International Journal of Applied Mathematical Research, 12 (1) (2023) 3-10

International Journal of Applied Mathematical Research

Website: www.sciencepubco.com/index.php/IJAMR

Research paper

Proximal Point Algorithm for Nonexpansive Mappings in Hadamard Spaces Based on SRJ Iteration Process

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Abstract

In this paper, We provide a new modified proximal point approach utilizing fixed point iterates of nonexpansive mappings in Hadamard space and show that the sequence created by our iterative process converges to a minimizer of a convex function and a fixed point of mappings. Finally, we present a numerical illustration for supporting our main result. Our results obtained in this paper improve, extend and unify results of Khan-Abbas [23], Cholamjiak et al. [10] and Dashputre et al. [11].

Keywords: *Proximal point algorithm; nonexpansive mappings; CAT(0) space; strong and Δ-convergence.*

1. Introduction

Let ϑ be a nonempty subset of a metric space (X,d) and $\phi: \vartheta \to \vartheta$ be a nonlinear mapping. The fixed point set of ϕ is denoted by $\digamma(\phi)$, that is, $\digamma(\phi) = \{x \in \vartheta : x = \phi x\}$.
Kirk [24] pioneered the study of fixed point theory in a CAT(0) space. Since then, there has been a lot of interest in fixed point theory for various types of mappings in CAT(0) spaces. In 2008, Dhompongsa and Panyanak [12] studied the convergence of nonexpansive mappings in CAT(0) spaces. Several writers then examined the convergence of nonexpansive mappings using various iteration approaches.
Recently, Dashputre at al. [11] used the SRJ iteration process to generate novel fixed point solutions in the setting of CAT(0) spaces, and they also used a numerical example to understand the effectiveness of the new three step iteration procedure. The SRJ iteration procedure is as follows:
Let ϑ be a nonempty, closed and convex subset of a complete CAT(0) space X and $\phi: \vartheta \to \vartheta$ be a mapping. Let $x_1 \in \vartheta$ be arbitrary and the sequence $\{x_n\}$ generated iteratively by

$$\begin{aligned} & x_1 \in \vartheta \\ & z_n = \phi((1-\alpha_n)x_n \oplus \alpha_n \phi x_n) \\ & y_n = \phi((1-\beta_n)z_n \oplus \beta_n \phi z_n) \\ & x_{n+1} = \phi((1-\gamma_n)y_n \oplus \gamma_n \phi y_n), n \geq 1 \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in (0,1).
As an example, [16, 26, 27] provides some intriguing findings for fixing a nonlinear mappings problem in the setting of CAT(0) spaces.
Let (X,d) be a metric space and $\acute{f}: X \to (-\infty, \infty]$ be a proper and convex function. One of the major problems in optimization is to find $x \in X$ such that

$$\acute{f}(x) = \min_{y \in X} \acute{f}(y).$$

The set of minimizers of f is denoted by $\text{argmin}_{y \in X} \acute{f}(y)$. Martinet [30] invented the well-known proximal point algorithm (also known as the PPA) in 1970, and it has shown to be an effective and strong strategy for addressing this problem. Rockafellar [34] investigated the convergence to a solution of the convex minimization problem in the setting of Hilbert spaces using PPA in 1976.
Indeed, let $\acute{f}$ be a proper, convex, and lower semi-continuous(lsc) function on a Hilbert space $\mathbb{H}$ that reaches its minimum. The PPA is defined by $x_1 \in \mathbb{H}$ and

$$X_{n+1} = arg \min_{y \in \mathbb{H}} \left(\acute{f}(y) + \frac{1}{2\mu_n} ||y - x_n||^2 \right),$$

for each $n \in \mathbb{N}$, where $\mu_n > 0$. It was proved that the sequence $\{x_n\}$ converges weakly to a minimizer of $\acute{f}$ provided $\sum_{n=1}^{\infty} \mu_n = \infty$. However, as Güler [19] has demonstrated, the PPA does not always converges strongly in general. In the year 2000, Kamimura and Takahashi [22] combined the PPA and Halpern's algorithm [20] to ensure strong convergence.
In 2013, Bacák [5] presented the PPA in a CAT(0) space (X,d), as follows: $x_1 \in X$ and

$$X_{n+1} = arg \min_{y \in \mathbb{H}} \left(\acute{f}(y) + \frac{1}{2\mu_n} d^2(y, x_n) \right),$$

for each $n \in \mathbb{N}$, where $\mu_n > 0$. It was proved that if $\acute{f}$ has a minimizer and $\sum_{n=1}^{\infty} \mu_n = \infty$, then the sequence $\{x_n\}$ Δ-converges to its minimizer based on the Fejer monotonicity idea. In 2014, Bacák [4] minimised a sum of convex functions using a split version of the PPA in complete CAT(0) spaces.
Many PPA convergence approaches have recently been extended to the setting of manifolds from traditional linear spaces such as Euclidean spaces, Hilbert spaces, and Banach spaces for tackling optimization issues [15, 28, 31, 33, 37]. Minimizers of the objective convex functional in nonlinear spaces play an important role in analysis and geometry. Many applications in computer vision, machine learning, electrical structure computation, system balancing, and robot manipulation can be thought of as addressing optimization problems on manifolds [1, 35, 36].
We provide a modified proximal point approach for two nonexpansive mappings in Hadamard spaces utilising the SRJ-type iteration process, and illustrate various convergence outcomes of the proposed process under several moderate conditions based on previous work. Our main findings extend Dashputre at el. [11] discovery from one nonexpansive mapping to two nonexpansive mappings in Hadamard spaces involving the convex and lower semi-continuous functions.

