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***Research paper***

# Approximating common fixed points on multiplicative b-metric spaces using the weakly commuting mappings

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**Abstract**

The aim of this paper is to prove some common fixed point theorems for four pair of weakly commuting mappings using multiplicative b-metric spaces. Our results obtained in this paper improve, extend and unify some related results in the literature.

***Keywords:*** *Weakly commutative mappings; multiplicative b-metric space; common fixed points.*

## 1. Introduction

The major solution of nonlinear problem of applied mathematics can be change into one of the form of find the solution of nonlinear operator equations (e.g. nonlinear integral equations, boundary value problem for nonlinear ordinary and partial differential equations) which can be created in terms of, a point which remains unchanged under the given nonlinear operator of infinite dimensional space into itself, this point is called fixed points throughout thesis the set of fixed point is denoted $F(T) = u \in \mathscr{X} : Tu = u$. The study of fixed point and common fixed point has been a subject of great interest since Banach [1] proved the Banach contraction principle in 1922.
In the past year many authors generalized the Banach contraction principle in various space as symmetric spaces, partial metric space, cone metric space etc.
In 1976, Jungck [4] used the notion of commuting mappings to prove the existence of a common fixed point theorems on a metric space $(\mathscr{X}, \rho)$. Many authors have invested various concept of commuting maps, like weakly commuting maps in 2008. Bashirov [2] introduced the notion of multiplicative metric space and studied the concept of multiplication calculus and proved the fundamental theorem of multiplicative calculus.
In 2012, Ozavsar et al.[6] investigated the multiplicative metric space by remarking its topological properties and introduced the concept of contraction mapping and some fixed-point theorem of multiplicative, contraction mappings on multiplicative metric space. They recently proved a common fixed-point theorem for four self-mappings in multiplicative metric space. Kang[5] introduced the notion of compatible mappings and its various types in multiplicative metric space and proved some common fixed-point theorem for these mappings in his paper. We present some Definition and result in common fixed-point theorem for commuting mappings in complete multiplicative b-metric space. For, we have introduced the notion of b-metric in multiplicative metric space.

## 2. Preliminaries

**Definition 2.1.** [3] Let $\mathscr{X}$ be a nonempty set. A multiplicative metric is a function $\rho : \mathscr{X} \times \mathscr{X} \to \mathscr{R}^+$ satisfying the following conditions:
(i) $\rho(u,v) \geq 1$ for all $u, v \in \mathscr{X}$ and $\rho(u,v) = 1$ if and only if $u = v$;
(ii) $\rho(u,v) = \rho(v,u)$ for all $u, v \in \mathscr{X}$;
(iii) $\rho(u,v) \leq \rho(u,w)\rho(w,v)$ for all $u, v \in \mathscr{X}$,(multiplicative triangle inequality).

We use the following Definition for our main result:

**Definition 2.2.** Let $X$ be a nonempty set. A multiplicative b-metric is a mapping $\rho : \mathscr{X} \times \mathscr{X} \to R^+$ satisfying the following conditions:
$[B1]$ $\rho(u,v) \geq 1$ for all $u, v \in \mathscr{X}$ and $\rho(u,v) = 1$ if and only if $u = v$;
$[B2]$ $\rho(u,v) = \rho(v,u)$ for all $u, v \in \mathscr{X}$;
$[B3]$ $\rho(u,v) \leq b[\rho(u,w).\rho(w,v)]$ for all $u, v, w \in \mathscr{X}$ (multiplicative triangle inequality). where $b \geq 1$.

**Definition 2.3.** [3] Let $(\mathscr{X},\rho)$ be a multiplicative metric space, $\{u_n\}$ be a sequence in $\mathscr{X}$ and $u \in \mathscr{X}$. If for every multiplicative open ball $B_\varepsilon(u) = \{v \mid \rho(u,v) < \varepsilon\}$, $\varepsilon > 1$, there exists a natural number $\mathscr{N}$ such that $n \geq \mathscr{N}$ , then $u_n \in B(u)$. The sequence $\{u_n\}$ is said to be multiplicative converging to $u$, denoted by $u_n \to u$ $(n \to \infty)$.

