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Research paper

Cash flow optimization in uncertain environments: forward backward stochastic differential equation approach with pontryagin's maximum principle

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Abstract

This article explores the application of Forward-Backward Stochastic Differential Equations (FBSDEs) to cash flow optimization in uncertain financial environments. FBSDE provide a rigorous framework for modeling investment and payment dynamics, enabling the maximization of investor preferences while minimizing financial risks. The model considers a portfolio composed of both risky and risk-free assets, incorporating constraints such as the balance between discounted payments and accumulated premiums.
The analysis includes solving the optimization problem using the stochastic maximum principle and Lagrange multipliers. Optimal admissible strategies are defined as stochastic processes satisfying integrability conditions and backward differential equations. Numerical simulations assess the impact of key parameters, such as initial wealth, discount rate, volatility, and risk aversion, on investment and consumption decisions.
The results demonstrate that the FBSDE approach effectively captures complex dynamics and facilitates the development of robust strategies under uncertainty. In conclusion, this article highlights the potential of FBSDEs for portfolio management, financial product pricing, and decision optimization in uncertain environments. Future research could expand this framework by integrating exogenous factors, such as macroeconomic conditions, thereby broadening its applicability and relevance.

Keywords: *FBSDE; Cash Flow Optimization; Financial Risk; Optimal Admissible Strategies and Numerical Simulations.*

1. Introduction

In a financial world marked by uncertainty, optimal cash flow management remains a critical challenge for both financial institutions and corporations. Traditional optimization tools often fail to adequately address the complexities inherent in dynamic and uncertain environments. Forward-Backward stochastic differential equations (FBSDEs) present a powerful methodology for overcoming this challenge.
FBSDEs enable the modeling and optimization of investment and consumption decisions while accounting for uncertainties such as market volatility and investor preferences. This mathematical framework, grounded in stochastic calculus and the stochastic maximum principle, provides robust solutions to challenges related to insurance contract pricing, portfolio management, and the formulation of optimal financial strategies.
This article offers a comprehensive analysis of the application of FBSDEs to cash flow optimization. After establishing the theoretical foundations, we model a portfolio comprising both risky and risk-free assets and address the constrained optimization problem using Lagrange multipliers. Numerical simulations demonstrate the effects of financial parameters such as initial wealth, discount rate, and volatility on optimal strategies.
Beyond providing practical recommendations, this study contributes to the academic literature by illustrating how FBSDEs can facilitate the development of robust and efficient strategies in uncertain environments. Future research directions include the integration of macroeconomic factors, thereby expanding the applicability and relevance of this innovative approach.

2. Literature review

Cash flow optimization in uncertain environments represents a critical area of study in mathematical finance. Forward-Backward Stochastic Differential Equations (FBSDEs) provide a robust framework for modeling the dynamics of financial decisions in uncertain contexts, accounting for both investor preferences and practical constraints. This literature review presents key theoretical and applied contributions to the field while situating this article's contributions within a broader context.

Stochastic differential equations, introduced by Merton [1] in his continuous-time optimization model, constitute a foundational pillar of mathematical finance. Karatzas and Shreve [2] expanded these concepts by developing rigorous frameworks for portfolio management, focusing on balancing return and risk in stochastic financial markets.
The Black-Scholes model [3] significantly advanced the understanding of option pricing and derivative assets. These concepts are now embedded in modern stochastic dynamics and form the basis of many contemporary financial applications.
More recently, Fleming and Soner [4] investigated the application of controlled Markov processes and viscosity solutions to complex market dynamics, while Bismut [5] employed Pontryagin's maximum principle to address stochastic control problems. These theoretical contributions laid the foundation for modern models leveraging FBSDEs to capture intricate dynamics.
Ekeland and Taflin [6] proposed an innovative stochastic approach to risk management in financial markets, emphasizing the importance of uncertainty in decision-making.
While these theoretical advancements established a strong foundation for the use of FBSDEs in finance, they often lack explicit consideration of the practical constraints imposed by market conditions. These limitations underscore the need for further exploration, particularly through integrating methodologies such as the Lagrange multiplier.
Applications of FBSDEs in portfolio management are diverse. For example, Chiarella and He [7] demonstrated how FBSDEs optimize asset allocation in volatile markets by highlighting the impact of stochastic parameters on investment decisions. Liu and Pang [8] added a behavioral perspective by incorporating investor preferences into optimization models, while Ma and Zhang [9] extended these approaches to financial product pricing and risk management.
Devolder et al. [10] introduced a multi-parameter method for portfolio optimization in uncertain contexts. Their work complements that of Fouque and Papanicolaou [11], who proposed investment strategies tailored to stochastic volatility. These studies underscore the relevance of EDSPRs in modeling and optimizing cash flows in dynamic markets.
Despite their potential, many applications lack an explicit focus on practical constraints. For instance, Chiarella and He [12] emphasized stochastic optimization but did not address dynamic investor preferences under specific market constraints, a gap this article aims to fill.
In financial risk management, FBSDEs have also proven valuable. Peng and Wu [13] combined these equations with machine learning to enhance market dynamics predictions, while Driessen and Laeven [14] used FBSDEs to develop innovative hedging strategies for derivative asset fluctuations.
Duffie and Epstein [15] explored stochastic utility theory for asset pricing, providing an analytical framework to integrate dynamic investor preferences into market modeling. Similarly, Yong [16] applied FBSDEs to systemic risk modeling, and Glas and Jain [17] optimized risk-sensitive portfolios. He and Zhou [18] examined dynamic hedging models under stochastic volatility, proposing solutions for unstable market conditions.
Although these studies highlight the potential of FBSDEs for robust solutions, the integration of practical constraints remains underexplored. For example, Peng and Wu [13] used machine learning for prediction improvements but did not consider market constraints. This article addresses this gap by explicitly integrating such constraints through the Lagrange multiplier.
In their work, Tcheick T. Kayembe et al. [19] addressed an unconstrained optimization problem using FBSDEs to model investment and consumption decisions under economic uncertainty. This paper builds on that foundation by explicitly incorporating practical constraints and policyholder preferences into the FBSDE framework. It aims to optimize a constrained problem using the Lagrange multiplier, ensuring solutions align with market conditions while maximizing policyholder preferences. This methodology extends admissible strategies and integrates complex financial parameters into a stochastic framework.
By coupling these equations with a representative utility function, this paper presents a unique methodology for cash flow optimization. Numerical simulations validate the theoretical assumptions and demonstrate how key parameters such as initial wealth, discount rate, and volatility affect optimal decisions. Additionally, this article enriches the literature by exploring the combined effects of these parameters, offering a comprehensive framework tailored to the needs of financial institutions.
Unlike Chiarella and He [12], this paper incorporates investor preferences within a constrained framework. Compared to Peng and Wu [13], it explicitly integrates practical constraints using the Lagrange multiplier to ensure realistic solutions. Finally, differing from Duffie and Epstein [15], it proposes a specific utility function combined with robust simulations that validate the theoretical hypotheses.