2. Preliminaries

This section contains some well-known concepts and results that will be referenced throughout the paper. Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a mapping K from a closed interval $[0,r] \subset \mathbb{R}$ to X such that

$$c(0) = x,\ c(r) = y,\ d(c(t), c(s)) = |t - s|$$

for all $s,t \in [0,r]$. In particular, K is an isometry and $d(x,y) = r$. The image of K is call a geodesic segment (or metric segment) joining x and y. When it is unique, this geodesic is denoted by $[x,y]$. We denote the point $w \in [x,y]$ such that $d(x,w) = \alpha d(x,y)$ by $w = (1-\alpha)x \oplus \alpha y$, where $\alpha \in [0,1]$.
The space (X,d) is called a geodesic space if any two points of X are joined by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x,y \in X$. A subset $D \subseteq X$ is said to be convex if D includes geodesic segment joining every two points of itself. A geodesic triangle $\Delta(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consist of three points (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for geodesic triangle (or $\Delta(x_1,x_2,x_3)$) in (X,d) is a triangle $\bar{\Delta}(x_1,x_2,x_3) = \Delta(\bar{x_1},\bar{x_2},\bar{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that

$$d_{\mathbb{R}^2}(\bar{x_i},\bar{x_j}) = d(x_i,x_j)$$

for $i,j \in \{1,2,3\}$. A geodesic metric space is said to be a CAT(0) space if all geodesic triangle of appropriate size satisfy the following CAT(0) comparison axiom:
Let Δ be a geodesic triangle in C and let $\bar{\Delta} \subset \mathbb{R}^2$ be comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) inequality if for all $x,y \in \Delta$ and all comparison points $\bar{x},\bar{y} \in \bar{\Delta}$,

$$d(x,y) \leq d_{\mathbb{R}^2}(\bar{x},\bar{y}).$$

If x,y_1,y_2 are points of a CAT(0) space and y_0 is the midpoint of the segment $[y_1,y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the CAT(0) inequality impels

$$d^2(x, \frac{y_1 \oplus y_2}{2}) \leq \frac{1}{2}d^2(x,y_1) + \frac{1}{2}d^2(x,y_2) - \frac{1}{4}d^2(y_1,y_2).$$

this inequality is the (CN) inequality of Bruhat and Tits [8]. In fact, a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.
It is well known that all complete, simply combined Riemannian manifold having non-positive section curvature is a CAT(0) space. For other examples, Euclidean buildings [7], Pre-Hilbert spaces, $\mathbb{R}$-trees [6], the complex Hilbert ball with a hyperbolic metric ([17]) is a CAT(0) space. Further, complete CAT(0) spaces are called Hadamard spaces.

Lemma 2.1. *[6] Let X be a CAT(0) space, $x,y,z \in X$ and $t \in [0,1]$. Then*

$$d(tx \oplus (1-t)y, z) \leq td(x,z) + (1-t)d(y,z).$$

Lemma 2.2. *[6] Let X be a CAT(0) space, $x,y,z \in X$ and $t \in [0,1]$. Then*

$$d^2(tx \oplus (1-t)y, z) \leq (1-t)d^2(x,z) + td^2(y,z) - t(1-t)d^2(x,y).$$

Remember that a function $\acute{f}\colon \vartheta \to (-\infty,\infty]$ defined on a convex subset ϑ of a CAT(0) space is convex if the function $\acute{f} o \Psi$ is convex for any geodesic $\Psi\colon [a,b] \to \vartheta$. We say that a function on ϑ is lower semi-continuous at a point $x \in \vartheta$ if

$$\acute{f}(x) \leq \liminf_{n\to\infty} \acute{f}(x_n),$$

for each sequence $x_n \to x$. A function $\acute{f}$ is said to be lower semi-continuous on ϑ if it is lower semi-continuous at any point in ϑ.
For any $\mu > 0$, define the Moreau-Yosida resolvent of $\acute{f}$ in CAT(0) spaces as

$$J_\mu = arg \min_{y \in X} \left(\acute{f}(y) + \frac{1}{2\mu_n} d^2(y,x) \right),$$

for all $x \in X$. The mapping J_μ is well defined for all $\mu > 0$ (see [18,29]).

