section 4 presents a numerical example to illustrate the effectiveness of the approach. Finally, Section 5 concludes the paper with a summary.

2. Governing equation

We consider the inverse coefficient problem for a time-fractional heat equation, where the objective is to determine the pair of unknown functions $(u(x,t), g(t))$. The governing equation is given by:

$$ {}_0^cD_t^\alpha u(x,t) - g(t)u_{xx}(x,t) = F(x,t), (x,t) \in (0,l) \times (0,T] \quad (1)$$

Subject to the initial condition

$$u(x,0) = \varphi(x), x \in [0,l] \quad (2)$$

And boundary conditions

$$u(0,t) = a(t), u(l,t) = b(t), t \in [0,T] \quad (3)$$

An additional overspecified boundary condition is given at x = 0:

$$g(t)u_x(0,t) = \mu(t), 0 \le t \le T \quad (4)$$

Where $F(x,t)$, $\varphi(x)$ and $\mu(t)$ are known functions, $u(x,t)$ is the unknown temperature distribution and $g(t) > 0$ is an unknown time-dependent thermal coefficient.
The Caputo fractional derivative of order $\alpha \in (0, 1]$ is defined as:

$$ {}_0^cD_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial u(x,s)}{\partial s} ds, 0 \le t \le T \quad (5)$$

3. Solution of the model via LHAM

Applying the Laplace transform to the governing equations (1) − (4), we obtain:

$$\mathcal{L}[{}_0^cD_t^\alpha u(x,t)] - \mathcal{L}[g(t)u_{xx}(x,t)] = \mathcal{L}[F(x,t)] \quad (6)$$

Where

$$\mathcal{L}[{}_0^cD_t^\alpha u(x,t)] = s^\alpha U(x,s) - s^{\alpha-1}U(x,0) \quad (7)$$

$$s^\alpha U(x,s) - s^{\alpha-1}U(x,0) - \mathcal{L}[g(t)u_{xx}(x,t)] = \mathcal{L}[F(x,t)] \quad (8)$$

And

$$U(x,s) = \frac{U(x,0)}{s} + \frac{\mathcal{L}[g(t)u_{xx}(x,t)] + \mathcal{L}[F(x,t)]}{s^\alpha} \quad (9)$$

Using the Homotopy Analysis Method (HAM), the zeroth-order deformation equation of
The Laplacian equation (9) has the form:

$$(1-q)\big(\phi(x,s;q) - U_0(x,s)\big) = qh\Big(\phi(x,s;q) - \frac{U(x,0)}{s} - \frac{\mathcal{L}[g(t)u_{xx}(x,t)] + \mathcal{L}[F(x,t)]}{s^\alpha} \quad (10)$$

Where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is the convergence-control parameter, $U_0(x,s)$ is the Laplace transform of the initial best guess $u_0(x,t)$ for the actual solution $u(x,t)$.
Expanding $\phi(x,s;q)$ in Taylor series for q we obtain

$$\phi(x,s;q) = U_0(x,s) + \sum_{m=1}^{\infty} U_m(x,s) q^m \quad (11)$$

Where

$$U_m(x,s) = \frac{1}{m!} \frac{\partial^m}{\partial x^m} \phi(x,s;q)\Big|_{q=0} \quad (12)$$

When $q = 0$ and $q = 1$ respectively, equation (11) becomes

$$\phi(x,s;0) = U_0(x,s), \phi(x,s;1) = U(x,s) \quad (13)$$

Differentiating equation (9) m times concerning q and then dividing it by m!; then setting $q = 0$, we obtain an $mth - order$ deformation equation

$$U_m(x,s) - \chi_m U_{m-1}(x,s) = hR_m[U_{m-1}(x,s)] \quad (14)$$

Where

$$R_m[U_{m-1}(x,s)] = \chi_m U_{m-1}(x,s) - h\left((U_{m-1}(x,s) - \frac{\mathcal{L}\left[g(t)\left(u_{m-1}(x,t)\right)_{xx}\right] + \mathcal{L}[F(x,t)}{s^\alpha} - (1-\chi_m)\frac{U_0(x,s)}{s}\right) \quad (15)$$

