International Journal of Applied Mathematical Research, 3 (4) (2014) 390-406
©Science Publishing Corporation
www.sciencepubco.com/index.php/IJAMR
doi: 10.14419/ijamr.v3i4.3283
Research paper

Auto-Bäcklund transformations and new exact soliton solutions of KdV equation for nonlinear dust acoustic solitary waves in dust plasma with variable dust charge

R. S. Ibrahim

Mathematics Department, Faculty of Science Beni-Suef University, Egypt
Email: rs_ibrahim67@yahoo.com

Abstract

The nonlinear properties of dust acoustic solitary waves in unmagnetized dust plasma consisting of negative charged dust particles , Boltzmann distributed electrons and non-thermal distributed ions with variable dust charge are investigated. By using the reductive perturbation theory, a Korteweg-de Veries (KdV) equation is derived. The Sagdeev's potential are obtained in terms of ion acoustic velocity by using the auto-Bäcklund (BT) transformation, the modified $\frac{\acute{G}}{G}$ expansion, the sine-cosine expansion method, the sinh-coshine expansion method, and the sech-tanh expansion method describing the nonlinear propagation of ion-acoustic solitary waves in unmagnetized dust plasma.

Keywords: *dust plasma, KdV equation, Reductive perturbation method, solitary waves, Bäcklund transformations, Traveling wave solutions.*

1. Introduction

The study of dusty plasmas represents one of the most rapidly growing branches of plasma physics. The growing interest in the physics of dusty plasmas has arisen not only from dust being an omnipresent ingredient of our universe, but also from its vital role in understanding collective processes in astrophysical and space environments, such as mode modication, new eigenmodes, coherent structures, etc [1]. Usually the dust grains are of micrometer or sub-micrometer size and their masses are very large. The consideration of charged dust grains in plasmas not only modifies the existing plasma wave spectra, but also introduces a number of new eigenmodes, such as dust-acoustic waves (DAWs), which were reported theoretically first by Rao et al. [2] and verified experimentally by Barkan et al [3]. The linear characteristics of DAWs have by now been well established both theoretically [4] and experimentally [5]. There is also an enormous theoretical literature on the topic of nonlinear DAWs [6, 7, 8, 9]. In the absence of dissipation (or if the dissipation is weak at the characteristic dynamical time scales of the system) the balance between nonlinear and dispersion effects can result in the formation of symmetrical solitary waves (a soliton). Shalini Bagchi [10] investigated the effects of obliquity and external magnetic field on the dust acoustic solitary waves in hot magnetized dusty plasmas with Boltzmann distributed electrons and two-temperature trapped ions. It was found that both compressive and rarefactive solitary waves as well as compressive and rarefactive double

layers exist. Brindaban Das and Prasanta Chatterjee [11] studied large amplitude double layers in dusty plasma with non-thermal electrons and two temperature isothermal ions. Xie et al. [12] investigated the small- and large-amplitude dust-acoustic solitary waves (DASWs) in dusty plasmas with variable dust charge and two-temperature ions. It was noticed that both compressive and rarefactive solitary waves as well as double layers exist.
The multispecies plasmas consisting of cold or warm positive and negative ions with usual Boltzmanns electrons have been the focus of research for the past few years. Most research work was based on the study of ion-acoustic solitons. KdV equation is derived in such plasmas using reductive perturbation method [13]. It was predicted that with the introduction of negative ions, there exists a critical ion concentration of negative ions below which compressive solitons exist and above which rarefactive solitons exist.
Lin and Daun also considered dust acoustic solitary wave in a dusty plasma with non-thermal ions [14]. In this paper we we considered unmagnetized dusty plasma which including variation dust charge, Boltzmann distributed electrons and non thermal distributed ions. We use the reductive perturbation technique to derive a KdV equation. would like to use the homogeneous balance method [15, 16] to construct an auto-Bäcklund transformation (BT)[17-23] , new exact soliton solutions of KdV equation are obtained and we have applied the modified $\frac{\acute{G}}{G}$ expansion method[24-31], the sine-cosine expansion method, the sinh-coshine expansion method [3-36],and the sech-tanh expansion method[33,37-41] and to obtain a new travelling wave solutions of KdV equation.
The paper is organized as follows :This introduction in Section 1. In Section 2, the mathematical analysis and, the KdV equation is derived by using reductive perturbation theory. In Section 3, the auto-Bäcklund transformation (BT) applying to construct new exact soliton solutions of KdV equation. In Section 4, the modified $\frac{\acute{G}}{G}$ expansion method, the sine-cosine expansion method, the sinh-coshine expansion method, and the sech-tanh expansion method are applied to construct new travelling wave solutions of KdV equation. Finally, the application of the solutions are given in Section 5. Conclusions are given in Section 6.

2. Mathematical analysis

Basic equation for consideration dust acoustic wave in unmagnetized dusty plasma with include variable dust charge , Boltzmann electrons and non-thermal ions , is given by:

$$\frac{\partial n_d}{\partial t}+\frac{\partial (n_d u_d)}{\partial x}=0 \tag{1}$$

$$n_d\frac{\partial u_d}{\partial t}+n_d u_d\frac{\partial u_d}{\partial x}+T\frac{\partial p_d}{\partial x}=-n_d z_d\frac{\partial \phi}{\partial x} \tag{2}$$

$$\frac{\partial p_d}{\partial t}+u_d\frac{\partial p_d}{\partial x}+3p_d\frac{\partial u_d}{\partial x}=0 \tag{3}$$

$$\frac{\partial^2 \phi}{\partial x^2}=z_d n_d+\mu n_e-\frac{1}{1-\mu}n_i \tag{4}$$

where n_d is the dust particle number density normalized to n_{d0}. u_d is dust particle velocity normalized to $C_d=(\frac{z_d T_i}{m_d})^{\frac{1}{2}}$. z_d, m_d and ϕ number of charge on dust particles, particle mass and electrostatic potential, respectively. z_d and ϕ are normalized to z_{d0} and $\frac{T_i}{e}$, respectively. Time and space variable are normalized respectively to the dust plasma period $w_{pd}=(\frac{4\pi e^2 n_{0d} z_d^2}{m_d})^{\frac{1}{2}}$ and the Debye length $\lambda_{De}=(\frac{T_i}{4\pi e^2 n_{0d} z_d^2})^{\frac{1}{2}}$. $T=\frac{T_i}{T_e}$ where T_i and T_e are temperature ions and electrons respectively and $\mu=\frac{n_{0e}}{n_{0i}}$, where n_{0e} and n_{0i} are number density of unperturbed ions and electrons in dust acoustic.
n_i is number density of ions and n_e is number density of electrons which normalized to n_{0i} and n_{0e} respectively. n_i and n_e is given by[42,43]:

$$n_i=[1+\frac{4\alpha}{1+3\alpha}(\phi+\phi^2)]exp(\phi) \tag{5}$$

$$n_e=\frac{1}{1-\mu}exp(T\phi), \tag{6}$$

α being a parameter defining the population of non-thermal ions [43,44]. In the case of $\alpha=0$, we can neglect the effect of non-thermal ions. Total charge neutrality at equilibrium is:

$$n_{0e}+n_{0d}z_{0d}=n_{0i}, \tag{7}$$

where n_{0e} ,n_{0i}, n_{0d} and z_{0d} are number density of unperturbed electrons, ions and the number density equilibrium values of unperturbed number of charge on the dust particles respectively. The variation of dust charge is because of collision between electrons and ions with dust particles. The collision of electrons and ions with plasma particles produces a charged current. The charge-current balance equation is given by :

$$\frac{\partial q_d}{\partial t} = I_e + I_i, \tag{8}$$

where I_e and I_i are current of electrons and ions respectively. If we consider that the streaming velocities of electrons and ions are much smaller than the thermal velocities. Therefore $\frac{dq_d}{dt} << I_i, I_e$ and charge- current balance equation (8) reduce to equation $I_e + I_i \approx 0$.
The current of ions and electrons are[45]:

$$I_e = -e\pi r^2 (\frac{8T_e}{\pi m_e})^{\frac{1}{2}} n_e exp(\frac{e\Phi}{T_e}), \tag{9}$$

$$I_i = e\pi r^2 (\frac{8T_i}{\pi m_i})^{\frac{1}{2}} n_i (1 - \frac{e\phi}{T_i}), \tag{10}$$

where Φ denotes the dust particle surface potential related to the plasma potential ϕ [12] and z_d is the normalized dust charge obtained from:

$$z_d = \frac{\psi}{\psi_0}, \tag{11}$$

where $\psi = \frac{e\Phi}{T_{eff}}$ and $\psi_0 = \psi(\phi = 0)$.