Lemma 2.3. *[3] Let $\acute{f}: X \to (-\infty,\infty]$ be a proper, convex and lsc function, where (X,d) is a Hadamard space. Then the set $F(J_\mu)$ of fixed points of the resolvent associated with $\acute{f}$ coincides with the set $\arg\min_{y\in X} \acute{f}(y)$ of minimizers of $\acute{f}$.*

Definition 2.4. A self map ϕ defined on a nonempty subset ϑ of a Hadamard space is said to be nonexpansive if

$$d(\phi x, \phi y) \leq d(x,y),$$

for all $x, y \in \vartheta$.

Lemma 2.5. *[25] For any $\mu > 0$, the resolvent J_μ of $\acute{f}$ is nonexpansive.*

Lemma 2.6. *[2] Let $\acute{f}: X \to (-\infty,\infty]$ be a proper, convex and lsc function, where (X,d) is a Hadamard space. Then $x, y \in X$ and $\mu > 0$, we have*

$$\frac{1}{2\mu}d^2(J_\mu x, y) - \frac{1}{2\mu}d^2(x,y) + \frac{1}{2\mu}d^2(x, J_\mu x) + \acute{f}(J_\mu x) \leq \acute{f}(y).$$

Lim [29] first proposed the concept of -convergence in a broad metric space in 1976. Kirk and Panyanak [25] extended Lim's approach to CAT(0) spaces in 2008 and demonstrated that it is analogous to the weak convergence in the Banach space setting. Since the concept of Δ-convergence has received a lot of attention. We will now define Δ-convergence and list some of its fundamental features.
Let $\{x_n\}$ be a bounded sequence in X, Hadamard spaces. For $x \in X$ set:

$$r(x,\{x_n\}) = \lim_{n\to\infty} sup\, d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ is given by

$$r(\{x_n\}) = inf\{r(x, x_n) : x \in \vartheta\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined as:

$$A(\{x_n\}) = \{x \in \vartheta : r(x, x_n) = r(\{x_n\})\}.$$

Remark 2.7. The cardinality of the set $A(\{x_n\})$ in any CAT(0) space is always equal to one, (see e.g., [12]).
The ([12], Proposition 2.1) tells us that in the setting of Hadamard spaces, for every bounded sequence, namely, $\{x_n\} \subset \vartheta$, the set $A(\{x_n\})$ is essentially the subset of ϑ provided that ϑ is convex and bounded. It is well-known that $\{x_n\}$ has a subsequence which Δ-converges to some point provided that the sequence is bounded.

Definition 2.8. [25] A sequence $\{x_n\}$ in Hadamard space is said to be Δ-converges to $x \in \vartheta$ if x is the unique asymptotic center for every subsequence $\{a_n\}$ of $\{x_n\}$. In this case we write $\Delta - lim_n x_n = x$ and read as x is the $\Delta - limit$ of $\{x_n\}$.

Notice that a bounded sequence $\{x_n\}$ in a Hadamard space is known as regular if and only if for every subsequence, namely, $\{a_n\}$ of $\{x_n\}$ one has $r(\{x_n\}) = r\{a_n\}$. It is wellknown that, in the setting of Hadamard spaces each regular sequence Δ-converges and consequently each bounded sequence has a Δ-convergent subsequence.

Lemma 2.9. *[25] Every bounded sequence in a Hadamard space admits a Δ-convergent subsequence.*

Lemma 2.10. *[13] Let X be a Hadamard space, ϑ be closed convex subset of X. If $\{x_n\}$ is a bounded sequence in ϑ, then the asymptotic center of $\{x_n\}$ is in ϑ.*

Lemma 2.11. *[12] Let ϑ be a closed and convex subset of a Hadamard space X and ϕ be a nonexpansive self mapping on ϑ. Let $\{x_n\}$ be a bounded sequence in ϑ such that $\lim_{n\to\infty} d(x_n, \phi x_n) = 0$ and $\Delta - \lim_{n\to\infty} x_n = x$. Then $x = \phi x$.*

Lemma 2.12. *[12] If $\{x_n\}$ is a bounded sequence in a Hadamard space with $A(\{x_n\}) = \{x\}$, $\{a_n\}$ is a subsequence of $\{x_n\}$ with $A(\{a_n\}) = \{a\}$ and the sequence $\{d(x_n, a)\}$ converges, then $x = a$.*

Lemma 2.13. *(The resolvent identity, [21]). Let (X,d) be a Hadamard space and $\acute{f}: X \to (-\infty,\infty]$ be proper convex and lower semi-continuous. Then, the following identity holds:*

$$J_\mu x = J_\eta \left(\frac{\mu-\eta}{\mu} J_\mu x \oplus \frac{\eta}{\mu} x\right),$$

for all $x \in X$ and $\mu > \eta > 0$.