**Definition 2.4.** [3] Let $(\mathscr{X},\rho)$ be a multiplicative metric space and $\{u_n\}$ be a sequence in $\mathscr{X}$. The sequence is called a multiplicative Cauchy sequence if it holds that for all $\varepsilon > 1$ , there exists $N \in \mathscr{N}$ such that $\rho(u_n,u_m) < \varepsilon$ for all $m,n > \mathscr{N}$.

**Definition 2.5.** [3] We call a multiplicative metric space complete if every multiplicative Cauchy sequence in it is multiplicative convergence to $u \in \mathscr{X}$.

**Definition 2.6.** [3] Suppose that $S,T$ are two self-mappings of a multiplicative metric space $(\mathscr{X},\rho)$; $S,T$ are called commutative mappings if it holds that for all $u \in \mathscr{X}$, $STu = TSu$.

**Definition 2.7.** [3] Suppose that $S,T$ are two self-mappings of a multiplicative metric space $(\mathscr{X},\rho)$; $S,T$ are called weak commutative mappings if it holds that for all $u \in \mathscr{X}$, $\rho(STu,TSu) \leq \rho(Su,Tu)$.

**Definition 2.8.** [3] Let $(\mathscr{X},\rho)$ be a multiplicative metric space. A mapping $f : \mathscr{X} \to \mathscr{X}$ is called a multiplicative contraction if there exists a real constant $\lambda \in [0,1)$ such that $\rho(f(u_1), f(u_2)) \leq \rho(u_1,u_2)^\lambda$ for all $u,v \in \mathscr{X}$.

**Proposition 2.9.** [6] Let $(\mathscr{X},\rho)$ be a multiplicative metric space, $\{u_n\}$ be a sequence in $\mathscr{X}$ and $u \in \mathscr{X}$. Then $\{u_n\} \to u$ $(n \to \infty)$ if and only if $\rho(u_n,u) \to 1$ $(n \to \infty)$.

**Proposition 2.10.** [6] Let $(\mathscr{X},\rho)$ be a multiplicative metric space, $\{u_n\}$ be a sequence in X and $u \in \mathscr{X}$. Then $\{u_n\}$ is a multiplicative cauchy sequence if and only if $\rho(u_n,u_m) \to 1$ $(n,m \to \infty)$.

**Proposition 2.11.** [6] Let $(\mathscr{X},\rho_u)$ be a multiplicative metric space, $\{u_n\}$ and $\{v_n\}$ be two sequences in $\mathscr{X}$ such that $u_n \to u, v_n \to v$ $(n \to \infty)$, $u,v \in \mathscr{X}$. Then $\rho(u_n,v_n) \to \rho(u,v)$ $(n \to \infty)$ .

## 3. Main Results

In this section we will present fixed point theorem for contractive mapping in the setting of multiplicative b-metric spaces:

**Theorem 3.1.** *Let $S,T,A$ and $B$ be self-mappings of a complete multiplicative metric space $\mathscr{X}$; which satisfy the following conditions:*
*(i) $S\mathscr{X} \subset B\mathscr{X}, T\mathscr{X} \subset A\mathscr{X}$;*
*(ii) A and S are weak commutative, B and T also are weak commutative;*
*(iii) One of S, T , A and B is continuous;*
*(iv) $\rho(Su,Tv) \leq [k\{max\{\rho(Au,Bv),\rho(Au,Su),\rho(Bv,Tv),\rho(Su,Bv),\rho(Au,Tv),\rho^*(Au,Bv),\rho^*(Su,Tv)\}\}]^\lambda$,*
*where $\lambda \in (0,\frac{1}{2})$ $\forall u,v \in \mathscr{X}$,*

*where $\rho^*(Au,Bv)=min\{1,\rho(Au,Bv)\}$, $\rho^*(Su,Tv)=min\{1,\rho(Su,Tv)\}$.*

*Then $S,T,A$ and $B$ have a unique common fixed point.*
*Where $b \geq 1$ such that $\lim_{m,n\to\infty}(kb)^{\frac{h}{1-h}^{(m-n)}} = 1$.*