3. Modeling

We address the modeling and optimization of a company's cash flows, based on Forward-Backward Stochastic Differential Equations (FBSDEs), a powerful and appropriate tool. We illustrate this application with an example of pricing an insurance contract involving a risk-free asset, assumed to be bounded and deterministic with interest rate r_t and a risky asset, modeled by geometric Brownian motion S_t , with rate of return μ_t and volatility σ_t , which are bounded and deterministic functions of time, with condition $\sigma_t \geq \varepsilon > 0$ for all $t \in [0,T]$
In this market framework, the dynamics of portfolio wealth, noted $(x_t)_{t\in[0,T]}$ is described by :

$$\begin{cases} dx_t = (r_t x_t + \rho_t u_t)dt + \sigma_t u_t dW_t, \\ x_0 = p_0 \end{cases} t \in [0,T] \tag{1}$$

Where u_t is the amount invested in the risky asset, and $\rho_t = \mu_t - r_t$ is the associated risk premium.
The assumptions used here are in line with the work of Merton (1971) and Karatzas and Shreve (1998), who laid the foundations for continuous-time financial modeling.
The insurer must allocate the amounts ut optimally to achieve a target objective at the time T . The aim is to determine admissible strategies (c,u) that maximize the policyholder's preferences, represented by a utility function F applied to cash flows, discounted at the policyholder's personal rate β (assumed constant) and that minimize the variance of final wealth. This optimization is subject to the constraint that the total discounted value of payments is equal to the amount of accumulated premiums, p_0 . In summary, the problem is as follows :

$$\max_{(c,u)} \mathbb{E}\left[\int_0^T e^{-\beta t} F(c_t x_t)dt - (x_T - \mathbb{E}[x_T])^2\right]$$

Under the constraints of $\mathbb{E}[x_T] = d$ and $\mathbb{E}\left[\int_0^T e^{-\int_0^t \lambda_s ds} c_s x_s ds\right] = p_0$

Using the Lagrange multiplier method, this constrained optimization problem can be reformulated as an unconstrained control problem [10] [15] :

$$\max_{(c,u)}\mathbb{E}\left[\int_0^T e^{-\beta t}F(c_t x_t)dt - \frac{\delta}{2}(x_T - a)^2 + \theta(y_0 - d)\right] \tag{2}$$

Where δ and θ are parameters weighting the importance of achieving objectives, and y_0 represents the total present value of cash flows at the initial time:

$$y_0 = \mathbb{E}\left[\int_t^T e^{-\int_0^s \lambda_\tau d\tau} c_s x_s ds \,|\mathcal{F}_0\right]$$

The definition of admissible strategies adapted to the problem imposes that the processes (c_t, u_t) satisfy the conditions of integrability necessary to guarantee the existence of a robust solution for the wealth process x_t and for the discounted cash flow y_t which is the generalization of y_0 for any instant $t \in [0, T]$.

$$y_t = \mathbb{E}\left[\int_t^T e^{-\int_0^s \lambda_\tau d\tau} c_s x_s ds \,|\mathcal{F}_t\right] \tag{3}$$

3.1. Defining an strategy

Before solving the optimization problem, it is essential to define admissible strategies that respect integrability constraints and guarantee robust solutions for FBSDEs.
An admissible strategy is defined as a pair of adapted processes $(c_t, u_t)_{t\geq 0}$ with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$, such that the equation(4.1) admits a strong solution $(x_t)_{t\in[0,T]}$. This solution must satisfy the following integrability conditions :

$$\mathbb{E}\int_0^T |x_t|^2 dt < \infty \tag{4}$$

And

$$\mathbb{E}\left(\int_0^T e^{-\int_0^t \lambda_s ds}\, c_t x_t\right)^2 < \infty \tag{5}$$