3. Main result

Theorem 3.1. *Consider $\acute{f}: X \to (-\infty,\infty]$ is a proper, convex and lsc function, where (X,d) is a Hadamard space. Let ϕ_1, ϕ_2 be nonexpansive self maps defined on X such that $\Theta = F(\phi_1) \cap F(\phi_2) \cap \arg\min_{y\in X} \acute{f}(y) \neq \emptyset$ Consider $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences for all $n \in \mathbb{N}$ and for some $\alpha, \beta, \gamma \in (0,1)$ and $\{\mu_n\}$ is a sequence such that $\mu_n \geq \mu > 0$ for all $n \in \mathbb{N}$ and for some μ. Let $\{x_n\}$ be generated in the following manner:*

$$\begin{aligned} u_n &= \arg\min_{y\in X}\left(\acute{f}(y) + \frac{1}{2\mu_n} d^2(y, x_n)\right), \\ z_n &= \phi_1((1-\alpha_n)x_n \oplus \alpha_n \phi_1 u_n), \\ y_n &= \phi_2((1-\beta_n)z_n \oplus \beta_n \phi_2 z_n), \\ x_{n+1} &= \phi_2((1-\gamma_n)y_n \oplus \gamma_n \phi_1 y_n), \end{aligned} \tag{3.1}$$

for each $n \in \mathbb{N}$. Then, we have the following:
(i) $\lim_{n\to\infty} d(x_n, q)$ exists for all $q \in \Theta$;
(ii) $\lim_{n\to\infty} d(x_n, u_n) = 0$;
(iii) $\lim_{n\to\infty} d(x_n, \phi_1 x_n) = \lim_{n\to\infty} d(x_n, \phi_2 x_n) = 0$.

Proof. Let $q \in \Theta$. Then $q = \phi_1 q = \phi_2 q$ and $\acute{f}(q) \leq \acute{f}(y)$ for all $y \in X$. It follows that

$$\acute{f}(q) + \frac{1}{2\mu_n} d^2(q,q) \leq \acute{f}(y) + \frac{1}{2\mu_n} d^2(y,q),$$

for all $y \in X$ and hence $q = J_{\mu_n} q$ for all $n \in \mathbb{N}$.
(i) First, we prove that $\lim_{n\to\infty} d(x_n,q)$ exists. Writing $u_n = J_{\mu_n}$ for all $n \in \mathbb{N}$. Using Lemma 2.5, we have

$$d(u_n,q) = d(J_{\mu_n} x_n, J_{\mu_n} q) \leq d(x_n,q).$$

Also, by Definition 2.4, Lemma 2.1 and (3.1), we get

$$\begin{aligned} d(z_n,p) &= d(\phi_1((1-\alpha_n)x_n \oplus \alpha_n \phi_1 u_n),q) \\ &\leq ((1-\alpha_n))d(x_n,q) + \alpha_n d(\phi_1 u_n,q) \\ &\leq (1-\alpha_n)d(x_n,q) + \alpha_n d(u_n,q) \\ &\leq (1-\alpha_n)d(x_n,q) + \alpha_n d(x_n,q) \\ &\leq d(x_n,q). \end{aligned} \tag{3.2}$$

By Definition 2.4, Lemma 2.1 and (3.1), (3.2), we get

$$\begin{aligned} d(y_n,p) &= d(\phi_2((1-\beta_n)z_n \oplus \beta_n \phi_2 z_n),q) \\ &\leq (1-\beta_n)d(z_n,q) + \beta_n d(\phi_2 z_n,q) \\ &\leq (1-\beta_n)d(z_n,q) + \beta_n d(z_n,q) \\ &\leq (1-\beta_n)d(x_n,q) + \beta_n d(x_n,q) \\ &\leq d(x_n,q). \end{aligned} \tag{3.3}$$

By Definition 2.4, Lemma 2.1 and (3.1), (3.2), (3.3), we get

$$\begin{aligned} d(x_{n+1},p) &= d(\phi_2((1-\gamma_n)y_n \oplus \gamma_n \phi_1 y_n),q) \\ &\leq (1-\gamma_n)d(y_n,q) + \gamma_n d(\phi_1 y_n,q) \\ &\leq (1-\gamma_n)d(y_n,q) + \gamma_n d(y_n,q) \\ &\leq (1-\gamma_n)d(x_n,q) + \gamma_n d(x_n,q) \\ &\leq d(x_n,q). \end{aligned} \tag{3.4}$$

Hence $\lim_{n\to\infty} d(x_n,q)$ exists and $\lim_{n\to\infty} d(x_n,q) = w$ for some w.
(ii) Now we prove $\lim_{n\to\infty} d(x_n,u_n) = 0$. Using Lemma 2.3, we see that

$$\begin{aligned} &\frac{1}{2\mu_n} d^2(J_{\mu_n}(x_n),q) - \frac{1}{2\mu_n} d^2(x_n,q) + \frac{1}{2\mu_n} d^2(x_n, J_{\mu_n}(x_n)) + \acute{f}(J_{\mu_n}(x_n)) \leq \acute{f}(q), \\ &\frac{1}{2\mu_n} d^2(u_n,q) - \frac{1}{2\mu_n} d^2(x_n,q) + \frac{1}{2\mu_n} d^2(x_n,u_n) + \acute{f}(u_n) \leq \acute{f}(q), \\ &\frac{1}{2\mu_n} d^2(u_n,q) - \frac{1}{2\mu_n} d^2(x_n,q) + \frac{1}{2\mu_n} d^2(x_n,u_n) \leq \acute{f}(q) - \acute{f}(u_n). \end{aligned}$$