In order to study the nonlinear propagation of the dust acoustic wave in a unmagnetized dust plasma, We now use the reductive perturbation method to obtain the KdV equation that governs the behavior of small amplitude dust acoustic waves. The independent variables are stretched as

$$\xi = \varepsilon^{\frac{1}{2}}(x - \lambda t) \quad and \quad \tau = \varepsilon^{\frac{3}{2}} t \tag{12}$$

where λ is the phase velocity and ε is a small dimensionless expansion parameter. By expanding the independent variables we have:

$$n_d = 1 + \varepsilon n_{d1} + \varepsilon^2 n_{d2} + \varepsilon^3 n_{d3} + ... \tag{13}$$

$$u_d = \varepsilon u_{d1} + \varepsilon^2 u_{d2} + \varepsilon^3 u_{d3} + ... \tag{14}$$

$$\phi = \varepsilon \phi_1 - \varepsilon^2 \phi_2 + \varepsilon^3 \phi_3 - ...... \tag{15}$$

$$p = 1 + \varepsilon p_{d1} + \varepsilon^2 p_{d2} + \varepsilon^3 p_{d3} + ... \tag{16}$$

$$z = 1 + \varepsilon^2 z_{d1} + \varepsilon^4 z_{d2} + ... \tag{17}$$

Substituting equations (13-17) into equations (1-4) and collecting terms with same powers of ϵ, from the cofficients of lowest order we have:

$$n_{d1} = \frac{1}{\lambda^2 - 3T}\phi_1; \quad u_{d1} = \frac{\lambda}{\lambda^2 - 3T}\phi_1; \quad P_{d1} = \frac{3}{\lambda^2 - 3T}\phi_1; \eta = \frac{1}{3T - \lambda^2} = \frac{\mu T + 3\alpha\mu T - \alpha + 1}{(1-\mu)(1+3\alpha)}. \tag{18}$$

And from the higher order coefficients of ε we have:

$$-\lambda \frac{\partial n_{d2}}{\partial \xi} + \frac{\partial n_{d1}}{\partial \tau} + \frac{\partial u_{d2}}{\partial \xi} + \frac{\partial (n_{d1} u_{d1})}{\partial \xi} = 0 \tag{19}$$

$$-\lambda\frac{\partial u_{d2}}{\partial\xi}-\lambda n_{d1}\frac{\partial u_{d1}}{\partial\xi}+\frac{\partial u_{d1}}{\partial\tau}+u_{d1}\frac{\partial u_{d1}}{\partial\xi}+T\frac{\partial p_{d2}}{\partial\xi}=\frac{\partial\phi_2}{\partial\xi}-n_{d1}\frac{\partial\phi_1}{\partial\xi} \tag{20}$$

$$-\lambda\frac{\partial p_{d2}}{\partial\xi}+\frac{\partial p_{d1}}{\partial\tau}+u_{d1}\frac{\partial p_{d1}}{\partial\xi}+3\frac{\partial u_{d2}}{\partial\xi}+3p_{d1}\frac{\partial u_{d1}}{\partial\xi}=0 \tag{21}$$

$$\frac{\partial^2\phi}{\partial\xi^2}=z_{d1}+n_{d2}-(\frac{\mu T+3\alpha\mu T-\alpha+1}{(1-\mu)(1+3\alpha)})\phi_2+\phi_1^2\{\frac{\mu T^2-1}{2(1-\mu)}\}. \tag{22}$$

By substituting equation (19-22) into (18) KdV equation is obtained:

$$\frac{\partial\phi_1}{\partial\tau}+C\phi_1\frac{\partial\phi_1}{\partial\xi}+D\frac{\partial^3\phi_1}{\partial\xi^3}=0, \tag{23}$$

where coefficient C can be positive or negative. With positive C, solitary wave is compressive and with negative C, solitary wave is rarefactive. The coefficients C and D are functions of T , α and μ and given by:

$$C=\frac{-\lambda}{-\mu+3\mu^2T+\mu^2\lambda^2}\ ,\quad D=\frac{3\mu^2\lambda+2\alpha\lambda-12\mu^3T\lambda}{-\mu+3\mu^2T+\mu^2\lambda^2}, \tag{24}$$

where T , α and μ are plasma parameters.

3. Auto-B$\ddot{a}$cklund transformation (BT) and new exact soliton solutions of KdV equation

By using the idea of the Homogeneous balance method [46], we seek for B$\ddot{a}$cklund transformation (BT) of equations (23) in the form

$$\phi_1(\xi,\tau)=\psi(\tau)\partial^2\xi f[\chi(\xi,\tau),\eta(\xi,\tau)]+\phi_0(\xi,\tau), \tag{25}$$

where $\psi(\tau)$ is a differentiable function, $f(\chi,\eta)$ is a function to be determined later and ϕ_0 be the special (old) solution of KdV equation (23). In the following analysis, we will stay with the following conditions

$$\psi(\tau)=1,\eta_\xi(\xi,\tau)=0\Rightarrow\eta(\xi,\tau)=\eta(\tau) \tag{26}$$

Substituting (25)and (26)into equation (23) yields

$$\begin{aligned}&(Cf_{\chi\chi}f_{\chi\chi\chi}+Df_{\chi\chi\chi\chi\chi})\chi_\xi^5+(\chi_{\xi\xi\tau}+C\chi_{\xi\xi}\phi_{0\xi}+C\chi_{\xi\xi\xi}\phi_0+D\chi_{\xi\xi\xi\xi\xi})f_\chi\\&+(2\chi_\xi\chi_{\xi\tau}+\chi_\tau\chi_{\xi\xi}+C\chi_\xi^2\phi_{0\xi}+3C\chi_\xi\chi_{\xi\xi}\phi_0+10D\chi_{\xi\xi}\chi_{\xi\xi\xi}+5D\chi_\xi\chi_{\xi\xi\xi\xi})f_{\chi\chi}\\&+(\chi_\tau\chi_\xi^2+C\chi_\xi^3\phi_0+15D\chi_\xi\chi_{\xi\xi}^2+10D\chi_\xi^2\chi_{\xi\xi\xi})f_{\chi\chi\chi}+(10D\chi_\xi^3\chi_{\xi\xi})f_{\chi\chi\chi\chi}+(C\chi_{\xi\xi}\chi_{\xi\xi\xi})f_\chi^2+(3C\chi_\xi^3\chi_{\xi\xi})f_{\chi\chi}^2\\&\quad+(C\chi_\xi^2\chi_{\xi\xi\xi}+3C\chi_\xi\chi_{\xi\xi}^2)f_\chi f_{\chi\chi}+(C\chi_\xi^3\chi_{\xi\xi})f_\chi f_{\chi\chi\chi}+\eta_\tau\chi_\xi^2f_{\chi\chi\eta}+\eta_\tau\chi_{\xi\xi}f_{\chi\eta}+\phi_{0\tau}+C\phi_0\phi_{0\xi}+D\phi_{0\xi\xi\xi}=0\end{aligned} \tag{27}$$

Setting the coefficients ofχ_ξ^5 in (27) to zero, we obtain the ordinary differential equation for f; namely

$$Cf_{\chi\chi}f_{\chi\chi\chi}+Df_{\chi\chi\chi\chi\chi}=0 \tag{28}$$

which admits the solution

$$f=c\ln(\chi)+\chi\sigma(\eta) \tag{29}$$

where $\sigma(\eta)$ is differential function and c is arbitrary constant. According to (27), we obtain

$$\begin{aligned}&f_\chi=\frac{c}{\chi}+\sigma,\ \ f_{\chi\chi}=\frac{-c}{\chi^2},\ \ f_{\chi\chi\chi}=\frac{2c}{\chi^3},f_{\chi\chi\chi\chi}=\frac{-6c}{\chi^4},f_{\chi\chi}^2=\frac{-c}{6}f_{\chi\chi\chi\chi},\ \ f_\chi^2=\frac{-1}{c}(c+\chi\sigma(\eta))^2f_{\chi\chi},\\&\qquad f_\chi f_{\chi\chi}=\frac{-1}{2}(c+\chi\sigma(\eta))f_{\chi\chi\chi},f_\chi f_{\chi\chi\chi}=\frac{-1}{3}(c+\chi\sigma(\eta))f_{\chi\chi\chi\chi}\end{aligned} \tag{30}$$