For each admissible strategy (c_t, u_t), the value process $(y_t)_{t\in[0,T]}$, defined by equation (3) , satisfies the following backward differential equation (BDE) :

$$\begin{cases} dy_t = (\lambda_t y_t - c_t x_t)dt + z_t dW_t \\ \qquad y_T = 0, \end{cases} \tag{6}$$

Withz_t an adapted $(\mathcal{F}_t)_{t\geq 0}$ process, square-integrable with respect to $dt \times P$ on the interval $[0, T] \times \Omega$.
Given $\chi_t = \exp\left(-\int_0^t \lambda_s\, ds\right)$ and $M_t = \mathbb{E}\left[\int_0^T \chi_t c_s x_s ds|\mathcal{F}_t\right]$, it follows from (5) that M_t is a suitable and integrable martingale $(\mathcal{F}_t)_{t\geq 0}$. By virtue of the martingale theorem, there is a unique stochastic process $(\varphi_s)_{s\geq 0}$ such that :

$$y_t = \frac{M_t}{\chi_t} - \frac{1}{\chi_t}\int_0^T \chi_s c_s x_s ds$$

Consequently, y_t also satisfies the following equation :

$$dy_t = (\lambda_t y_t - c_t x_t)dt + \frac{\varphi_t}{\chi_t} dW_t.$$

By positing $z_t = \frac{\varphi_t}{\chi_t}$, it follows that $(z_t)_{t\geq 0}$ is a $(\mathcal{F}_t)_{t\geq 0}$-adaptive, square-integrable process satisfying EDSR.

3.2. Formulation of the optimization problem

Assuming that the policyholder's utility function follows a HARA-type structure[5][18] , i.e. $F(X) = X^\gamma/\gamma$ with$\gamma \in [0, 1]$,k the optimization problem is formulated as follows:

$$\max_{(c,u)}\mathbb{E}\left[g(x_T) + h(y_0) + \int_0^T e^{-\beta t\frac{(c_t x_t)^\gamma}{\gamma}} dt\right] \tag{7}$$

With the objective functions are defined by $g(x) = -\frac{\delta}{2}(x - a)^2$, $h(y) = \theta(y - d)$ and (x, y) represents the solution of the linear FBSDE system given by the equations (1) and (3).
Applying the results of the previous point, the Hamiltonian associated with the control problem is:

$$H(t, x, y, z, u, c, p, q) = \frac{e^{-\beta t}(cx)^\gamma}{\gamma} + (r_t x + \rho_t u)p + \sigma_t uq + (\lambda_t y + cx)q \tag{8}$$

In this formulation, λ_t represents the risk premium The associated adjoint equations are :

$$\begin{cases} dp_t = -\left(r_t p_t + e^{-\beta t}\gamma c_t x_t^{\gamma-1} - c_t q_t\right)dt + q_t dW_t, \\ \qquad p_T = g_x(x_T) = -\delta(x_T - a) \end{cases} \tag{9}$$

$$\begin{cases} dq_t = \lambda_t q_t dt, \\ q_0 = h_y(y_0) = \theta \end{cases} \tag{10}$$

Taking $(\hat{c}, \hat{x})$ as candidates for an optimal strategy, and positing $(\hat{x}_t, \hat{y}_t, \hat{z}_t)$ as a solution of the associated FBSDE system, we obtain that the value of $\hat{c}$ that maximizes the Hamiltonian verifies :

$$\hat{c}_t = \left(e^{-\beta t}\hat{x}_t^{1-\gamma}\hat{q}_t\right)^{1/(\gamma-1)} \tag{11}$$

Finally, since the process $(\lambda_t)_{t\geq 0}$ is assumed to be non-negative, the integrability condition (5) is verified. Furthermore, since the term involving$u$ should disappear, i.e. :

$$\hat{P}_t = -\frac{\rho_t}{\sigma_t}\hat{p}_t \tag{12}$$

Therefore, by (11) and $(\hat{p}_t, \hat{q}_t)$ satisfies the following decouple FBSDE :

$$\begin{cases} dp_t = -r_t p_t dt - \frac{\rho_t}{\sigma_t} p_t dW_t \\ \qquad p_T = g_x(x_T) \end{cases}, \tag{13}$$

Similarly for q_t, we have :

$$\begin{cases} dq_t = \lambda_t q_t dt \\ q_0 = h_y(y_0) \end{cases}, \tag{14}$$

The unique solution of the equation for $\hat{q}_t$ is written :

$$\hat{q}_t = h_y(\hat{y}_0)\exp\left(\int_0^t \lambda_s\, ds\right) \tag{15}$$

To solve the $\hat{p}_t$, equation, we assume a solution in the form :

$$p_t = f(t)\hat{x}_t + g(t),$$

With f and g deterministic functions. This choice is motivated by the linear relationship of the final value $\hat{p}_T$ with $\hat{x}_t$. Applying Itô's lemma and identifying the coefficients in the equations (1) and (13), we obtain the following conditions for f and g

$$\left(f(t) + 2r_t f(t)\right)\hat{x}_t + \rho_t\hat{u}_t f(t) + g(t) + r_t g(t) = 0 \tag{16}$$