But $\acute{f}(q) \leq \acute{f}(u_n)$ $\forall n \in \mathbb{N}$, hence

$$\begin{aligned} &d^2(u_n,q) - d^2(x_n,q) + d^2(x_n,u_n) \leq 0, \\ &d^2(x_n,u_n) \leq d^2(x_n,q) - d^2(u_n,q). \end{aligned}$$

To prove $\lim_{n\to\infty} d(x_n,u_n) = 0$, suppose that $\lim_{n\to\infty} d(u_n,q) = w$ for $w > 0$. Now,

$$d(x_{n+1},q) \leq d(y_n,q).$$

So, we have

$$w = \liminf_{n\to\infty} d(x_n,q) = \liminf_{n\to\infty} d(x_{n+1},q) \leq \liminf_{n\to\infty} d(y_n,q),$$

and also,

$$\limsup_{n\to\infty} d(y_n,q) \leq \limsup_{n\to\infty} d(x_n,q) = w.$$

Thus,

$$\lim_{n\to\infty} d(y_n,q) = w$$

and

$$\begin{aligned} &d(z_n,q) \leq (1-\alpha_n)d(x_n,q) + \alpha_n d(u_n,q) \\ &d(x_n,q) \leq \frac{1}{\alpha}[d(x_n,q) - d(z_n,q)] + d(u_n,q), \end{aligned}$$

It gives that

$$w = \liminf_{n\to\infty} d(x_n,q) \leq \liminf_{n\to\infty} d(u_n,q).$$

Also,

$$\limsup_{n\to\infty} d(u_n,q) \le w.$$

It shows that

$$\lim_{n\to\infty} d(x_n,u_n) = 0.$$

(iii) To show

$$\lim_{n\to\infty} d(x_n,\phi_1 x_n) = \lim_{n\to\infty} d(x_n,\phi_2 x_n) = 0.$$

We observe that

$$\begin{aligned} d^2(z_n,p) &= d^2(\phi_1((1-\alpha_n)x_n \oplus \alpha_n \phi_1 u_n),q) \\ &\le (1-\alpha_n)d^2(x_n,q) + \alpha_n d^2(\phi_1 u_n,q) - (1-\alpha_n)\alpha_n d^2(x_n,\phi_1 u_n) \\ &\le d^2(x_n,q) - \alpha(1-\beta)d^2(x_n,\phi_1 u_n), \end{aligned}$$

$$\begin{aligned} d^2(x_n,\phi_1 u_n) &\le \frac{1}{\alpha(1-\beta)}(d^2(x_n,q) - d^2(z_n,q)) \\ &\to 0 \;\; as \;\; n \to \infty. \end{aligned}$$

Hence

$$\lim_{n\to\infty} d(x_n,\phi_1 u_n) = 0.$$

It follows that

$$\begin{aligned} d(x_n,\phi_1 x_n) &\le d(x_n,\phi_1 u_n) + d(\phi_1 u_n,\phi_1 x_n) \\ &\to 0 \;\; as \;\; n \to \infty. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} d^2(y_n,p) &= d^2(\phi_2((1-\beta_n)z_n \oplus \beta_n \phi_2 z_n),q) \\ &\le (1-\beta_n)d^2(z_n,q) + \beta_n d^2(\phi_2 z_n,q) - (1-\beta_n)\beta_n d^2(z_n,\phi_1 u_n) \\ &\le d^2(z_n,q) - \alpha(1-\beta)d^2(z_n,\phi_1 z_n), \end{aligned}$$

$$\begin{aligned} d^2(z_n,\phi_2 z_n) &\le \frac{1}{\alpha(1-\beta)}(d^2(x_n,q) - d^2(y_n,q)) \\ &\to 0 \;\; as \;\; n \to \infty. \end{aligned}$$

This implies

$$\lim_{n\to\infty} d(\phi_1 x_n \phi_2 z_n) = 0.$$

Also,

$$\begin{aligned} d(u_n,\phi_1 u_n) &\le d(u_n,x_n) + d(x_n,\phi_1 u_n) \\ &\to 0 \;\; as \;\; n \to \infty. \end{aligned}$$

$$\begin{aligned} d(z_n,u_n) &= d(\phi_1((1-\alpha_n)x_n \oplus \alpha_n \phi_1 u_n),u_n) \\ &\le (1-\alpha_n)d(x_n,u_n) + \alpha_n d(\phi_1 u_n,u_n) - (1-\alpha_n)\alpha_n d(x_n,\phi_1 u_n) \\ &\le d(x_n,u_n) - \alpha(1-\beta)d(x_n,\phi_1 u_n), \\ &\to 0 \;\; as \;\; n \to \infty. \end{aligned}$$

and

$$\begin{aligned} d(x_n,z_n) &\le d(x_n,u_n) + d(u_n,x_n) \\ &\to 0 \;\; as \;\; n \to \infty. \end{aligned}$$