Substituting (28)and(30)into (27)yields a linear polynomial of f_{χ},$f_{\chi\chi}$,$f_{\chi\chi\chi}$ and $f_{\chi\chi\chi\chi}$. Equating the coefficients of f_{χ},$f_{\chi\chi}$and $f_{\chi\chi\chi\chi}$ to zero, holds

$$\chi_{\xi\xi\tau} + C\chi_{\xi\xi}\phi_{0\xi} + C\chi_{\xi\xi\xi}\phi_0 + D\chi_{\xi\xi\xi\xi\xi} = 0 \tag{31}$$

$$2\chi_\xi\chi_{\xi\tau} + \chi_\tau\chi_{\xi\xi} + C\chi_\xi^2\phi_{0\xi} + 3C\chi_\xi\chi_{\xi\xi}\phi_0 + 10D\chi_{\xi\xi}\chi_{\xi\xi\xi} + 5D\chi_\xi\chi_{\xi\xi\xi\xi} - \frac{1}{c}(c+\chi\sigma(\eta))^2(C\chi_{\xi\xi}\chi_{\xi\xi\xi}) = 0 \tag{32}$$

$$\chi_\tau\chi_\xi^2 + C\chi_\xi^3\phi_0 + 15D\chi_\xi\chi_{\xi\xi}^2 + 10D\chi_\xi^2\chi_{\xi\xi\xi} - \frac{1}{2}(c+\chi\sigma(\eta))(C\chi_\xi^2\chi_{\xi\xi\xi} + 3C\chi_\xi\chi_{\xi\xi}^2) = 0 \tag{33}$$

$$10D\chi_\xi^3\chi_{\xi\xi} - \frac{c}{6}(3C\chi_\xi^3\chi_{\xi\xi}) - \frac{1}{3}(c+\chi\sigma(\eta))(C\chi_\xi^3\chi_{\xi\xi}) = 0 \tag{34}$$

$$\phi_{0\tau} + C\phi_0\phi_{0\xi} + D\phi_{0\xi\xi\xi} = 0 \tag{35}$$

The next crucial step is the assumption that

$$\chi(\xi,\tau) = 1 + \exp^{\lambda(\tau)\pm k\xi} and \phi_0(\xi,\tau) = \phi_0 \tag{36}$$

where k is arbitrary constant. Substituting into equations (31-35), results in

$$\lambda'(\tau) + Ck\phi_0 + k^3 D = 0 \tag{37}$$

$$3\lambda'(\tau) + 3Ck\phi_0 + 15k^3 D - C\frac{(c+\chi\sigma(\eta))^2}{c}k^3 = 0 \tag{38}$$

$$\lambda'(\tau) + Ck\phi_0 + 25k^3 D - 2C(c+\chi\sigma(\eta))k^3 = 0 \tag{39}$$

$$10D - \frac{5}{6}Cc - \frac{c}{3}\chi\sigma(\eta) = 0 \tag{40}$$

Solving the above system , we have

$$\lambda = \frac{1}{30}(-c^2k^3 - 12ck\phi_0)\tau + c_0, D = \frac{c^2}{30}, \sigma = 0, C = \frac{2c}{5} \tag{41}$$

where c_0 is an integration constant, and let $c_0 = 0$. Substituting from equation (41) into equation (36), we have

$$\chi(\xi,\tau) = 1 + \exp^{\frac{1}{30}(-c^2k^3-12ck\phi_0)\tau\pm k\xi} and \phi_0(\xi,\tau) = \phi_0 \tag{42}$$

Substituting from equation (42) into equation (29), we have

$$f = c\ln(1 + \exp^{\frac{1}{30}(-c^2k^3-12ck\phi_0)\tau\pm k\xi}) \tag{43}$$

Substituting from equation (43) into equation (25), we have, the new exact soliton solution of KdV equation (23) in the form

$$\phi_1 = ck^2\left(\frac{\exp^{\frac{1}{30}(-c^2k^3-12ck\phi_0)\tau\pm k\xi}}{1+\exp^{\frac{1}{30}(-c^2k^3-12ck\phi_0)\tau\pm k\xi}} - \frac{\exp^{2(\frac{1}{30}(-c^2k^3-12ck\phi_0)\tau\pm k\xi)}}{(1+\exp^{\frac{1}{30}(-c^2k^3-12ck\phi_0)\tau\pm k\xi})^2}\right) + \phi_0 \tag{44}$$

where ϕ_0 be special (old) solution of KdV equation (23).

4. Travelling wave solutions of the KdV equation by applying the modified $\frac{\acute{G}}{G}$ expansion method, the sine-cosine expansion method, the sinh-coshine expansion method, the sech-tanh expansion method

4.1. The modified $\frac{\acute{G}}{G}$ expansion method and new travelling wave solutions of KdV equation

Suppose that NPDE is given by

$$F(\phi, \phi_\xi, \phi_\tau, \phi_{\xi\xi}, \phi_{\tau\tau}, \phi_{\xi\tau}, ...) = 0, \tag{45}$$

where $\phi = \phi(\xi, \tau)$ is unknown function and F is a polynomial in and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the modified $\frac{\acute{G}}{G}$ Expansion method:

Step 1. The travelling wave variable

$$\phi(\xi, \tau) = \phi(\chi), \chi = l\xi - \rho\tau, \tag{46}$$

where ρ and l are constants, permits us to reduce (45)into ODE;

$$F(\phi, \phi', \phi'', \phi''', ...) = 0. \tag{47}$$

Step 2. Suppose that the solution of (47) can be expressed by a polynomial $\frac{\acute{G}}{G}$ in as follows;

$$\phi(\chi) = \alpha_0 + \sum_{i=1}^{m}[\alpha_i \left(\frac{\acute{G}}{G}\right)^i + \alpha_{-i}\left(\frac{\acute{G}}{G}\right)^{-i}], \tag{48}$$

where $G = G(\xi)$ satisfies the second order linear ODE

$$G'' + \gamma G = 0, \tag{49}$$

where $\alpha_0, \alpha_i, \alpha_{-i}$ and γ are constants to be determined.

Step 3. The parameter "m" in (48) can be found by balancing the highest order derivatives term and the highest nonlinear term in (47) ;
(i)if m is a positive integer then go to step 4;
(ii)if m is not positive integer, we put $\phi = v^m$ and then return to step 1.

Step4. Substituting (48) into (47) and using (49), collecting all terms with the same powers of $\frac{\acute{G}}{G}$ together,and then equating each coefficient of the resulted polynomial to zero, yield a system of algebraic equations for $\alpha_0, \alpha_i, \alpha_{-i}$ and γ.