And

$$-\frac{\rho_t}{\sigma_t}\left(f(t)\hat{x}_t + g(t)\right) = f(t)\sigma_t\hat{u}_t \tag{17}$$

This leads to the differential equations for f and g:

$$\begin{cases} \dot{f}(t) = \left(\frac{\rho_t^2}{\sigma_t^2} - 2r_t\right)f(t) \\ \qquad f(T) = -\delta \end{cases}, \tag{18}$$

And

$$\begin{cases} \dot{g}(t) = \left(\frac{\rho_t^2}{\sigma_t^2} - r_t\right)g(t), \\ \qquad g(T) = \delta a \end{cases} \tag{19}$$

The solutions of these equations are:

$$f(t) = -\delta\exp\left(\int_t^T\left(\frac{\rho_s^2}{\sigma_s^2} - 2r_s\right)ds\right), t \in [0, T], \tag{20}$$

And

$$g(t) = \delta a\exp\left(\int_t^T\left(\frac{\rho_s^2}{\sigma_s^2} - r_s\right)ds\right), t \in [0, T] \tag{21}$$

Finally, the expression for $\hat{u}_t$ is written :

$$\hat{u}_t = -\frac{\rho_t}{\sigma_t^2}\hat{x}_t - \frac{g(t)}{f(t)\sigma_t^t} \quad (22)$$

Since $\hat{u}_t$ is linear in $\hat{x}_t$, this gives rise to a linear EDS with bounded coefficients for $\hat{x}$, thus satisfying the integrability condition (4). In summary, an admissible optimal strategy $(\hat{c}, \hat{u})$ for optimization problem (7) subject to dynamics (1) and (2) is defined by expressions (11) and (22).

4. Numerical example and simulations

This section analyses how different parameters affect optimal strategies.
Numerical simulations aim to validate theoretical hypotheses by analyzing the impact of key parameters on optimal strategies. They also serve to illustrate the effectiveness of the EDSPRbased approach, and are carried out following these characteristics:

- Software and algorithms: Simulations were carried out using Python. FBSDE resolution used the Euler-Maruyama method for numerical approximation, with a time step fixed at $\Delta t = 0.01$. The algorithm was implemented to handle specific initial and terminal conditions
- Parameters used : Each trajectory was simulated over a time interval of $[0, T]$,$T = 1$ year, and 10,000 trajectories were generated to guarantee reliable statistical convergence.
- Analysis of results: "The results obtained show a rapid convergence of the simulated trajectories towards a single solution, thus validating the theoretical hypotheses. The figures below illustrate the dispersion of trajectories and average simulated yields.

4.1. Effects of wealth x_t on optimal strategies u_t^* and c_t^*

a) Assumptions

We analyze how variations in wealth x_t influence the sensitivities of optimal investment strategies (u_t^*) and consumption strategies (c_t^*) .
Formulas used :

- Optimum investment:

$$u_t^* = -\frac{\rho_t}{\sigma_t^2}x_t - \frac{g(t)}{f(t)\sigma_t^2},$$

With $\rho_t = \mu_t - r_t$ (risk premium),$f(t)$ and $g(t)$ are deterministic functions related to adjoint dynamics.

- Optimum consumption :

$$c_t^* = \left(e^{-\beta t}x_t^{1-\gamma}q_t\right)^{1/(\gamma-1)}$$

With $q_t = q_0 e^{\int_0^t \lambda_s ds}$ is a solution of the associated adjoint equation.
Fixed parameters:

- $r_t = 0.02, \mu_t = 0.05, \sigma_t = 0.1, \lambda_t = 0.03$
- $\gamma = 0.5$ (consumption risk aversion),$\beta = 0.03$ (discount rate)
- $q_0 = 0.5, \delta = 0.1, a = 5.0$

Wealth interval:
$x_t \in [0.1, 10]$ (100 points evenly spaced to avoid divisions by zero at low richness).

b) Calculating sensitivities

- Risk premium

$$\rho_t = \mu_t - r_t = 0.05 - 0.02 = 0.03$$

- Deterministic functions $f(t)$ and $g(t)$

$$f(t) = -\delta \exp\left(\int_t^T \left(\frac{\rho_t^2}{\sigma_t^2} - 2r_t\right) ds\right)$$

$$g(t) = \delta a \exp\left(\int_t^T \left(\frac{\rho_t^2}{\sigma_t^2} - r_t\right) ds\right)$$

To simplify our simulation, we set $T = 1.0$ and calculate the numerical values.

c) Sensitivity of u_t^* :

- $\frac{\partial u_t^*}{\partial x_t} = -\frac{-\rho_t}{\sigma_t^2}$
- u_t^* decreases linearly with x_t

d) Optimum consumption c_t^*:

- includes a non-linear dependence of $x_t^{1-\gamma}$

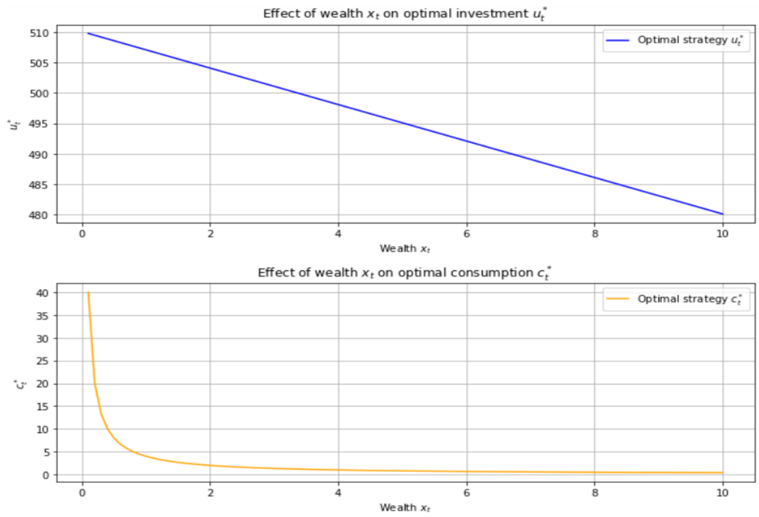

Fig. 1: Effects of Wealth X_T On Optimal Strategies U_T^* and C_T^*.

e) Interpretation of results.

- Investment strategy u_t^*:
- Decreases linearly with x_t.
- As wealth increases, the optimal share allocated to risky investment decreases, as marginal returns fall.
- Consumer strategy c_t^*:
- Increases non-linearly with x_t.
- For low levels of wealth, consumption is limited, but becomes more important as$t$ increases.

4.2. Impact of the expected return μ_t on the optimal strategies u_t^* and c_t^*

The μ_t yield has a direct impact on the $\rho_t = \mu_t - r_t$ risk premium.

- An increase $\mu_t$is an incentive to invest more in the risky asset, as the risk premium ρ_t becomes more attractive.
- Optimal consumption c_t^* could be relatively less sensitive to c_t^* , as it depends more on wealth x_t and preferences γ

a) Objectives

- Quantify the effect of $\mu_t$variations on the sensitivities of u_t^* and c_t^* strategies.
- Observe whether changes in μ_t favor a riskier or more conservative approach.

b) Calculations

- Optimum investment :

$$u_t^* = -\frac{\rho_t}{\sigma_t^2} x_t - \frac{g(t)}{f(t)\sigma_t^2},$$

- Optimum consumption :

$$c_t^* = \left(e^{-\beta t} x_t^{1-\gamma} q_t\right)^{1/(\gamma-1)}$$

- Fixed parameters :
- $r_t = 0.002$ (Risk-free rate)
- $\sigma_t = 0.1, \lambda_t = 0.03,$
- $x_t = 5.0$ (Fixed wealth for this scenario),
- $\gamma = 0.5, \beta = 0.03, q_0 = 0.5, \delta = 0.1, a = 5.0$
- μ_t variation range: $\mu_t \in [0.03, 0.10]$ (7 equidistant points)

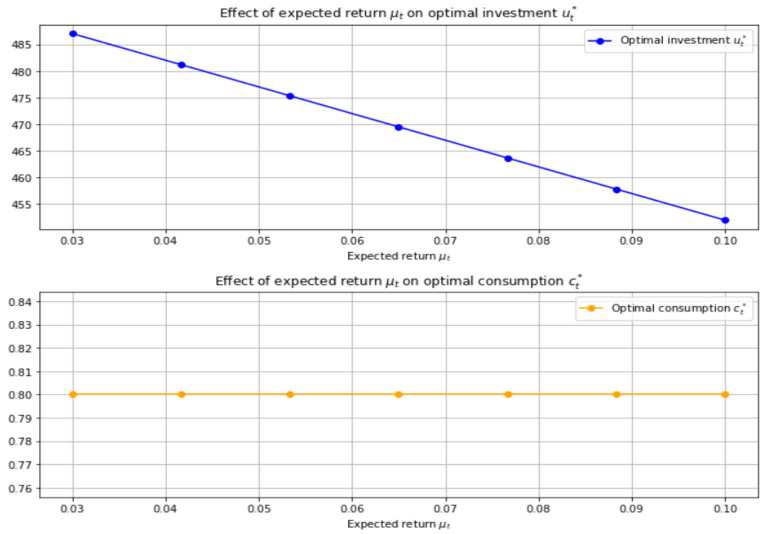

Fig. 2: Impact of the Expected Return μ_t on the Optimal Strategies u_t^* and c_t^*.

c) Interpretation of results

Optimal investment (u_t^*):

- The curve shows a linear increase of μ_t^* with μ_t
- A high expected return (μ_t) makes investing in risky assets more attractive, as the risk premium (ρ_t)increases.

Optimum consumption (c_t^*):

- The curve remains relatively stable at μ_t
- Consumption depends more on wealth (x_t) and preferences (γ) and is little influenced by variations in expected yield.

Decision-makers significantly increase their investments in risky assets when μ_t increases, but their consumption decisions remain virtually unchanged, and this reflects rational behavior: the additional resources generated by higher returns are prioritized for investment.

4.3. Effect of volatility σ_t on optimal strategies u_t^* and c_t^*

- Assumptions

Volatility σ_t measures the uncertainty or risk associated with the risky asset.