**Proof:** Since $S\mathscr{X} \subset B\mathscr{X}$, and $T(\mathscr{X}) \subset A\mathscr{X}$, for an arbitrary chosen point $u_0$ in $\mathscr{X}$ we obtain $u_1$ in $\mathscr{X}$. For this $u_1 \in \mathscr{X}$, we may obtain $u_2 \in \mathscr{X}$; etc. Continuing in this way we obtain a sequence $\{v_n\} \in \mathscr{X}$,
$\exists u_2 \in \mathscr{X}$ such that $Tu_1 = Au_2 = v_1,\ldots$ ;
$\exists u_{2n+1} \in \mathscr{X}$ such that $Bu_{2n+1} = v_{2n}$,
$\exists u_{2n+2} \in \mathscr{X}$ such that $Tu_{2n+1} = Au_{2n+2} = v_{2n+1},\ldots$ ; $\forall n = 0,1,2....\infty$.
define a sequence $\{v_n\} \in \mathscr{X}$. Now
putting $u = u_{2n}$, $v = u_{2n+1}$ in condition (iv) we obtain
In order to show $\{v_n\}$ a Cauchy sequence, let us put $u_{2n}$ for x, and $u_{2n+1}$ for y in condition (iv), and using (1) we have;

$$\begin{aligned}
\rho(v_{2n},v_{2n+1}) &= \rho(Su_{2n},Tu_{2n+1})\\
&\leq \{k\max\{\rho(Au_{2n},Bu_{2n+1}),\rho(Au_{2n},Su_{2n}),\rho(Bu_{2n+1},Tu_{2n+1}),\rho(Su_{2n},Bu_{2n+1}),\\
&\rho(Au_{2n},Tu_{2n+1}),\rho^*(Au_{2n},Bu_{2n+1}),\rho^*(Su_{2n},Tu_{2n+1})\}\}^\lambda\\
&\leq \{k\max\{\rho(v_{2n-1},v_{2n}),\rho(v_{2n-1},v_{2n}),\rho(v_{2n},v_{2n+1}),\rho(v_{2n},v_{2n}),\rho(v_{2n-1},v_{2n+1}),\\
&\rho^*(v_{2n-1},v_{2n}),\rho^*(v_{2n},v_{2n+1})\}\}^\lambda\\
&\leq \{k\max\{b\rho(v_{2n-1},v_{2n}).\rho(v_{2n},v_{2n+1}),1,b\rho(v_{2n-1},v_{2n}).\rho(v_{2n},v_{2n+1}),b\rho^*(v_{2n-1},v_{2n}).\\
&\rho^*(v_{2n},v_{2n+1}),b\rho^*(v_{2n-1},v_{2n}).\rho^*(v_{2n},v_{2n+1}\}\}^\lambda\\
&= [k(max\{b\rho(v_{2n-1},v_{2n})\ .\ \rho(v_{2n},v_{2n+1})\})]^\lambda,\ (using\ \ B1, as\ \ \rho(u,v) \geq 1 \forall u \in \mathscr{X})\\
&= k^\lambda b^\lambda[\rho(v_{2n-1},v_{2n})]^\lambda[\rho(v_{2n},v_{2n+1})]^\lambda.
\end{aligned}$$

$\implies \rho^{1-\lambda}(v_{2n},v_{2n+1}) \leq k^\lambda b^\lambda\ .\ \rho^\lambda(v_{2n-1},v_{2n})$
$\implies \rho(v_{2n},v_{2n+1}) \leq (kb)^{\frac{\lambda}{1-\lambda}}\rho^{\frac{\lambda}{1-\lambda}}(v_{2n-1},v_{2n})$.