Step 5. Since the general solutions (48) are well known to us, then substituting $\alpha_0, \alpha_i, \alpha_{-i}$ and γ and the general solutions (48) into (47), we have the traveling wave solutions of the given NPDE.
For KdV equation (23), to transform it to ODE using the wave variable $\chi = l\xi - \rho\tau$, so these equation became

$$-\rho\phi_1' + lC\phi_1\phi_1' + l^3 D\phi_1''' = 0 \tag{50}$$

integerating these equation once and setting the constant of integration equal to zero

$$-\rho\phi_1 + \frac{lC}{2}\phi_1^2 + l^3 D\phi_1'' = 0 \tag{51}$$

$$\phi_1 = \alpha_0 + \alpha_1\left(\frac{\acute{G}}{G}\right) + \alpha_{-1}\left(\frac{\acute{G}}{G}\right)^{-1} + \alpha_2\left(\frac{\acute{G}}{G}\right)^2 + \alpha_{-2}\left(\frac{\acute{G}}{G}\right)^{-2}. \tag{52}$$

Substituting from (52) into (50), setting the coefficients of $\frac{\acute{G}}{G}$together to zero, we have the following set of over determined equations in the unknowns $\alpha_0, \alpha_1, \alpha_2, \alpha_{-1}, \alpha_{-2} and \quad l$. Solve the set of equations of coefficients of $\frac{\acute{G}}{G}$together , by using Mathematica we obtain the following solution:
I)$\alpha_0 = -\frac{4Dl^2\gamma}{C}, \alpha_1 = 0, \alpha_2 = 0, \alpha_{-1} = 0, \alpha_{-2} = -\frac{12Dl^2\gamma^2}{C}, \rho = \pm 4Dl^3\gamma$
So that, the generalized solution can be written as

$$\phi_1 = -\frac{4Dl^2\gamma}{C} - \frac{12Dl^2\gamma^2}{C}\left(\frac{c_1 cos\sqrt{\gamma}\chi + c_2 sin\sqrt{\gamma}\chi}{c_2\sqrt{\gamma}cos\sqrt{\gamma}\chi - c_1\sqrt{\gamma}sin\sqrt{\gamma}\chi}\right)^2. \tag{53}$$

II)$\alpha_2 = \frac{4\gamma\alpha_0}{3+\gamma^2}, k = 8Dl^3\gamma$
So that, the generalized solution can be written as

$$\phi_1 = \alpha_0 + \frac{4\gamma\alpha_0}{3+\gamma^2}\left(\frac{c_2\sqrt{\gamma}cos\sqrt{\gamma}\chi - c_1\sqrt{\gamma}sin\sqrt{\gamma}\chi}{c_1 cos\sqrt{\gamma}\chi + c_2 sin\sqrt{\gamma}\chi}\right)^2, \gamma = -2\left(\frac{G'}{G}\right)^2 - \sqrt{-3+4\left(\frac{G'}{G}\right)^4} \tag{54}$$

4.2. The sine-cosine expansion method and new travelling wave solutions of KdV equation

Suppose that a nonlinear partial differential equation (NPDE) is given by

$$F(\phi, \phi_\xi, \phi_\tau, \phi_{\xi\xi}, \phi_{\tau\tau}, \phi_{\xi\tau}, ...) = 0. \tag{55}$$

where $\phi = \phi(\xi, \tau)$ is unknown function and F is a polynomial in and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the Sine-Cosine expansion method:

Step 1. The traveling wave variable

$$\phi(\xi, \tau) = \phi(\chi), \chi = l\xi - \rho\tau, \tag{56}$$

where ρ and l are constants, permits us to reduce(55)into the following ordinary differential equation (ODE);

$$F(\phi, \phi', \phi'', \phi''', ...) = 0. \tag{57}$$

Step 2. Suppose that the solution of (57) can be expressed in the form as follows;

$$\phi(\chi) = A_0 + \sum_{i=1}^{n} cos^{i-1}(w(\chi))[A_i sin(w(\chi))] + B_i cos(w(\chi))], \tag{58}$$

whereA_i (i = 0, 1, ..., n) and B_i (i =1, ..., n) are constants to be determined.

Step 3. The parameter ”n” in (58) can be found by balancing the highest order derivatives term and the highest nonlinear term in (57);
(i)if n is a positive integer then go to step 4;
(ii)if n is not positive integer, we put $\phi = v^n$ and then return to step 1.

Step 4. Substitute (58) with the fixed parameter n into the obtained ODE using

$$\frac{d\omega(\chi)}{d\chi} = \gamma\sqrt{a + bsin^2\omega(\chi)}, \gamma = \pm 1, \tag{59}$$

where a and b are constants. collecting all terms with the same powers of $w'^s sin^i w cos^j w$ together. Set to zero the coefficients of $w'^s sin^i w cos^j w$(i = 0, 1; s = 0, 1; j = 0,1, 2,.,n) to get a set of algebraic equations C_{ijs} $(A_S, B_S) = 0$ with respect to the unknowns A_0, τ, A_i(i = 1, ..., n) and B_i (i=1, ..., n).

Step 5. Solve the set of algebraic equations to get A_0, A_i and B_i, then we have the traveling wave solutions of the given nonlinear equation. In The sine-cosine expansion method there are three cases of (59):

Case I
In these case a=0, b=1, then the equation reduces to the first-order ODE

$$\frac{dw(\chi)}{d\chi} = sin(w(\chi)), \tag{60}$$

which has the solutions:
$sin(w(\chi)) = sech(\chi), or cos(w(\chi)) = -tanh(\chi)$
and $sin(w(\chi)) = icsch(\chi), or cos(w(\chi)) = -coth(\chi)$,
where $i = \sqrt{-1}$

Case II
In these case $a = 1, b = -m^2$, then the equation reduces to the first-order ODE

$$\frac{dw(\chi)}{d\chi} = \pm\sqrt{1 - m^2 sin^2(w(\chi))}, \tag{61}$$

where m is the modulus of Jacobi elliptic functions, which has the solutions
$sin(w(\chi)) = sn(\chi; m), or cos(w(\chi)) = cn(\chi; m)$
and $sin(w(\chi)) = \frac{ns(\chi;m)}{m}, or cos(w(\chi)) = \frac{ids(\chi;m)}{m}$

Case III
In these case $a = m^2, b = -1$, then the equation reduces to the first-order ODE,

$$\frac{dw(\chi)}{d\chi} = \pm\sqrt{m^2 - sin^2(w(\chi))}, \tag{62}$$

which has the solutions,
$sin(w(\chi)) = msn(\chi; m), or cos(w(\chi)) = dn(\chi; m)$.
and $sin(w(\chi)) = ns(\chi; m), or cos(w(\chi)) = ics(\chi; m)$.

For KdV equation (23), to transform it to ODE using the wave variable $\chi = \rho\tau - l\xi$, so these equation became

$$-\rho\phi_1' + lC\phi_1\phi_1' + l^3 D\phi_1''' = 0 \tag{63}$$

integerating these equation once and setting the constant of integration equal to zero

$$-\rho\phi_1 + \frac{lC}{2}\phi_1^2 + l^3 D\phi_1'' = 0 \tag{64}$$

according to steps ,we know that n=2 and suppose that the solution take the form

$$\phi_1 = A_0 + A_1 sin(w(\chi)) + B_1 cos(w(\chi)) + A_2 sin(w(\chi))cos(w(\chi)) + B_2 cos^2(w(\chi)) \tag{65}$$

Substituting from (65) into (64), setting the coefficients of $sin^i w cos^j w$(i = 0, 1; j = 0, 1, ..., n) to zero, we have the following set of over determined equations in the unknowns $A_0, A_1, A_2, B_1, B_2 and\ l$. Solve the set of equations of coefficients of $sin^i w cos^j w$, by using Mathematica we obtain the following solutions:
$I)A_0 = \frac{a^2\rho}{C(a^2+ab+b^2)l} + \frac{ab\rho}{C(a^2+ab+b^2)l} + \frac{b^2\rho}{C(a^2+ab+b^2)l} + \frac{aDl^2\rho}{C\sqrt{(a^2+ab+b^2)D^2l^6}} + \frac{2bDl^2\rho}{C\sqrt{(a^2+ab+b^2)D^2l^6}}$, $A_1 = 0$, $A_2 = 0$,$B_1 = 0$,$B_2 = -\frac{3bDl^2\rho}{C\sqrt{(a^2+ab+b^2)D^2l^6}}$, $\gamma = -\frac{\sqrt{\rho}}{2(a^2D^2l^6+abD^2l^6+b^2D^2l^6)^{\frac{1}{4}}}$
So that, the generalized solution can be written as

$$\phi_1 = \frac{a^2\rho}{C(a^2+ab+b^2)l} + \frac{ab\rho}{C(a^2+ab+b^2)l} + \frac{b^2\rho}{C(a^2+ab+b^2)l} + \frac{aDl^2\rho}{C\sqrt{(a^2+ab+b^2)D^2l^6}} + \frac{2bDl^2\rho}{C\sqrt{(a^2+ab+b^2)D^2l^6}} - \frac{3bDl^2\rho}{C\sqrt{(a^2+ab+b^2)D^2l^6}}cos^2(w(\chi)). \tag{66}$$