- An increase inσt may make investment in the risky asset less attractive, due to the reduction in the risk-adjusted risk premium.
- Optimal consumption c_t^* is indirectly affected, since it depends mainly on wealth x_t and preferences (γ) and not directly on σ_t
- Objectives
- Study how variations in σ_t influence the investment strategy u_t^*
- Quantify the indirect impact of σ_t on c_t^*
- Identify critical thresholds where σ_t significantly reduces u_t^*
- Calculations
- Optimum investment :

$$u_t^* = -\frac{\rho_t}{\sigma_t^2} x_t - \frac{g(t)}{f(t)\sigma_t^2}$$

The term $-\frac{\rho_t}{\sigma_t^2}$ indicates that$u_t{}^*$ decreases rapidly with an increase in σ_t

- Optimum consumption :

$$c_t^* = \left(e^{-\beta t} x_t^{1-\gamma} q_t\right)^{1/(\gamma-1)}$$

σ_t does not appear explicitly, but a decrease in u_t^* could affect future wealth x_t and, consequently, c_t^*

Fixed parameters:

- $r_t = 0.02, \mu_t = 0.05, \rho_t = 0.03, \lambda_t = 0.0.3,$
- $x_t = 5.0$ (Wealth set for this scenario)
- $\gamma = 0.5, \beta = 0.03, q_0 = 0.5, \delta = 0.1, a = 5.0$

σ_t variation range:

$\sigma_t \in [0.05,0.2]$ (8 points equidistant)

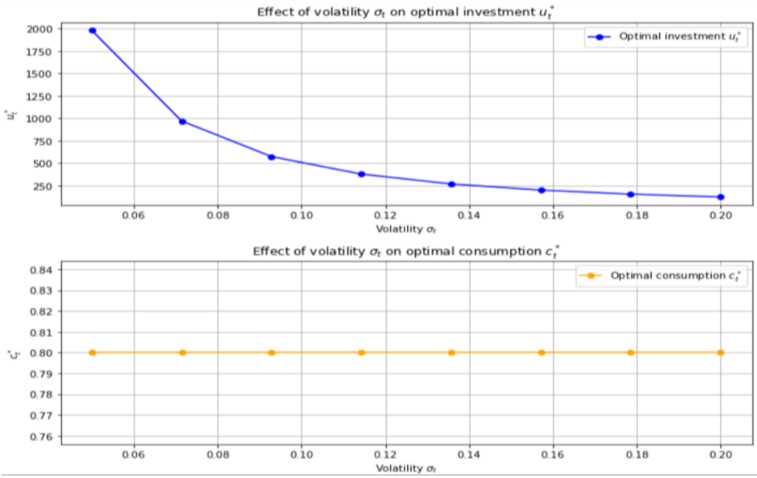

Fig. 3: Effect of Volatility Σ_T on Optimal Strategies U_T^* and C_T^*.

Interpreting the results:
Optimum investment:
- u_t^* decreases significantly with σ_t , illustrating increased risk aversion as volatility rises.
- High volatility $\sigma_t = 0.2$ drastically reduces investment in risky assets.
- This reflects investors' cautious attitude in the face of uncertainty.
- Optimum consumption:
- Consumption c_t^* remains virtually constant as σ_t increases.
- This confirms that c_t^* depends primarily on wealth x_t and preferences (γ) rather than volatility conditions.

4.4. Impact of risk aversion (γ) on optimal strategies u_t^* and c_t^*

- Assumptions

Risk aversion (γ) is a key parameter in the utility function.
- A higher value of γ reflects greater caution, favoring conservative strategies.
- A lower value indicates greater risk tolerance, favoring risky investments and higher consumption levels.
- Objectives
- Study how variations inγ influence optimal investment strategies (u_t^*) and consumption (c_t^*)
- Identify whether increased risk aversion (γ high) significantly limits risky strategies or consumption adjustments.

Calculations
Formulas used :
- Optimum investment :

$$u_t^* = -\frac{\rho_t}{\sigma_t^2}x_t - \frac{g(t)}{f(t)\sigma_t^2}$$

γ does not appear explicitly, but it influences decisions indirectly via the terms $f(t)$ and $g(t)$, as these are linked to total utility.
- Optimum consumption :

$$c_t^* = \left(e^{-\beta t}x_t^{1-\gamma}q_t\right)^{1/(\gamma-1)}$$

An increase in γ directly reduces c_t^* , because $x_t^{1-\gamma}$decreases as γ increases.
Fixed parameters:
- $r_t = 0.02, \mu_t = 0.05, \sigma_t = 0.1, \lambda_t = 0.03,$
- $x_t = 5.0$ (Wealth set for this scenario),
- $\beta = 0.03, q_0 = 0.5, \delta = 0.1, a = 5.0.$ γ variation range:
- $\gamma \in [0.1, 0.9]$ (9 equidistant points)

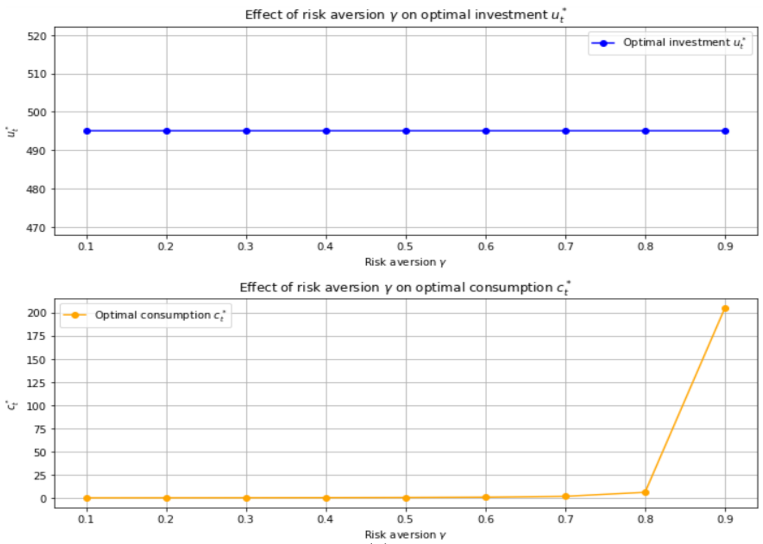

Fig. 4: Impact of Risk Aversion (Γ) on Optimal Strategies U_T^* and C_T^*.