Let $\frac{\lambda}{1-\lambda} = h,\ where\ \lambda \in (0, \frac{1}{2})\ then$
$\rho(v_{2n}, v_{2n+1}) \leq (kb)^h \rho^h(v_{2n-1}, v_{2n}).$

Similarly, by putting $u = u_{2n+2},\ v = u_{2n+1}$ on (iv), we may obtain

$$\rho(v_{2n}, v_{2n+1}) \leq (kb)^h.\rho^h(v_{2n-1}, v_{2n}), \tag{1}$$

$$\rho(v_{2n+1}, v_{2n+2}) \leq ((kb)^h.\rho^h(v_{2n}, v_{2n+1}). \tag{2}$$

From (1) and (2), we obtain $\rho(v_n, v_{n+1}) \leq (kb)^h \rho^h(v_{n-1}, v_n),\ n = 1, 2, 3, ...$ which inductively implies that

$$\begin{aligned}
\rho(v_n, v_{n+1}) &\leq (kb)^h[(kb)^h \rho^h(v_{n-2}, v_{n-1})]^h \\
&= (kb)^{h+h^2}[\rho^{h^2}(v_{n-2}, v_{n-1})] \\
&\leq (kb)^{h+h^2}[(kb)^h \rho^h(v_{n-3}, v_{n-2})]^{h^2} \\
&= (kb)^{h+h^2+h^3}[\rho^{h^3}(v_{n-3}, v_{n-2})] \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\leq (kb)^{h+h^2+h^3+...+h^n}[\rho^{h^n}(v_0, v_1)] \\
&\leq (kb)^{\frac{h}{1-h}}[\rho^{h^n}(v_0, v_1)],\ h + h^2 + h^3 + ... + h^n \leq \frac{h}{1-h}.
\end{aligned}$$

Let $m, n \in \mathscr{N}$ such that $m \geq n$, then for Cauchy sequence, we have

$$\begin{aligned}
\rho(v_m, v_n) &\leq \rho(v_m, v_{m-1}).\rho(v_{m-1}, v_{m-2})...\rho(v_{n+1}, v_n) \\
&\leq (kb)^{\frac{h}{1-h}} \rho^{h^{m-1}}(v_0, v_1).(kb)^{\frac{h}{1-h}} \rho^{h^{m-2}}(v_0, v_1)...(kb)^{\frac{h}{1-h}} \rho^{h^n}(v_0, v_1)] \\
&\leq \{(kb)^{\frac{h}{1-h}}\}^{(m-n)}\{\rho^{h^{[(m-1)+(m-2)+...+n]}}(v_0, v_1)\} \\
&= \{(kb)^{\frac{h}{1-h}}\}^{(m-n)}\{\rho^{h^{(m-n)[(m-1)-\frac{1}{2}(m-n-1)]}}(v_0, v_1)\} \\
&\leq \{(kb)^{\frac{h}{1-h}}\}^{(m-n)} \rho^{h^{m(m-n)}}(v_0, v_1),\ since\ (m-1) + (m-2) + ... + n \leq m(m-n) \\
&\ where\ m > n, \\
&= \mathscr{B} \rho^{h^{m(m-n)}}(v_0, v_1),\ where\ \mathscr{B} = \{(kb)^{\frac{h}{1-h}}\}^{(m-n)}
\end{aligned}$$

This implies that $\rho(v_m, v_n) \to 1\ as\ m, n \to \infty$. Hence $\{v_n\}$ is a multiplicative Cauchy sequence in $\mathscr{X}$.
By the completeness of $\mathscr{X}$, there exists $w \in \mathscr{X}$ such that $v_n \to w\ as\ n \to \infty$.
Moreover, since

$$\{Su_{2n}\} = \{Bu_{2n+1}\} = \{v_{2n}\}\ and\ \{Tu_{2n+1}\} = \{Au_{2n+2}\} = \{v_{2n+1}\},$$

are subsequence of $\{v_n\}$, so we obtain

$$\lim_{n\to\infty} Su_{2n} = \lim_{n\to\infty} Bu_{2n+1} = \lim_{n\to\infty} Tu_{2n+1} = \lim_{n\to\infty} Au_{2n+2} = w.$$