Note that

$$\chi_m = \begin{cases} 0, m \leq 1 \\ 1, m > 1 \end{cases} \quad (16)$$

And

$$g_0(t) = \frac{\mu(t)}{\frac{\partial}{\partial x}u_0(x,t)\Big|_{x=0}} \quad (17)$$

When $m \geq 1$

$$g_m(t) = \frac{{}_0^C D_t^\alpha u_m(x,t) - F(x,t)}{(u_m(x,t))_{xx}} \quad (18)$$

Finally, applying the inverse Laplace Transform to both sides of equation (15), we have the following power series

$$u(x,t) = \sum_{m=0}^{N} u_m(x,t) \quad (19)$$

4. LHAM solution to some examples

Solving equations (1) – (4) with the following input data;
Example 1

$$\varphi(x) = x - x^2, x \in [0,1] \quad (20)$$

$$\mu(t) = (t+1)^2, t \in [0,1] \quad (21)$$

$$F(x,t) = \frac{1}{\Gamma(2-\alpha)}(x - x^2)t^{1-\alpha} + 2(t+1)^2, (x,t) \in [0,1] \times [0,1] \quad (22)$$

where

$$u_0(x,t) = \varphi(x) \quad (23)$$

Going through the solution process in Section 3, the following solutions were obtained

$$u(x,t) = (x - x^2)(t+1) \quad (24)$$

$$g(t) = (t+1) \quad (25)$$

Equations (24) and (25) correspond to the exact solutions presented by Q. W. Ibrahim and M. S. Hussein [16], [17].

Example 2: We have the following

$$\varphi(x) = x - x^2, x \in [0,1] \quad (26)$$

$$\mu(t) = (1 + |t - 0.5|)(t+1), t \in [0,1] \quad (27)$$

$$F(x,t) = \frac{1}{\Gamma(2-\alpha)}(x - x^2)t^{1-\alpha} + (2|t - 0.5| + 2)(1+t), (x,t) \in [0,1] \times [0,1] \quad (28)$$

Going through the solution process in Section 3, the following solutions were obtained

$$u(x,t) = (x - x^2)(t+1) \quad (29)$$

$$g(t) = 1 + |t - 0.5| \quad (30)$$

Equations (29) and (30) correspond to the exact solutions presented by Q.W. Ibrahim and M.S. Hussein [16], [17].

Example 3: We have the following properties

$$\varphi(x) = x^3, x \in [0,1] \quad (31)$$

$$b(t) = \frac{1}{8} + t^2, t \in [0,1] \quad (32)$$

where

$$l = \frac{1}{2} \tag{33}$$

And equation (4) is replaced with the following condition

$$g(t)u_{xx}(l,t) = \omega(t) \tag{34}$$

Where

$$\omega(t) = 3 + 3t, t \in [0,1] \tag{35}$$

And $g_0(t)$ is calculated as follows

$$g_0(t) = \frac{\omega(t)}{u_0xx(l,t)} \tag{36}$$

$$F(x,t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - 6x(1+t), (x,t) \in [0,1] \times [0,1] \tag{37}$$

Going through the solution process in Section 3, the following solutions were obtained

$$u(x,t) = x^3 + t^2 \tag{38}$$

$$g(t) = 1 + t \tag{39}$$

Equations (38) and (39) correspond to the exact solutions presented by Mimi Li, Gongsheng Li, Zhiyun Li, and Xianzheng Jia [18].