So, it follows that

$$\begin{aligned} d(x_n,\phi_2 x_n) &\le d(x_n,\phi_1 x_n) + d(\phi_1 x_n,\phi_2 z_n) + d(z_n,x_n) \\ &\to 0 \;\; as \;\; n \to \infty. \end{aligned}$$

Hence

$$\lim_{n\to\infty} d(x_n,\phi_1 x_n) = \lim_{n\to\infty} d(x_n,\phi_2 x_n) = 0.$$

This completes the proof. □

Theorem 3.2. *Consider $\acute{f}: X \to (-\infty,\infty]$ is a proper, convex and lsc function, where (X,d) is a Hadamard space. Let ϕ_1,ϕ_2 be nonexpansive self maps defined on X such that $\Theta = F(\phi_1) \cap F(\phi_2) \cap \arg\min_{y\in X} \acute{f}(y) \neq \emptyset$ Consider $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences for all $n \in \mathbb{N}$ and for some $\alpha,\beta,\gamma \in (0,1)$ and $\{\mu_n\}$ is a sequence such that $\mu_n \ge \mu > 0$ for all $n \in \mathbb{N}$ and for some μ. Let $\{x_n\}$ be generated by (3.1), then $\{x_n\}$ Δ-converges to an element of Θ.*

Proof. In fact, it follows from Lemma 2.13 and Theorem 3.1(ii), that

$$\begin{aligned} d(x_n,J_\mu x_n) &\le d(x_n,u_n)+d(u_n,J_\mu x_n)\\ &\le d(J_\mu x_n,J_{\mu_n}x_n)+d(x_n,u_n)\\ &\le d\left(J_\mu x_n,J_\mu\left(\frac{\mu_n-\mu}{\mu_n}J_{\mu_n}x_n\oplus\frac{\mu}{\mu_n}x_n\right)\right)+d(x_n,u_n)\\ &\le d\left(x_n,\left(1-\frac{\mu}{\mu_n}\right)J_{\mu_n}x_n\oplus\frac{\mu}{\mu_n}x_n\right)+d(x_n,u_n)\\ &\le \left(1-\frac{\mu}{\mu_n}\right)d(x_n,J_{\mu_n}x_n)+\frac{\mu}{\mu_n}d(x_n,x_n)+d(x_n,u_n)\\ &\le \left(1-\frac{\mu}{\mu_n}\right)d(x_n,u_n)+d(x_n,u_n)\\ &\to 0 \;\; as \;\; n\to\infty. \end{aligned}$$

Theorem 3.1(i) shows that $\lim n\to\infty d(x_n,q)$ exists for all $q\in\Theta$ and Theorem 3.1(iii) also implies that $\lim n\to\infty d(x_n,\phi_i x_n)=0$ for all $i=1,2$.

Next, we show that $W_\Delta(x_n)\subset\Theta$. Let $a\in W_\Delta(x_n)$. Then there exists a subsequence $\{a_n\}$ of $\{x_n\}$ such that $A(\{a_n\})=\{a\}$. From Lemma 2.11, there exists a subsequence $\{b_n\}$ of $\{a_n\}$ such that $\Delta-\lim n\to\infty b_n=b$ for some $b\in\Theta$. So, $a=b$ by Lemma 2.12. This shows that $W_\Delta(x_n)\subset\Theta$.

Finally, we show that the sequence $\{x_n\}$ Δ-converges to a point in Θ. To this end, it suffices to show that $W_\Delta(x_n)$ consists of exactly one point. Let $\{a_n\}$ be a subsequence of $\{x_n\}$ with $A(\{a_n\})=\{a\}$ and let $A(\{x_n\})=\{x\}$. Since $a\in W_\Delta(x_n)\subset\Theta$ and $\{d(x_n,a)\}$ converges, by Lemma 2.12, we have $x=a$. Hence $W_\Delta(x_n)=\{x\}$. This completes the proof. □

Corollary 3.3. *Consider $\acute{f}\colon X\to(-\infty,\infty]$ is a proper, convex and lsc function, where (X,d) is a Hadamard space. Let ϕ_1,ϕ_2 be nonexpansive self maps defined on X such that $\Theta=F(\phi)\cap\arg\min_{y\in X}\acute{f}(y)\neq\emptyset$ Consider $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences for all $n\in\mathbb{N}$ and for some $\alpha,\beta,\gamma\in(0,1)$ and $\{\mu_n\}$ is a sequence such that $\mu_n\ge\mu>0$ for all $n\in\mathbb{N}$ and for some μ. Let $\{x_n\}$ be generated in the following manner:*

$$\begin{aligned} u_n &= arg\min_{y\in X}\left(\acute{f}(y)+\frac{1}{2\mu_n}d^2(y,x_n)\right),\\ z_n &= \phi((1-\alpha_n)x_n\oplus\alpha_n\phi u_n),\\ y_n &= \phi((1-\beta_n)z_n\oplus\beta_n\phi z_n),\\ x_{n+1} &= \phi((1-\gamma_n)y_n\oplus\gamma_n\phi y_n), \end{aligned} \tag{3.5}$$

for each $n\in\mathbb{N}$, then $\{x_n\}$ Δ-converges to an element of Θ.