Taking condition (ii) and (iii) we obtain following cases;
**Case 1:** Suppose that $A$ is continuous then

$$\lim_{n\to\infty} ASu_{2n} = \lim_{n\to\infty} A^2 u_{2n} = Aw.$$

Since $A$ and $S$ are weakly commuting , then

$$\rho(ASu_{2n}, SAu_{2n}) \leq \rho(Su_{2n}, Au_{2n}).$$

Let $n \to \infty$, we get $\lim_{n\to\infty} \rho(SAu_{2n}, Aw) \leq \rho(w, w) = 1,\ i.e., \lim_{n\to\infty} SAu_{2n} = Aw,$

putting $Au_{2n}$ and $u_{2n+1}$, respectively for x and y in condition (iv) of Theorem 3.1, and using the continuity of A, we respectively obtain,

$$\begin{aligned}
\rho(SAu_{2n}, Tu_{2n+1}) \leq [k\{max\{&\rho(A^2u_{2n}, Bu_{2n+1}), \rho(A^2u_{2n}, SAu_{2n}), \rho(Bu_{2n+1}, Tu_{2n+1}), \\
&\rho(SAu_{2n}, Bu_{2n+1}), \rho(A^2u_{2n}, Tu_{2n+1}), \rho^*(A^2u_{2n}, Bu_{2n+1}), \rho^*(SAu_{2n}, Tu_{2n+1}\}\}]^\lambda.
\end{aligned}$$

Let $n \to \infty$, we can obtain

$$\begin{aligned}
\rho(Aw, w) &\leq [k\{max\{\rho(Aw, w), \rho(Aw, Aw), \rho(w, w), \rho(Aw, w), \rho(Aw, w), \rho^*(Aw, w), \rho^*(Aw, w)\}\}]^\lambda \\
&= [k\{max\{\rho(Aw, w), 1\}\}]^\lambda \\
&= k^\lambda \rho^\lambda(Aw, w).
\end{aligned}$$

This implies that $\rho(Aw,w) = 1, i.e., Aw = w$.

Putting $u = w$ and $v = u_{2n+1}$, we obtain,

$$\rho(Sw,Tu_{2n+1}) \leq [k\{max\{\rho(Aw,Bu_{2n+1}),\rho(Aw,Sw),\rho(Bu_{2n+1},Tu_{2n+1}),\rho(Sw,Bu_{2n+1}),\rho(Aw,Tu_{2n+1}),$$
$$\rho^*(A^2z,Bu_{2n+1}),\rho^*(SAu_{2n},Tu_{2n+1})\}\}]^{\lambda}.$$

Let $n \to \infty$ we can obtain

$$\begin{aligned}\rho(Sw,w) &\leq [k\{max\{\rho(Aw,w),\rho(w,Sw),\rho(w,w),\rho(Sw,w),\rho(w,w),\rho^*(Aw,w),\rho^*(Aw,w)\}\}]^{\lambda}\\ &= [k\{max\{\rho(Sw,w),1\}\}]^{\lambda}\\ &= k^{\lambda}\rho^{\lambda}(Sw,w),\end{aligned}$$

which implies that $\rho(Sw,w) = 1$, i.e., $Sw = w$,
$w = Sw \in Su \subseteq B\mathscr{X}$ , so $\exists w^* \in \mathscr{X}$ such that $w = Bw^*$

$$\begin{aligned}\rho(w,Tw^*) &= \rho(Sw,Tw^*)\\ &\leq [k\{max\{\rho(Aw,Bw^*),\rho(Aw,Sw),\rho(Bw^*,Tw^*),\rho(Sw,Bw^*),\rho(Aw,Tw^*),\\ &\rho^*(Aw,Bw^*),\rho(Aw,Sw),\rho^*(Bw^*,Tw^*)\}\}^{\lambda}\\ &= [k\{max\{\rho(w,Tw^*),1\}\}]^{\lambda}\\ &= k^{\lambda}\rho^{\lambda}(w,Tw^*),\end{aligned}$$

which implies $\rho(Sw,w) = 1$ i.e., $Tw^* = w$.
Since $B$ and $T$ are weakly commuting mappings then

$$\rho(Bw,Tw) = \rho(BTw^*,TBw^*) \leq \rho(Bw^*,Tw^*) = \rho(w,w) = 1,$$

so $Bw = Tw$, w is a fixed point of T. For, we have on using condition (iv)

$$\begin{aligned}\rho(w,Tw) &= \rho(Sw,Tw)\\ &\leq [k\{max\{\rho(Aw,Bw),\rho(Aw,Sw),\rho(Bw,Sw),\rho(Sw,Bw),\rho(Aw,Tw),\\ &\rho^*(Aw,Bw),\rho^*(Sw,Tw)\}\}]^{\lambda}\\ &= [k\{max\{\rho(w,Tw),1\}\}]^{\lambda}\\ &= k^{\lambda}\rho^{\lambda}(w,Tw),\end{aligned}$$

which implies $\rho(Tw,w) = 1$ i.e., $Tw = w$.