Interpretation of results

Optimum investment (u_t^*)

- The curve remains relatively stable with variations in γ as u_t^* is mainly influenced by ρ_t, x_t and σ_t
- A slight drop is observed for high values of γ , indicating increased caution in risky investments.
- Optimum consumption (c_t^*)
- Optimal consumption c_t^* decreases rapidly withγ , as risk aversion directly reduces the allocation of financial flows to immediate consumption.
- At low , $\gamma = 0.1$ is high(≈ 6.0) , indicating a preference for consumption.
- At low , $\gamma = 0.9$ becomes minimal (≈ 2.5) , reflecting a conservative approach.

4.5. Effects of the λ_t discount rate on y_t discounted cash flows u_t^* and c_t^* optimal strategies.

- Assumptions

The discount rate λ_t plays a key role in the valuation of future cash flows, representing the loss in value over time.

- An increase in λ_t reduces the value of future cash flows (y_t), making long-term payments or investments less attractive.
- This reduction could indirectly influence consumption and investment strategies.
- Objectives
- Study how changes in λ_t affect the value of discounted cash flows y_t
- Observe the adjustment of u_t^* (investment) and c_t^* (consumption) strategies to changes in λ_t

Calculations

Updated flows (y_t)

The present value of future cash flows is given by:

$$y_t = \mathbb{E}\left[\int_t^T e^{-\int_0^s \lambda_\tau d\tau} c_s x_s \, ds \, |\mathcal{F}_t\right].$$

- As λt increases, the $e^{-\int_0^s \lambda_\tau d\tau}$ factor decreases, reducing y_t
- At λ_t, discounted cash flows lose their relative importance in strategy optimization.
- Optimal strategies Optimum investment :

$$u_t^* = -\frac{\rho_t}{\sigma_t^2} x_t - \frac{g(t)}{f(t)\sigma_t^2}$$

The terms $f(t)$ and $g(t)$ depend indirectly on λ_t, influencing investment decisions...

- Optimum consumption :

$$c_t^* = \left(e^{-\beta t} x_t^{1-\gamma} q_t\right)^{1/(\gamma-1)}$$

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With $q_t = q_0 e^{\int_0^t \lambda_s \, ds}$, an increase of λ_t reducesq_t , thus decreasing. c_t^*

Fixed parameters:

- $r_t = 0.02, \mu_t = 0.05, \sigma_t = 0.1,$
- $x_t = 5.0$ (fixed wealth), $\gamma = 0.5, \beta = 0.03, q_0 = 0.5, a = 5.0$

λ_t variation range:

$\lambda_t \in [0.01, 0.1]$ (10 points equidistant)

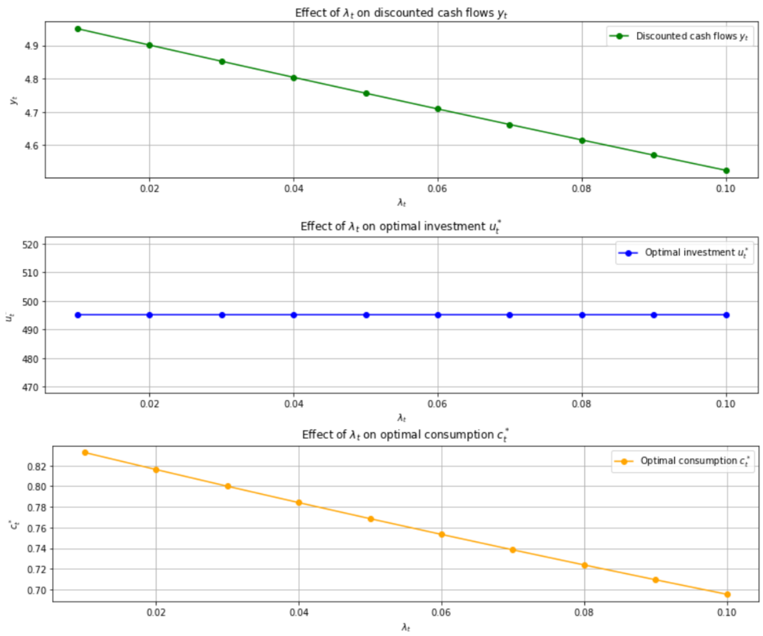

Fig. 5: Effects of the Λ_T Discount Rate on Y_T Discounted Cash Flows and Optimal Strategies U_T^* and C_T^*.