Since every Hilbert space is a Hadamard space, we obtain directly the following result.

Corollary 3.4. *Let $\mathbb{H}$ be a Hilbert space and $\acute{f}\colon\mathbb{H}\to(-\infty,\infty]$ is a proper, convex and lower semi-continuous function. Let ϕ_1,ϕ_2 be nonexpansive self maps defined on $\mathbb{H}$ such that $\Theta=F(\phi_1)\cap F(\phi_2)\cap\arg\min_{y\in\mathbb{H}}\acute{f}(y)\neq\emptyset$ Consider $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences for all $n\in\mathbb{N}$ and for some $\alpha,\beta,\gamma\in(0,1)$ and $\{\mu_n\}$ is a sequence such that $\mu_n\ge\mu>0$ for all $n\in\mathbb{N}$ and for some μ. Let $\{x_n\}$ be generated in the following manner:*

$$\begin{aligned} u_n &= arg\min_{y\in X}\left(\acute{f}(y)+\frac{1}{2\mu_n}||y-x_n||^2\right),\\ z_n &= \phi_1((1-\alpha_n)x_n+\alpha_n\phi_1 u_n),\\ y_n &= \phi_2((1-\beta_n)z_n+\beta_n\phi_2 z_n),\\ x_{n+1} &= \phi_2((1-\gamma_n)y_n+\gamma_n\phi_1 y_n), \end{aligned} \tag{3.6}$$

for each $n\in\mathbb{N}$, then $\{x_n\}$ weakly converges to an element of Θ.

Next, we establish the strong convergence theorems of our iteration.

Theorem 3.5. *Consider $\acute{f}\colon X\to(-\infty,\infty]$ is a proper, convex and lsc function, where (X,d) is a Hadamard space. Let ϕ_1,ϕ_2 be nonexpansive self maps defined on X such that $\Theta=F(\phi_1)\cap F(\phi_2)\cap\arg\min_{y\in X}\acute{f}(y)\neq\emptyset$ Consider $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences for all $n\in\mathbb{N}$ and for some $\alpha,\beta,\gamma\in(0,1)$ and $\{\mu_n\}$ is a sequence such that $\mu_n\ge\mu>0$ for all $n\in\mathbb{N}$ and for some μ. Let $\{x_n\}$ be generated by (3.1), then $\{x_n\}$ strongly-converges to an element of Θ if and only if $\liminf_{n\to\infty}d(x_n,\Theta)=0$, where $d(x,\Theta)=inf\{d(x,q^*)\colon q^*\in\Theta\}$.*

Proof. The necessity is obvious from Theorem 3.1. Conversely, let

$$\liminf_{n\to\infty}d(x_n,\Theta)=0.$$

Since

$$d(x_{n+1},q^*)\le d(x_n,q^*),$$

for all $q^*\in\Theta$. Hence

$$d(x_{n+1},\Theta)\le d(x_n,\Theta).$$

Hence $\liminf_{n\to\infty}d(x_n,\Theta)$ exists and $\liminf_{n\to\infty}d(x_n,\Theta)=0$. Following the proof of Theorem 2 of [23], we can show that $\{x_n\}$ is a Cauchy sequence in X. This implies that $\{x_n\}$ converges to a point q^* in X and so $d(q^*,\Theta)=0$. Since Θ is closed, $q^*\in\Theta$. This completes the proof. □

A family $\{S,T,U\}$ of mappings is said to satisfy the condition (Θ) if there exists a nondecreasing function $\acute{f}\colon [0,\infty)\to[0,\infty)$ with $\acute{f}(0)=0$, $\acute{f}(r)>0$ for all $r\in(0,\infty)$ such that $d(x,Sx)\geq \acute{f}(d(x,F))$ or $d(x,Tx)\geq \acute{f}(d(x,F))$ or $d(x,Ux)\geq \acute{f}(d(x,F))$ for all $x\in X$. Here $F=F(S)\cap F(T)\cap F(U)$.