For three cases

Case I:

$$\phi_1 = \frac{a^2\rho}{C(a^2+ab+b^2)l} + \frac{ab\rho}{C(a^2+ab+b^2)l} + \frac{b^2\rho}{C(a^2+ab+b^2)l} + \frac{aDl^2\rho}{C\sqrt{(a^2+ab+b^2)D^2l^6}} + \frac{2bDl^2\rho}{C\sqrt{(a^2+ab+b^2)D^2l^6}} + \frac{3bDl^2\rho}{C\sqrt{(a^2+ab+b^2)D^2l^6}} tanh^2(\chi) \quad (67)$$

or

$$\phi_1 = \frac{a^2\rho}{C(a^2+ab+b^2)l} + \frac{ab\rho}{C(a^2+ab+b^2)l} + \frac{b^2\rho}{C(a^2+ab+b^2)l} + \frac{aDl^2\rho}{C\sqrt{(a^2+ab+b^2)D^2l^6}} + \frac{2bDl^2\rho}{C\sqrt{(a^2+ab+b^2)D^2l^6}} + \frac{3bDl^2\rho}{C\sqrt{(a^2+ab+b^2)D^2l^6}} coth^2(\chi) \quad (68)$$

Case II:

$$\phi_1 = \frac{a^2\rho}{C(a^2+ab+b^2)l} + \frac{abr}{C(a^2+ab+b^2)l} + \frac{b^2\rho}{C(a^2+ab+b^2)l} + \frac{aDl^2r}{C\sqrt{(a^2+ab+b^2)D^2l^6}} \frac{2bDl^2\rho}{C\sqrt{(a^2+ab+b^2)D^2l^6}} - \frac{3bDl^2\rho}{C\sqrt{(a^2+ab+b^2)D^2l^6}} cn^2(\chi;m) \quad (69)$$

or

$$\phi_1 = \frac{a^2\rho}{C(a^2+ab+b^2)l} + \frac{ab\rho}{C(a^2+ab+b^2)l} + \frac{b^2\rho}{C(a^2+ab+b^2)l} + \frac{aDl^2r}{C\sqrt{(a^2+ab+b^2)D^2l^6}} + \frac{2bDl^2\rho}{C\sqrt{(a^2+ab+b^2)D^2l^6}} - \frac{3bDl^2\rho}{C\sqrt{(a^2+ab+b^2)D^2l^6}} \frac{ids^2(\chi;m)}{m} \quad (70)$$

Case III:

$$\phi_1 = \frac{a^2\rho}{C(a^2+ab+b^2)l} + \frac{ab\rho}{C(a^2+ab+b^2)l} + \frac{b^2\rho}{C(a^2+ab+b^2)l} + \frac{aDl^2r}{C\sqrt{(a^2+ab+b^2)D^2l^6}} + \frac{2bDl^2\rho}{C\sqrt{(a^2+ab+b^2)D^2l^6}} - \frac{3bDl^2\rho}{C\sqrt{(a^2+ab+b^2)D^2l^6}} dn^2(\chi;m) \quad (71)$$

or

$$\phi_1 = \frac{a^2\rho}{C(a^2+ab+b^2)l} + \frac{ab\rho}{C(a^2+ab+b^2)l} + \frac{b^2\rho}{C(a^2+ab+b^2)l} + \frac{aDl^2\rho}{C\sqrt{(a^2+ab+b^2)D^2l^6}} + \frac{2bDl^2\rho}{C\sqrt{(a^2+ab+b^2)D^2l^6}} - \frac{3bDl^2\rho}{C\sqrt{(a^2+ab+b^2)D^2l^6}} ics^2(\chi;m) \quad (72)$$

4.3. The sinh-coshine expansion method and new travelling wave solutions of KdV equation

Suppose that NPDE is given by

$$F(\phi, \phi_\xi, \phi_\tau, \phi_{\xi\xi}, \phi_{\tau\tau}, \phi_{\xi\tau}, ...) = 0, \tag{73}$$

where $\phi = \phi(\xi, \tau)$ is unknown function and F is a polynomial in and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the sinh-coshine expansion method:

Step 1. The traveling wave variable

$$\phi(\xi, \tau) = \phi(\chi), \chi = l\xi - \rho\tau, \tag{74}$$

where ρ and l are constants, permits us to reduce (73)into ODE;

$$F(\phi, \phi', \phi'', \phi''', ...) = 0. \tag{75}$$

Step 2. Suppose that the solution of (75) can be expressed in the form as follows;

$$\phi(\chi) = A_0 + \sum_{i=1}^{n} cosh^{i-1}(\chi)[A_i sinh(\chi) + B_i cosh(\chi)], \tag{76}$$

whereA_i (i = 0, 1, ..., n) and B_i (i = 1, ..., n) are constants to be determined.

Step 3. The parameter "n" in (76) can be found by balancing the highest order derivatives term and the highest nonlinear term in (75) ;
(i)if n is a positive integer then go to step 4;
(ii)if n is not positive integer, we put $\phi = v^n$ and then return to step 1.

Step 4. Substitute (76) with the fixed parameter n into the obtained ODE. Collecting all terms with the same powers of $sinh^i\chi cosh^j\chi$ together. Set to zero the coefficients of $sinh^i\chi cosh^j\chi$(i = 0, 1; j = 0, 1, 2,.,n) to get a set of algebraic equations C_{ijs} $(A_S, B_S) = 0$ with respect to the unknowns A_0, τ, A_i(i = 1, ..., n) and B_i (i = 1, ..., n).

Step 5. Solve the set of algebraic equations to get A_0, A_i and B_i then we have the traveling wave solutions of the given NPDE.

$$\phi_1 = A_0 + A_1 sinh(\chi) + B_1 cosh(\chi) + A_2 sinh(\chi)cosh(\chi) + B_2 cosh^2(\chi) \tag{77}$$

Substituting from (77) into (64), setting the coefficients of $sinh^i\chi cosh^j\chi$(i = 0, 1; j = 0, 1, ..., n) to zero, we have the following set of over determined equations in the unknowns $A_0, A_1, A_2, B_1, B_2 and\ l$. Solve the set of equations of coefficients of $sinh^i\chi cosh^j\chi$, by using Mathematica we obtain the following solution:
$A_1 = \pm\frac{\sqrt{D^2l^6-\rho^2}}{Cl}$, $A_2 =$, $B_1 =$, $B_2 = 0$
So that, the generalized solution can be written as

$$\phi_1 = A_0 \pm \frac{\sqrt{D^2l^6 - \rho^2}}{Cl} sinh(\chi), A_0 = \frac{-Cl\rho+Cl\sqrt{D^2l^6-\rho^2}sinh(\chi)+\sqrt{C^2l^2(\rho^2-2Dl^3\sqrt{D^2l^6-\rho^2}sinh(\chi))}}{C^2l^2} \tag{78}$$

4.4. The sech-tanh expansion method and new travelling wave solutions of KdV equation

Suppose that NPDE is given by

$$F(\phi, \phi_\xi, \phi_\tau, \phi_{\xi\xi}, \phi_{\tau\tau}, \phi_{\xi\tau}, ...) = 0, \tag{79}$$

where $\phi = \phi(\xi, \tau)$ is unknown function and F is a polynomial in and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the sech-tanh expansion method:

Step 1. The traveling wave variable

$$\phi(\xi, \tau) = \phi(\chi), \chi = l\xi - \rho\tau, \tag{80}$$

where ρ and l are constants, permits us to reduce (80)into ODE;

$$F(\phi, \phi', \phi'', \phi''', ...) = 0. \tag{81}$$

Step 2. Suppose that the solution of (81) can be expressed in the form as follows;

$$\phi(\chi) = A_0 + \sum_{i=1}^{n} sech^{i-1}(\chi)[A_i tanh(\chi)] + B_i sech(\chi)], \tag{82}$$

whereA_i (i = 0, 1, ..., n) and B_i (i = 1, ..., n) are constants to be determined.