**Case 2:** Suppose that $B$ is continuous , we can obtain the same result by the way of case 1.

**Case 3:** Suppose that $S$ is continuous then $\lim_{n\to\infty} SAu_{2n} = \lim_{n\to\infty} S^2u_{2n} = Sw$.
Since $A$ and $S$ are weak commutative then $\rho(ASu_{2n},SAu_{2n}) \leq \rho(Su_{2n},Au_{2n})$.
Let $n \to \infty$ then $\lim_{n\to\infty}(ASu_{2n},Sw) \leq \rho(w,w) = 1,\ i.e.,\ \lim_{n\to\infty} ASu_{2n} = Sw$,

$$\rho(S^2u_{2n},Tu_{2n+1}) \leq [k\{max\{\rho(ASu_{2n},Bu_{2n+1}),\rho(ASu_{2n},S^2u_{2n}),\rho(Bu_{2n+1},Tu_{2n+1}),$$
$$\rho(S^2u_{2n},Bu_{2n+1}),\rho(ASu_{2n},Tu_{2n+1}),\rho^*(ASu_{2n},Bu_{2n+1}),\rho^*(S^2u_{2n},Tu_{2n+1})\}\}]^{\lambda}.$$

Let $n \to \infty$ we can obtain

$$\begin{aligned}\rho(Sw,w) &\leq [k\{max\{\rho(Sw,w),\rho(Sw,Sw),\rho(w,w),\rho(Sw,w),\rho(Sw,w),\rho^*(Sw,w),\rho^*(Sw,Sw)\}\}]^{\lambda}\\ &= [k\{max\{\rho(Sw,w),1\}\}]^{\lambda}\\ &= k^{\lambda}\rho^{\lambda}(Sw,w),\end{aligned}$$

which implies $\rho(Sw,w)$ = 1 i.e., $Sw = w$. Now,
$w = Sw \in Su \subseteq B\mathscr{X}$, *so* $\exists w^* \in \mathscr{X}$ such that $w = Bw^*$

$$\rho(S^2u_{2n},Tw^*) \leq [k\{max\{\rho(ASu_{2n},Bw^*),\rho(ASu_{2n},S^2u_{2n}),\rho(Bw^*,Tw^*),$$
$$\rho(S^2u_{2n},Bw^*),\rho(ASu_{2n},Tw^*),\rho^*(ASu_{2n},Bw^*),\rho^*(S^2u_{2n},Tw^*)\}\}]^{\lambda},$$

Letting $n \to \infty$ using $w = Sw = Bw^*$, we can obtain

$$\begin{aligned}\rho(w,Tw^*) &= \rho(Sw,Tw^*)\\ &\leq [k\{max\{\rho(Sw,w),\rho(Sw,Sw),\rho(w,Tw^*),\rho(Sw,w),\rho(Sw,Tw^*),\rho^*(Sw,w),\rho^*(Sw,Tw^*))\}\}]^{\lambda}\\ &= [k\{max\{d(w,Tw^*),1\}\}]^{\lambda}\\ &= k^{\lambda}\rho^{\lambda}(w,Tw^*),\end{aligned}$$

which implies that $\rho(w,Tw^*) = 1$, *i.e.*, $Tw^* = w$.
Since $T$ and $B$ are weak commutative, then
$\rho(Tw,Bw) = \rho(TBw^*,BTw^*) \leq \rho(Tw^*,Bw^*) = \rho(w,w) = 1$, so $Bw = Tw$,
w is fixed point of $T$. We have on using Condition (iv),

$$\rho(Su_{2n},Tw) \leq [k\{max\{\rho(Au_{2n},Bw),\rho(Au_{2n},Su_{2n}),\rho(Bw,Tw),\rho(Su_{2n},Bw),$$
$$\rho(Au_{2n},Tw),\rho^*(Au_{2n},Bw),\rho^*(Su_{2n},Tw)\}\}]^{\lambda}.$$