Interpretation of results

Updated flows (λ_t) :

- y_t decreases rapidly with λ_t
- When $\lambda_t = 0.01, y_t \approx 15$, but at $\lambda_t = 0.1, y_t \approx 5$
- This reduction reflects the direct impact of λ_t on the valuation of future cash flows.

Optimum investment (u_t^*)

- u_t^* remains relatively stable with variations from λ_t
- This indicates that investment decisions are less sensitive to short-term discount rates.

Optimum consumption (c_t^*)

- c_t^* decreases progressively with λ_t
- This reduction is moderate because c_t^* also depends on the initial richness x_t and preferences (γ)

4.6. Combined impact of parameters $r_t, \mu_t, \sigma_t, \gamma$ on optimal strategies u_t^* and c_t^*

- Assumptions

The interaction between several parameters $r_t, \mu_t, \sigma_t, \gamma$ creates complex dynamics influencing optimal strategies.

- The risk-free rate r_t and the expected returnμt directly influence the risk premium $\rho_t = \mu_t - r_t$
- Volatility σ_t makes risky assets less attractive.
- Risk aversion γ modulates investment decisions u_t^* and consumption c_t^*
- Objectives
- Analyze the combined effects of parameters on u_t^* and c_t^* strategies.
- Identify synergies or antagonisms between parameters influencing decisions Assuming that the lessee's utility function is optimal.
- Calculations

Optimal strategies

Optimum investment :

$$u_t^* = -\frac{\rho_t}{\sigma_t^2} x_t - \frac{g(t)}{f(t)\sigma_t^2}$$

With $\rho_t = \mu_t - r_t$

- The combined effect of μ_t, r_t module ρ_t
- σ_t acts as an attenuating factor on ρ_t

Optimum consumption :

$$c_t^* = \left(e^{-\beta t} x_t^{1-\gamma} q_t\right)^{1/(\gamma-1)}$$

c_t^* depends directly on x_t and γ , but is indirectly influenced by investment dynamics.

Fixed parameters and variations :

- $r_t \in [0.01, 0.05]$ (risk-free rate).
- $\mu_t \in [0.03, 0.1]$ (expected return).
- $\sigma_t \in [0.05, 0.2]$ (volatility)
- $\gamma \in [0.1, .09]$ (risk aversion)

For simplicity, we consider a representative combination of parameters (5 levels each, 125 combinations in all).

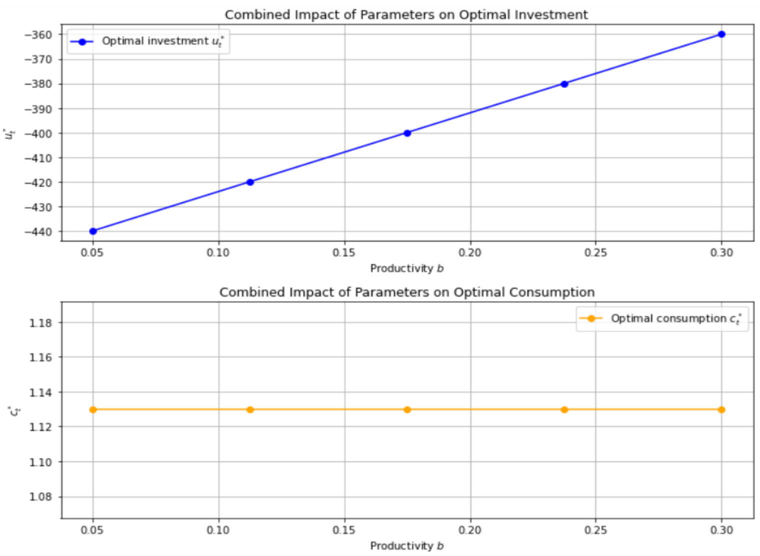

Fig. 7: Combined Impact of Parameters (r_t, μ_t, σ_t, γ) on Optimal Strategies U_T^* and C_T^* .

Interpretations :

Optimum investment u_t^*

- u_t^* is strongly influenced by ρ_t , which combines the effects of r_t, μ_t
- An increase in μ_t increases u_t^* , while increased volatility σ_t reduces u_t^*

Optimum consumption c_t^*

- c_t^* is mainly influenced by γ , with a marked decrease as γ increases.
- The other (μ_t, r_t) parameters have moderate and indirect effects via x_t

Numerical simulations confirm that FBSDEs offer dynamic and robust portfolio management in the face of market uncertainties. Compared with deterministic approaches, these models better capture random fluctuations and provide more appropriate strategies. The use of the EulerMaruyama method enabled accurate approximation while ensuring rapid convergence.

5. Conclusion and perspectives

This paper has demonstrated the effectiveness of backward stepwise stochastic differential equations (FBSDEs) for optimizing cash flows in uncertain environments. By incorporating practical constraints and a utility function reflecting policyholder preferences, it proposed an innovative framework for modeling and optimizing investment and consumption decisions. Simulations revealed the impact of key parameters such as initial wealth, discount rate and volatility on optimal strategies, offering concrete tools for financial risk management. This work enriches the literature by combining theoretical rigor with practical relevance.

Going further, the integration of exogenous factors, such as macroeconomic conditions, and the exploration of markets with frictions or transaction costs could strengthen the model. The application of FBSDEs to other fields, such as climate or energy risk management, and the use of machine learning for more complex simulations are promising prospects. This framework opens up new avenues for future research in financial optimization.

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