5. Analysis and conclusion

The results demonstrate that the choice of the initial guess $u_0(x,t)$plays a significant role in the convergence behavior of the homotopy series solution. When $u_0(x,t)$ is selected judiciously—typically based on the initial or boundary conditions—the convergence-control parameter h does not need to be explicitly computed, as it becomes inherently absorbed during the solution process. The absorption of h occurs because of he residual term $R_m(x,s)$ the deformation equations becomes structured in such a way that the contributions of h can be factored or normalized during recursion. As a result, no explicit tuning of h is necessary; the method still converges, and we avoid the need for h-curve plotting or parameter sweeping. This property reduces computational complexity and simplifies the application of the Laplace Homotopy Analysis Method (LHAM). Moreover, LHAM proves to be an effective semi-analytical tool for accurately determining both the primary solution $u(x,t)$ and the time-dependent coefficient $g(t)$, even in the presence of a fractional source term.

References

[1] Podlubny, I. (1999). *Fractional differential equations*. Academic Press.

[2] Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and applications of fractional differential equations*. Elsevier.

[3] Magin, R. L. (2006). *Fractional calculus in bioengineering*. Begell House.

[4] Mainardi, F. (2010). *Fractional calculus and waves in linear viscoelasticity*. World Scientific. https://doi.org/10.1142/9781848163300.

[5] Diethelm, K. (2010). *The analysis of fractional differential equations*. Springer. https://doi.org/10.1007/978-3-642-14574-2.

[6] Sun, H., Li, Z., Zhang, Y., & Chen, W. (2017). Fractional and fractal derivative models for transient anomalous diffusion: Model comparison. *Chaos, Solitons & Fractals, 102*(1), 346–356. https://doi.org/10.1016/j.chaos.2017.03.060.

[7] Sun, L., Yan, X., & Wei, T. (2019). Identification of time-dependent coefficients in time-fractional diffusion equations. *Journal of Computational Physics, 346*, 505–517. https://doi.org/10.1016/j.cam.2018.07.029.

[8] Tikhonov, A. N., & Arsenin, V. Y. (1977). *Solutions of ill-posed problems*. Winston & Sons.

[9] Engl, H. W., Hanke, M., & Neubauer, A. (1996). *Regularization of inverse problems*. Kluwer Academic. https://doi.org/10.1007/978-94-009-1740-8.

[10] Liu, F., Anh, V., & Turner, I. (2004). Numerical solution of the space fractional Fokker–Planck equation. *Journal of Computational and Applied Mathematics, 166*(1), 209–219. https://doi.org/10.1016/j.cam.2003.09.028.

[11] Lin, Y., & Xu, C. (2007). Finite difference/spectral approximations for the time-fractional diffusion equation. *Journal of Computational Physics, 225*(2), 1533–1552. https://doi.org/10.1016/j.jcp.2007.02.020.

[12] Adomian, G. (1994). *Solving frontier problems of physics: The decomposition method*. Springer. https://doi.org/10.1007/978-94-015-8289-6.

[13] He, J. H. (1999). Homotopy perturbation technique. *Computers & Mathematics with Applications, 40*(178), 257–262. https://doi.org/10.1016/S0045-7825(99)00018-3.

[14] Liao, S. (2003). *Beyond perturbation: Introduction to the homotopy analysis method*. Chapman & Hall/CRC.

[15] Gupta, V. G., & Kumar, P. (2012). Application of Laplace transform homotopy analysis method to fractional diffusion equations. *Communications in Nonlinear Science and Numerical Simulation, 12*(6), 4518–4530.

[16] Ibraheem, Q. W., & Hussein, M. S. (2023). Determination of time-dependent coefficient in time fractional heat equation. *Partial Differential Equations in Applied Mathematics, 7*, 100492. https://doi.org/10.1016/j.padiff.2023.100492.

[17] Ibraheem, Q. W., & Hussein, M. S. (2024). Determination of timewise-source coefficient in time-fractional reaction–diffusion equation from first order heat moment. *Iraqi Journal of Science, 65*(3), 1612–1628. https://doi.org/10.24996/ijs.2024.65.3.35.

[18] Li, M., Li, G., Li, Z., & Li, X. (2020). Determination of time-dependent coefficients in time-fractional diffusion equations by variational iteration method. *Journal of Mathematics Research, 12*(1), Article 74. https://doi.org/10.5539/jmr.v12n1p74.