Step 3. The parameter "n" in (82) can be found by balancing the highest order derivatives term and the highest nonlinear term in (81) ;
(i)if n is a positive integer then go to step 4;
(ii)if n is not positive integer,we put $\phi = v^n$ and then return to step 1.

Step 4. Substitute (82) with the fixed parameter n into the obtained ODE. Collecting all terms with the same powers of $sech^i\chi tanh^j\chi$ together. Set to zero the coefficients of $sech^i\chi tanh^j\chi$ (i = 0, 1; j = 0, 1, 2,.,n) to get a set of algebraic equations C_{ijs} $(A_S, B_S) = 0$ with respect to the unknowns A_0, τ, A_i(i = 1, ..., n) and B_i (i = 1, ..., n).

Step 5. Solve the set of algebraic equations to get A_0, A_i and B_i then we have the traveling wave solutions of the given NPDE .

$$\phi_1 = A_0 + A_1 tanh(\chi) + B_1 sech(\chi) + A_2 tanh(\chi)sech(\chi) + B_2 sech^2(\chi) \tag{83}$$

Substituting from (83) into (64), setting the coefficients of $sech^i\chi tanh^j\chi$ (i = 0, 1; j = 0, 1, ..., n) to zero, we have the following set of over determined equations in the unknowns $A_0, A_1, A_2, B_1, B_2 and\ l$. Solve the set of equations of coefficients of $sech^i\chi tanh^j\chi$, by using Mathematica we obtain the following solution:
I)$A_0 = -\frac{8Dl^2}{C}, A_1 = 0, B_1 = 0, A_2 = 0, B_2 = \frac{12Dl^2}{C}, \rho = -4Dl^3$
So that, the generalized solution can be written as

$$\phi_1 = -\frac{8Dl^2}{C} + \frac{12Dl^2}{C} sech^2(\chi) \tag{84}$$

II)$A_0 = 0, A_1 = 0, B_1 = 0, A_2 = 0, B_2 = \frac{12Dl^2}{C}, \rho = 4Dl^3$
So that, the generalized solution can be written as

$$\phi_1 = \frac{12Dl^2}{C} sech^2(\chi) \tag{85}$$

III)$A_0 = 0, A_1 = 0, B_1 = 0, A_2 = \pm\frac{6i\rho}{Al}, B_2 = \pm\frac{6\rho}{Cl}, D = \pm\frac{\rho}{l^3}$
So that, the generalized solution can be written as

$$\phi_1 = \pm\frac{6i\rho}{Cl} tanh(\chi)sech(\chi) + \pm\frac{6\rho}{Cl} sech^2(\chi) \tag{86}$$

5. Applications

Now, we shall look at three explicit physical applications of the solutions given above. We investigate Sagdeev potential ϕ_1 corresponding to the solutions of KdV equation (23).The detailed application of these solutions requires a judicious of the free parameters occurring in the solutions.

5.1. First application by using auto-Bäcklund transformation

Now on inserting the special (old) solution ϕ_0 [47,48] of the KdV equation (23) given by

$$\phi_0 = \frac{3\lambda}{C} Sech^2(\frac{\xi - \lambda\tau}{\omega}), \omega = 2\sqrt{\frac{D}{\lambda}} \tag{87}$$

The new exact soltion solution (44)of the KdV equation (23) can be given as

$$\phi_1 = ck^2 \frac{\exp^{\frac{1}{30}(-c^2k^3 - 12ck\frac{3\lambda}{C}Sech^2(\frac{\xi-\lambda\tau}{\omega}))\tau \pm k\xi}}{1 + \exp^{\frac{1}{30}(-c^2k^3 - 12ck\frac{3\lambda}{C}Sech^2(\frac{\xi-\lambda\tau}{\omega}))\tau \pm k\xi}} \left[1 - \frac{\exp^{(\frac{1}{30}(-c^2k^3 - 12ck\frac{3\lambda}{C}Sech^2(\frac{\xi-\lambda\tau}{\omega}))\tau \pm k\xi)}}{(1 + \exp^{\frac{1}{30}(-c^2k^3 - 12ck\frac{3\lambda}{C}Sech^2(\frac{\xi-\lambda\tau}{\omega}))\tau \pm k\xi)}}\right] + \frac{3\lambda}{C} Sech^2(\frac{\xi - \lambda\tau}{\omega}), \qquad c = \frac{5C}{2} \tag{88}$$

Applicable to some relevant values of C, D, λ and k, the Sagdeev potential ϕ_1 is displayed in Figures 1.

5.2. Second application by using the modified $\frac{\acute{G}}{G}$ expansion method

Now, using the backward substitution of the solution (53) through the backward transformations (46), we obtain the travelling wave solution of the KdV equation (23) in the form

$$\phi_1 = -\frac{4Dl^2\gamma}{C} - \frac{12Dl^2\gamma^2}{C}\left(\frac{c_1 cos\sqrt{\gamma}(l\xi - \rho\tau) + c_2 sin\sqrt{\gamma}(l\xi - \rho\tau)}{c_2\sqrt{\gamma}cos\sqrt{\gamma}(l\xi - \rho\tau) - c_1\sqrt{\gamma}sin\sqrt{\gamma}(l\xi - \rho\tau)}\right)^2. \tag{89}$$

Applicable to some relevant values of C,D, l, ρ, c_1, c_2, and γ, the Sagdeev potential ϕ_1 is displayed in Figures 2.

5.3. Third application by using the sech-tanh expansion method

Now, using the backward substitution of the solution (85) through the backward transformation (80), we obtain the travelling wave solution of the KdV equation (23) in the form

$$\phi_1 = \frac{12Dl^2}{C} sech^2(l\xi - \rho\tau), \rho = 4Dl^3. \tag{90}$$

Applicable to some relevant values of A,,l,and ρ, the Sagdeev potential ϕ_1 is displayed in Figures 3

From the above travelling wave solution (90) of KdV equation (23), the amplitude take the form $\frac{12Dl^2}{C}$ and $\rho = 4Dl^3$. This solution also stands for n_1. It should be noted here that the perturbation method, which is only valid for small but finite amplitude limit, is not valid for large propagation angle θ, which makes the wave amplitude large enough to break the condition $1 > \epsilon n_1$. For $\rho = 4Dl^3 > 0$, there exist solitary waves with positive density only. It is seen that as ρ increases, the amplitude increases, from equation (24) since $C = \frac{-\lambda}{-\mu + 3\mu^2 T + \mu^2\lambda^2}$, the amplitude decreases for $\lambda < 0$ and the amplitude increases for $\lambda > 0$.

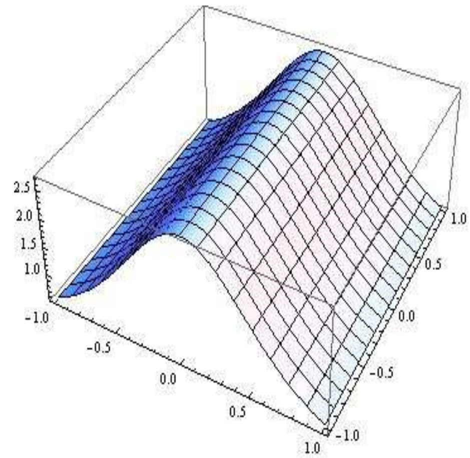

Fig.1.1. The Sagdeev potential ϕ_1 (88) in the surface graphic for $C = 1.2$, $D = 0.1$, $k = 0.5$, $\lambda = 1$.

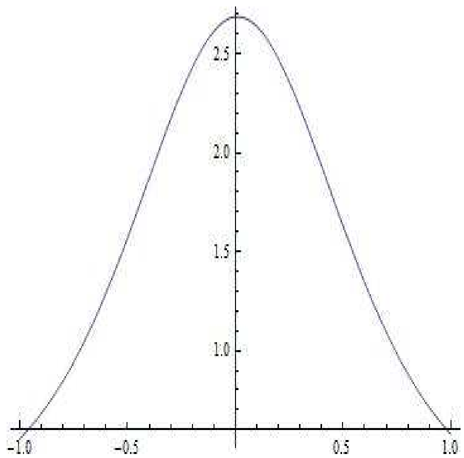

Fig.1.2. The Sagdeev potential ϕ_1 (88) in the plane graphic at $\tau = 0.01$ for $C = 1.2$, $D = 0.1$, $k = 0.5$, $\lambda = 1$.