Theorem 3.6. *Consider $\acute{f}\colon X\to(-\infty,\infty]$ is a proper, convex and lsc function, where (X,d) is a Hadamard space. Let ϕ_1,ϕ_2 be nonexpansive self maps defined on X such that $\Theta=F(\phi_1)\cap F(\phi_2)\cap\arg\min_{y\in X}\acute{f}(y)\neq\emptyset$ Consider $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences for all $n\in\mathbb{N}$ and for some $\alpha,\beta,\gamma\in(0,1)$ and $\{\mu_n\}$ is a sequence such that $\mu_n\geq\mu>0$ for all $n\in\mathbb{N}$ and for some μ. If $\{J_\mu,\phi_1,\phi_2\}$ satisfies the condition (Θ), then the sequence $\{x_n\}$ generated by (3.1) strongly converges to a point of Θ.*

Proof. From Theorem 3.1, we know that $\lim_{n\to\infty}d(x_n,q^*)$ exists for all $q^*\in\Theta$. This implies that $\lim_{n\to\infty}d(x_n,q^*)$ exists.
Also, by the condition (Θ), we have

$$\lim_{n\to\infty}\acute{f}(d(x_n,\Theta))\leq\lim_{n\to\infty}d(x_n,\phi_1x_n)=0,$$

or

$$\lim_{n\to\infty}\acute{f}(d(x_n,\Theta))\leq\lim_{n\to\infty}d(x_n,\phi_2x_n)=0,$$

or

$$\lim_{n\to\infty}\acute{f}(d(x_n,\Theta))\leq\lim_{n\to\infty}d(x_n,J_\mu x_n)=0.$$

Thus, we have

$$\lim_{n\to\infty}\acute{f}(d(x_n,\Theta))=0.$$

By using the property of $\acute{f}$, we obtain $\lim_{n\to\infty}d(x_n,\Theta)=0$. Thus, the proof follows from Theorem 3.5. □

A mapping $\phi\colon\vartheta\to\vartheta$ is said to be semi-compact if any sequence $\{x_n\}$ in ϑ satisfying $d(x_n,\phi x_n)\to0$ has a convergent subsequence.

Theorem 3.7. *Consider $\acute{f}\colon X\to(-\infty,\infty]$ is a proper, convex and lsc function, where (X,d) is a Hadamard space. Let ϕ_1,ϕ_2 be nonexpansive self maps defined on X such that $\Theta=F(\phi_1)\cap F(\phi_2)\cap\arg\min_{y\in X}\acute{f}(y)\neq\emptyset$ Consider $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences for all $n\in\mathbb{N}$ and for some $\alpha,\beta,\gamma\in(0,1)$ and $\{\mu_n\}$ is a sequence such that $\mu_n\geq\mu>0$ for all $n\in\mathbb{N}$ and for some μ. Suppose that J_μ,ϕ_1 and ϕ_2 is semi-compact, then the sequence $\{x_n\}$ generated by (3.1) strongly converges to a point of Θ.*

Proof. Suppose that ϕ_1 is semi-compact. By Theorem 3.1(iii), we have

$$d(x_n,\phi_1x_n)\to0,$$

as $n\to\infty$. So, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\to q^*\in X$. Since $d(x_n,J_\mu x_n)\to0$ and $d(x_n,\phi_ix_n)\to0$ for all $i\in\{1,2\}$, $d(q^*,J_\mu q^*)=0$, and $d(q^*,\phi_1q^*)=d(q^*,\phi_2q^*)=0$, which shows that $q^*\in\Theta$. In other cases, we can prove the strong convergence of $\{x_n\}$ to a point of Θ. This completes the proof. □

4. Numerical example

Now we present a numerical example to demonstrate the convergence of our iteration technique and to support our main theorem in a real-number space.

Example 4.1. [32] Let $X=\mathbb{R}$ with the Euclidean norm and $\vartheta=\{x: x\in[-4,4]\}$. For each $x\in\vartheta$, we define mappings ϕ_1 and ϕ_2 on ϑ as follows:

$$\phi_1x=x\ \ and\ \ \phi_2x=\frac{x}{5}.$$

Clearly, ϕ_1 and ϕ_2 are nonexpansive mappings.
Also, for each $x\in\vartheta$, we define $\acute{f}\colon\vartheta\to(-\infty,\infty]$ by

$$\acute{f}(x)=x^2.$$

We can easily check that $\acute{f}$ is a proper, convex and lower semi-continuous function.
We choose $\alpha_n=1-\frac{n}{3n+1}$, $\beta_n=\frac{n}{16n+1}$ and $\gamma_n=\frac{n}{n+5}$. We set $\mu=\frac{1}{2}$ forall n. It can be observed that all the assumption of Theorem 3.5 are satisfied. Hence the sequence $\{x_n\}$ generated by (3.1) converges to 0 which is the fixed point of ϕ_1,ϕ_2 and minimizer of $\acute{f}(x)$.

5. Conclusion

Our main results generalizes of Khan-Abbas [23], Cholamjiak et al. [10] and Dashputre et al. [11] from one nonexpansive mapping to two nonexpansive mappings involving the convex and lower semi-continuous function, we present a new modified proximal point algorithm for solving the convex minimization problem as well as the fixed point problem of nonexpansive mappings in Hadamard spaces. Finally, we provided a numerical illustration to support our main point.

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