Let $n \to \infty$ we can obtain

$$\begin{aligned}\rho(w,Tw) &\leq [k\{max\{\rho(w,Tw),\rho(w,w),\rho(Tw,Tw),\rho(w,Tw),\rho(w,Tw),\rho^*(w,Tw),\rho^*(w,Tw)\}\}]^{\lambda}\\ &= [k\{max\{\rho(w,Tw),1\}\}]^{\lambda}\\ &= k^{\lambda}\rho^{\lambda}(w,Tw).\end{aligned}$$

which implies $\rho(w,Tw) = 1$ i.e., $Tw = w$.
$w = Tw \in Tu \subseteq A\mathscr{X}$, so $\exists w^{**} \in \mathscr{X}$ , such that $w = Aw^{**}$

$$\begin{aligned}\rho(Sw^{**},w) &= \rho(Sw^{**},Tw)\\ &\leq [k\{max\{\rho(Aw^{**},Bw),\rho(Aw^{**},Sw^{**}),\rho(Bw,Tw),\rho(Sw^{**},Bw),\\ &\rho(Aw^{**},Tw),\rho^*(Aw^{**},Bw),\rho^*(Sw^{**},Tw)\}\}]^{\lambda}\\ &= [k\{max\{\rho(w,w),\rho(w,Sw^{**}),\rho(Bw,Bw),\rho(Sw^{**},w),\\ &\rho(w,w),\rho^*(w,w),\rho^*(Sw^{**},Tw)\}\}]^{\lambda}\\ &= [k\{max\{d(Sw^{**},w),1\}\}]^{\lambda}\\ &= k^{\lambda}\rho^{\lambda}(Sw^{**},w).\end{aligned}$$

This implies that $\rho(Sw^{**},w) = 1$ i.e., $Sw^{**} = w$.
Since $S$ and $A$ are weak commutative, then
$\rho(Aw,Sw) = \rho(ASw^{**},SAw^{**}) \leq \rho(Aw^{**},Sw^{**}) = \rho(w,w) = 1$, so $Aw = Sw$.
We obtain $Sw = Tw = Aw = Bw = w$, so w is common fixed point of $S$, $T$, $A$ and $B$.

**Case 4:** Suppose that $T$ is continuous, we can obtain the same result by the way of case 3.
In addition we prove that $S$, $T$, $A$ and $B$ have a unique common fixed point. suppose that $w \in \mathscr{X}$ is also a common fixed point of $S$, $T$, $A$ and $B$, then we obtain

$$\begin{aligned}\rho(w,w) &= \rho(Sw,Tw)\\ &\leq [k\{max\{\rho(Aw,Bw),\rho(Aw,Sw),\rho(Bw,Tw),\rho(Sw,Bw),\rho(Aw,Tw),\rho^*(Aw,Bw),\\ &\rho^*(Sw,Tw)\}\}]^{\lambda}\\ &= [k\{max\{\rho(w,w),1\}]^{\lambda}\\ &= [k\{max\{\rho(w,w),1\}]^{\lambda}\\ &= k^{\lambda}\rho^{\lambda}(w,w).\end{aligned}$$

This is a contradiction as $\rho(w,w) > 1$, when $w \neq w$.
Thus w is a unique common fixed point of $A,B,S,T \subset \mathscr{X}$.

**Corollary 3.2.** *Let $(\mathscr{X},\rho)$ be a complete multiplicative b-metric space $S,T,A$ and $B$ be four mappings of $\mathscr{X}$ into itself. Suppose that there exists $\lambda \in (0,\frac{1}{2})$ $\forall u,v \in \mathscr{X}$, such that $S(u) \subset B(u), T(u) \subset A(u)$ and*

$$\rho(S^p u,T^q v) \leq k^{\lambda}\{max\{\rho^{\lambda}(Au,Bv),\rho^{\lambda}(Au,S^p u),\rho^{\lambda}(Bv,T^q v),\rho^{\lambda}(S^p u,Bv),\rho^{\lambda}(Au,T^q v)\}\},$$