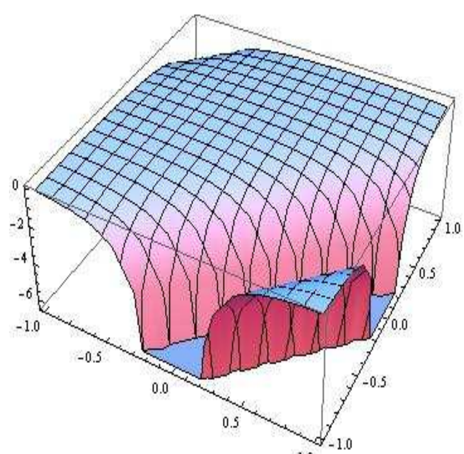

Fig.2.1. The Sagdeev potential ϕ_1 (89) in the surface graphic for $C = 2.1$, $D = 0.9$, $l = 0.4$, $c_1 = 2$, $c_2 = 3$, $\rho = 1$, $\gamma = 1$.

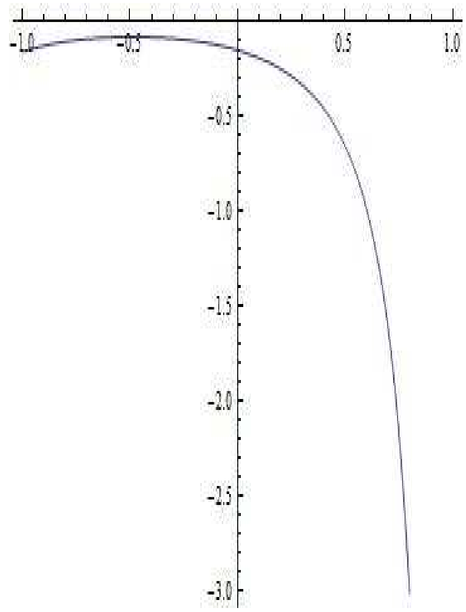

Fig.2.2. The Sagdeev potential ϕ_1 (89) in the plane graphic at $\tau = 0.1$ for $C = 2.1$, $D = 0.9$, $l = 0.4$, $c_1 = 2$, $c_2 = 3$, $\rho = 1$, $\gamma = 1$.

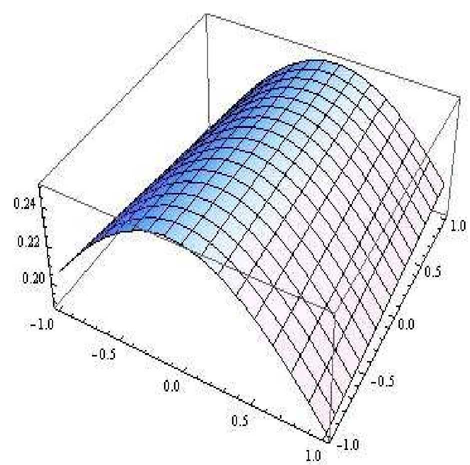

Fig.3.1. The Sagdeev potential ϕ_1 (90) in the surface graphic for $C = 1.2$, $D = 0.1$, $l = 0.5$.

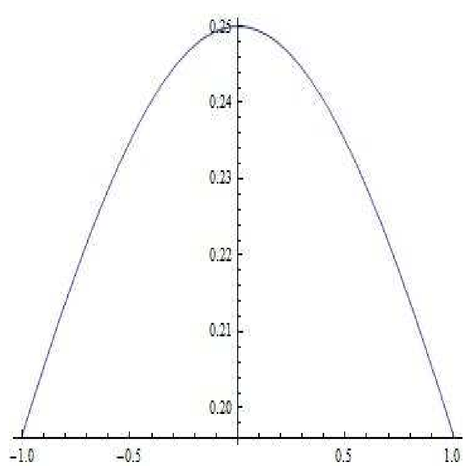

Fig.3.2. The Sagdeev potential ϕ_1 (90) in the plane graphic at $\tau = 0.01$ for $C = 1.2$, $D = 0.1$, $l = 0.5$.

6. Conclusions

In this Work, We have investigated the propagation of nonlinear dust-acoustic waves in unmagnetized dusty plasma with Boltzmann distributed electrons and nonthermal distributed ions considering variable dust charge. By using the reductive perturbation technique, a KdV equation with a nonlinearity proportional to three-half power of the wave potential is derived to investigate the nonlinear propagation of dust acoustic solitary waves. The stationary new soliton solution and travelling wave solutions of the KdV equation are derived. The auto-Bäcklund (BT) transformations, the modified $\frac{\acute{G}}{G}$ expansion method, the sine-cosine expansion method, sinh-coshine expansion method and the sech-tanh expansion method have been successfully applied to find the Sagdeev potential ϕ_1 for KdV equation which described unmagnetized plasma with Boltzmann distributed and nonthermal distributed ions.

References

[1] S. K. El-Labany , and W. F. El-Taibany, "Dust acoustic solitary waves and double layers in a dusty plasma with an arbitrary streaming ion beam", *Phys. Plasmas*, vol.10, (2003), pp.989-998.

[2] N. N. Rao, P. K. Shukla, and M. Y. Yu, "Dust-acoust waves in dusty plasmas", *Planet. Space Sci.*, Vol.38,(1990), pp.543-546.

[3] A. Barkan, R. L. Merlino, and N. D'Angelo, "Laboratory observation of the dust-acoustic wave mode", *Phys. Plasmas*, Vol.2, (1995), pp.3563-3565.

[4] P.K. Shukla, and A. A. Mamun," Introduction to Dusty Plasma Physics", *Institute of Physics, Bristol,UK,* (2002).

[5] R. L. Merlino, A. Barkan, C. Thompson, and and N. D'Angeloet,"Experiments on waves and instabilities in dusty plasmas ", *Plasma Phys. Control. Fusion*, Vol.39, (1997), pp. A421-429.

[6] N. N. Rao, "Dust-acoustic KDV Solitons in Weakly Non-ideal Dusty Plasmas", *Physica Scripta*, Vol.75, (1998), pp.1179-1181.

[7] J. K. Xue, and L. P. Zhang, "Non-linear waves in complex plasma", *Chaos, Solitons and Fractals*, Vol.32, (2007), pp. 592-597.

[8] H. Alinejad, A. A. Mamun, "Dust-acoustic solitary waves in a strongly coupled inhomogeneous dusty plasma in the presence of polarization force and effective dust-temperature", *Phys. Plasmas*, Vol.18, (2011), pp.073706-073712

[9] P. K. Shukla, and A. A. Mamun,"Solitons, shocks and vortices in dusty plasmas", *New J. Phys.*, Vol.5, (2003),pp.17-37.

[10] S. Bagchi, A. P. Misra, K. R. Chowdhury,and A. R. Chowdhury,"Nonlinear oscillations in a magnetized dusty plasma with two-temperature trapped ions", *Chaos, Solitons and Fractals*, Vol.40, (2009), pp.758-765.

[11] B. Das, and P. Chatterjee," Large amplitude double layers in dusty plasma with non-thermal electrons and two temperature isothermal ions" *Phys. Lett. A*, Vol.373, (2009),pp.1144-1147.

[12] B. S. Xie, K. F. He, and Z. Q. Huang Z Q, "Dust-acoustic solitary waves and double layers in dusty plasma with variable dust charge and two-temperature ions ", *Phys. Plasmas*, Vol.6, (1999), pp.3808-3817.

[13] R. Z. Sagdeev, "Cooperative Phenomena and Shock Waves in Collisionless Plasmas", *Reviews of Plasma Physics*, Vol. 4, 1966, pp. 23-91.

[14] M.M. Lin, W. S. Duan, "Dust acoustic solitary waves in a dusty plasmas with nonthermal ions", *Chaos,Solitos and Fractals*,Vol.33, (2007), pp.1189-1196.

[15] M. H. M. Moussa and M. Rehab, "Auto-Bäcklund transformation and similarity reductions to the variable coefficients variant Boussinesq system" *Phys Lett A.*, Vol.372(9),(2008), pp. 1429-1434.

[16] M.H. M. Moussa, and M. Rehab, "Two applications of the homogeneous balance method for solving the generalized Hirota-Satsume Coupled KdV system with variable coefficient", *International journal of nonlinear science*, Vol.7, (2009), pp. 29-38.

[17] T. S. Gill,N. S. Saini and H. Kaur, "The Kadomstev-Petviashvili equation in dust plasma with variable dust charge and two temperature ions", *Chaos, Soliton and Fractals*, Vol. 28, (2006), pp. 1106-1111.

[18] M. M. Lin and W.S.Duan, "The Kadomstev-Petviashvili (KP), MKP, and coupled KP equations for two-ion-temperature dusty plasmas", *Chaos, Soliton and Fractals*, Vol. 23,(2005), pp. 929-937.

[19] Y. Q. Yang, Y.H. Wang, Lixin and X. P. Cheng, "Riccati-type Bäcklund transformations of nonisospectral and generalized variable-coefficient KdV equations", *Chin. Phys. B*, Vol.23(3),(2014) 030506-030508.

[20] A. H. Khater, R.S. Ibrahim , A. B. Shamardan and D. K. Callebaut, "Bäcklund transformations and Painlevé analysis : Exact solutions for a Grad-Shafranov type Magnetohydrodynamic Equilibrium', *IMA J. Applied Math.*, Vol. 58,(1997), pp.51-69.

[21] A. H. Khater, R.S. Ibrahim , O. H. El-Kalaawy and D. K. Callebaut, "Bäcklund Transformations and Exact Soliton Solutions for Some Nonlinear Evolution equations of ZS/AKNS system", *Chaos Solitons and Fractals*, Vol.9, (1998), pp.1847-1855.

[22] A. Khater, D. Callebaut and R. S. Ibrahim, "Bäcklund transformations and Painlevé analysis:exact solutions for the unstable nonlinear Schödinger equation modeling electron beam plasma", *Phys. Plasmas*,Vol.5(2), (1998), pp. 395-400.

[23] O. H. EL-Kalaawy and R. B. Aldenari,"Painlevé analysis, Auto-Bäcklund transformation and new exact solutions for improved modifed KdV equation", *International Journal of Applied Mathematical Research*, Vol.3(3), (2014), pp. 265-272.

[24] A. R.Shehata , "The traveling wave solutions of the perturbed nonlinear Schrödinger equation and the cubic-quintic Ginzbury Landau equation using the modified $(\frac{\acute{G}}{G})$ expansion method", *Applied Mathematics and Computation*, Vol. 217,(2010) 1-10.

[25] S.Khaleghizadeh and M.A.Mirzazadeh, "Exact complex solutions of the generalized Zakharov equation", *World Applied Sciences Journal*, Vol.21,(Special Issue of Applied Math) (2013), pp. 39-45.

[26] Sh.Javadi, M. Miri Karbasaki and M. Jani, "Traveling wave solutions of a biological reaction-convection-diffusion equation model by using $\frac{\acute{G}}{G}$ expansion method", *Communications in Numerical Analysis*, Vol.2013,(2013), pp.1-5.

[27] A. Bekir and özkan Güner," Exact solutions of nonlinear fractional differential equations by $(\frac{\acute{G}}{G})$-expansion method", *Chin. Phys. B*, Vol.22(11), (2013) 110202-110207.

[28] E.M.E.Zayed and M.A.S. EL-Malky , "The generalized $(\frac{\acute{G}}{G})$ - expansion method for solving nonlinear paratial differential equations in mathematical physics", *Int. J. Contemp. Math. Sciences*, Vol.6,(2011), pp. 263-278.

[29] Harun-Or-Roshid, "N .Rahman and M.A.Akbar, "Traveling wave solutions of nonlinear Klein-Gordon equation by extended $(\frac{\acute{G}}{G})$ -expansion method", *Annals of Pure and Applied Mathematics*, Vol.3,(2013), pp. 10-16.

[30] A.I.Singh , I.Tomba and K.S. Bhamra, "Solutions of Burger type nonlinear partial differential equations using $\frac{\acute{G}}{G}$ method, *International Journal of Advance Research, IJOAR. org.*, Vol.3,(2013),pp. 1-6.

[31] Z. Ayati and J. Biazar, "Application of $\frac{\acute{G}}{G}$-expansion method to two concert problems", *International Journal of Applied Mathematical Research*, Vol.2(1),(2013), pp. 49-54.

[32] Z. T. Fu, S. K. Liu and S. D. Liu, "New transformations and new approach to find exact solutions to nonlinear equations", *Phys.Lett.A*, Vol. 299, (2002), pp. 507-512.

[33] Z.Yan , "A new sine-Gordan equation expansion algorithm to investigate some special nonlinear differential equations", *MMRC, AMSS, Academia Sinica*, Vol. 23,(2004), pp.300-310.

[34] L.Z.Chang , Y.T.Pan and X.M.Ma,"New Exact traveling wave solutions of Davey-Stewartson equation", *Journal of Computational Information Systems*, Vol. 4,(2013), pp. 1687-1693.

[35] L.Zhong and S.Tang, "Exact explicit traveling solutions for a (2+1)-dimensional generalized KP-BBME equation", *British Journal of Mathematics and Computer Science*, Vol. 3,(2013), pp. 655-663.

[36] R. S. Ibrahim and O. H. El-Kalaawy, "Traveling Wave Solution for a Modified Kortewegde Vries Equation Modeling to Nonlinear Ion-Acoustic Waves in a Collisionless Plasma", *Adv. Studies Theor. Phys.*, Vol. 2,(2008), pp.919 - 928.

[37] A.J. Jawad, "Soliton solutions for the Boussinesq equations", *J. Math. Comput. Sci.*,Vol.3, (2013), pp.254-265.

[38] A. R.Seadawy and A. Sayed, "Traveling wave solution of two dimensional nonlinear KdV-Burgers equation", *Applied Mathematical Sciences*, Vol.7,(2013), pp. 3367-3377.

[39] R. S. Ibrahim "Tanh-travelling wave solutions and reduction of variable-coefficient unstable nonlinear Schr?dinger equation to a quadrature in electron beam plasma", *Applied Mathematical Sciences*, Vol.2, (2008),pp.2157-216.

[40] R. S. Ibrahim and O. H. El-Kalaawy "Extended tanh-function method and reduction of nonlinear Schrodinger-type equation to a quadrature", *Chaos Solitons and Fractals*, Vol.31, (2007), pp.1001-1008.

[41] K. S. Al-Ghafri, "Analytic solutions of the Thomas equation by generalized tanh and travelling wave hypothesis methods",International *Journal of Applied Mathematical Research*, Vol.2(2),(2013), pp.274-278

[42] A. A. Mamun, S. M. Russell, C. A. Mendoza-Briceno, M. N. Alam MN, Datta TK, A. K. Das, "Multi-dimensional instability of electrostatic solitary structures in magnetized nonthermal dusty plasmas", *Planet Space Sci*, Vol. 48, (2000), pp. 163-173.

[43] R. A. Cairns, A. A. Mamum, R. Bingham, R. Bostrom, R. O. Dendy, C. M. C. Nairn and P. K. Shukla,"Electrostatic solitary structures in nonthermal plasmas" *Geophys Res Lett.*, Vol.22, (1995), pp. 2709-2712.

[44] A. A. Mamun, "Effects of ion temperature on electrostatic solitary structures in nonthermal plasmas" *Phys. Rev. E*, Vol.55,(1997), pp. 1852-1857.

[45] W. S. Daun, "The Kadomtsev-Petviashvili (KP) equation of dust acoustic waves for hot dust plasmas", *Chaos, Solitons and Fractals*, Vol.14, (2002), pp.503-506.

[46] R. Hirota, "Exact solution of the Korteweg de Vries equation for multiple collisions of solitons", *Phy. Rev. Lett.*, Vol. 27,(1971), pp.1192-1194.

[47] P. G. Drazin, and R. S. Johnson, *Solitons: an Introduction, Cambridge Univ. Press, Cambridge*, (1989).

[48] M. Remoissenet, Waves Called Solitons: Concepts and experiments, *Springer-Verlag, Berlin*, (1996).