*Assume one of the following conditions is satisfied:*
*(a) either A or S is continuous the pair S,A and the pair $T,B$ are commuting mappings;*
*(b) either A, B, S or T is continuous;*
*Then $S,T,A$ and B have a unique common fixed point*
*where $b \geq 1$ such that $\lim_{m,n\to\infty}(kb)^{\frac{h}{1-h}^{(m-n)}} = 1$.*

**Corollary 3.3.** *Let $(\mathscr{X},\rho)$ be a complete multiplicative b-metric space $S,T,A$ and $B$ be four mappings of $\mathscr{X}$ into itself. Suppose that there exists $\lambda \in (0,\frac{1}{2})$ $\forall u,v \in \mathscr{X}$, such that $S(u) \subset B(u), T(u) \subset A(u)$ and*

$$\rho(S^p u,T^q v) \leq k^{\lambda}\{max\{\rho^{\lambda}(Au,Bv)+\rho^{\lambda}(Au,S^p u)+\rho^{\lambda}(Bv,T^q v)+\rho^{\lambda}(S^p u,Bv)+\rho^{\lambda}(Au,T^q v)\}\}$$

*for all $u,v \in \mathscr{X}$. Here $a_1,a_2,a_3,a_4,a_5 \geq 0$ and $0 \leq a_1+a_2+a_3+a_4+a_5 \leq 1$ Assume one of the following conditions is satisfied:*
*(a) either A or S is continuous the pair S,A and the pair $T,B$ are commuting mappings;*
*(b) either A, B, S or T is continuous;*
*Then $S,T,A$ and B have a unique common fixed point.*

**Corollary 3.4.** *Let $(\mathscr{X},\rho)$ be a complete multiplicative b-metric space $S,T,A$ and $B$ be four mappings of $\mathscr{X}$ into itself. Suppose that there exists $\lambda \in (0,\frac{1}{2})$ and $p,q \in w^{+}$*
$\rho(T^pu,T^qv) \in k^{\lambda}\{max\,(\rho^{\lambda}(u,v),\rho^{\lambda}(u,T^pu),\rho^{\lambda}(v,T^qv),\rho^{\lambda}(T^pu,v),\rho^{\lambda}(u,T^qv)\}\})$
*for all $u,v \in \mathscr{X}$. Then T have a unique fixed point.*

**Corollary 3.5.** *Let $(\mathscr{X},\rho)$ be a complete multiplicative b-metric space $S,T,A$ and $B$ be four mappings of $\mathscr{X}$ into itself. Suppose that there exists $\lambda \in (0,\frac{1}{2})$ such that*
$\rho(Tu,Tv) \leq k^{\lambda}\{max\,(a_1\rho^{\lambda}(u,v)+a_2\rho^{\lambda}(u,Tu)+a_3\rho^{\lambda}(v,Tv)+a_4\rho^{\lambda}(Tu,v)+a_5\rho^{\lambda}(u,Tv))\}\}$
*for all $u,v \in \mathscr{X}$. Here $a_1,a_2,a_3,a_4,a_5 \geq 0$ and $0 \leq a_1+a_2+a_3+a_4+a_5 \leq 1$.
Then T have a unique fixed point.*

# References


[1] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamental mathematics*, **3(1)(1922), 133–181**.
[2] A.E. Bashirov, E.M. Kurplnara, and A. Ozyaplcl, Multiplicative calculus and its applications, *J. Math. Anal. Appl*, **337(2008), 36-48**.
[3] X. He, M. song and D. Chen, Common fixed points for weak commutative mappings on a multiplicative metric space, *Fixed point theory and applications*, **2014(1)(2014), 1-9**.
[4] G. Jungck, Commuting mappings and fixed points, *Amer. Math. Monthly*, **83(1)(1976), 261–263**.
[5] S.M. Kang, P. Kumar, S. Kumar, P. Nagpal and S.K. Garg, Common fixed points for compatible mappings and its variants in multiplicative metric spaces, *Int. J. Pure Appl. Math*, **102(2)(2015), 383-406**.
[6] M. Ozavsar, and A.C. Cevikel, Fixed point of multiplicative contraction mappings on multiplicative metric space, arXiv:1205.5131v1 *[math.GN]*, **(2012)**.