International Journal of Applied Mathematical Research, 10 (2) (2021) 22-27

International Journal of Applied Mathematical Research

Website: www.sciencepubco.com/index.php/IJAMR

Research paper

Resistances of Infinite Electrical Networks

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Abstract

It is interesting to find the equivalent resistance between two nodes of an infinite electrical network. In this paper, we consider an infinite electrical network that can be described as a series of squares whose edges are resistors with resistance R and whose corresponding vertices are joined successively by resistors with resistance R as well. Our major work is to find the equivalent resistance between the diagonal vertices of the base square of this infinite network. First, we apply the techniques of balanced bridges and symmetry of voltages to convert each iteration of the network to a parallel circuit that includes the previous iteration. Then, we evaluate the equivalent resistance of each iteration of the network and derive a recursive sequence of equivalent resistances with iterations. After that, we prove that the recursive sequence is convergent using the contraction theorem in real analysis. Finally, we claim that the limit of the recursive sequence is the equivalent resistance of the infinite network.

Keywords: *Equivalent resistance; infinite network; recursive sequence; convergent sequence; contraction principle.*

1. Introduction

The first thorough mathematical description of electrical circuits goes back to Gustav Kirchhoff [5]. Ever since, the topic has attracted many researchers and produced a great number of pedagogical articles. On one hand, networks are a source of elegant problems and solutions or interesting experiments. On the other hand, electrical networks can be used to visually and intuitively illustrate some complicated electrodynamical concepts.

Important question is determining the equivalent resistance between two nodes of finite or infinite electrical networks. Many work has been done on finding equivalent resistance of different infinite electrical networks; see [1, 2, 3, 4, 9]. However, none of these studies deal with a three dimensional infinite network constructed in this way: a series of squares whose edges are resistors with resistance R and whose corresponding vertices are joined successively by resistors with resistance R as well; see Figure 1. It is interesting to find the equivalent resistance between two nodes a and b of the base square.

Of course, infinite electrical networks do not exist in real world, but an infinite electrical network can be used as a model of an electrical network with long-distance transmission lines.

If a node is only shared by two edges, such edges are said to be connected in series, and the equivalent resistance is the sum of individual resistances. If two resistors are on edges connected across the same pair of nodes, such a connection is called in parallel, and the equivalent resistance is the reciprocal of the sum of reciprocals of resistances. Both of these situations can be generalized straightforwardly for more than two resistors.

It is clear that not every network can be simplified in series or parallel, for example, a bridge structure, as depicted in Figure 2(a).

If a bridge network, such as the one depicted in Figure 2(a), satisfies that $\frac{R_1}{R_2} = \frac{R_3}{R_4}$, then the bridge is called a balanced bridge. In this case, the voltage at node c equals the voltage at node d and no current flows through the resistor R_5. Hence, the resistance between a and b does not depend on the value of R_5. Therefore, R_5 can be replaced with a short; see Figure 2(b), or entirely removed from the network; see Figure 2(c). We will use this feature to simplify the infinite network in Figure 1.

By the way, whenever two nodes of any network have the same voltage, there is no current flow between these two nodes. Consequently, introducing a short between these two nodes will have no impact on the overall network. Further discussions of circuit theory can be found in [7].

To complete the process, we need to introduce a few mathematics concepts. Please refer to [6] for detailed discussion.

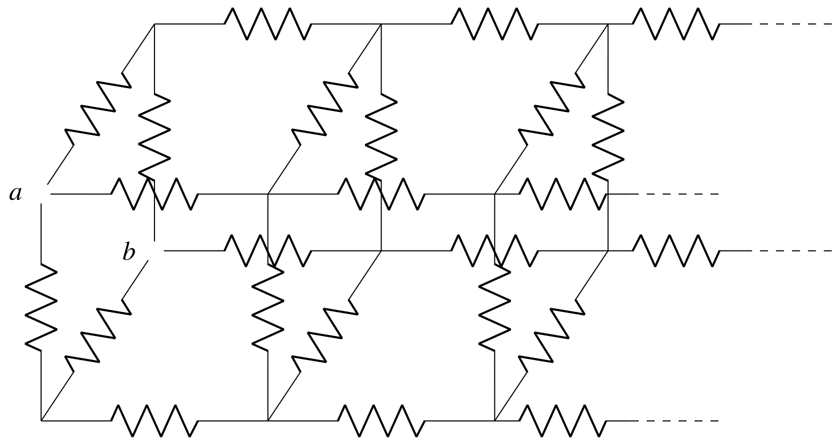
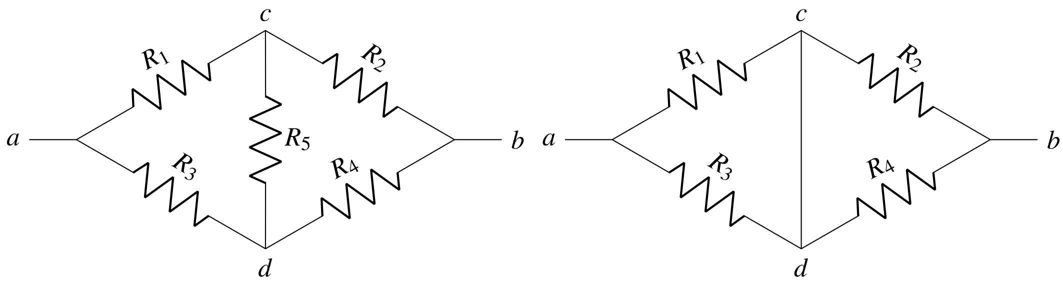
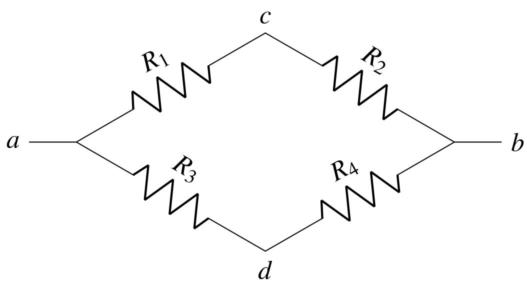


Figure 1: Infinite electrical network



(a) A bridge network **(b)** The network with the bridge shorted



(c) The network with the bridge removed

Figure 2: Circuit with bridge structure

A sequence $\{a_n\}$ of real numbers converges to a real number A if for each real number $\varepsilon > 0$, there exists a positive integer n^* such that $|a_n - A| < \varepsilon$ for all $n \geq n^*$.
A sequence $\{a_n\}$ is said to be a contractive sequence if there exists a constant k with $k \in (0,1)$, such that

$$|a_{n+2} - a_{n+1}| \leq k|a_{n+1} - a_n|$$

for all $n \in \mathbb{N}$.
The well known contraction principle states that every contractive sequence is convergent in $\mathbb{R}$.

2. Main Results

Note that while the infinite network depicted in Figure 1 is three dimensional, it can be converted into a planar network, as depicted in Figure 3. We attempt to find the equivalent resistance between nodes a and b using this planar network.

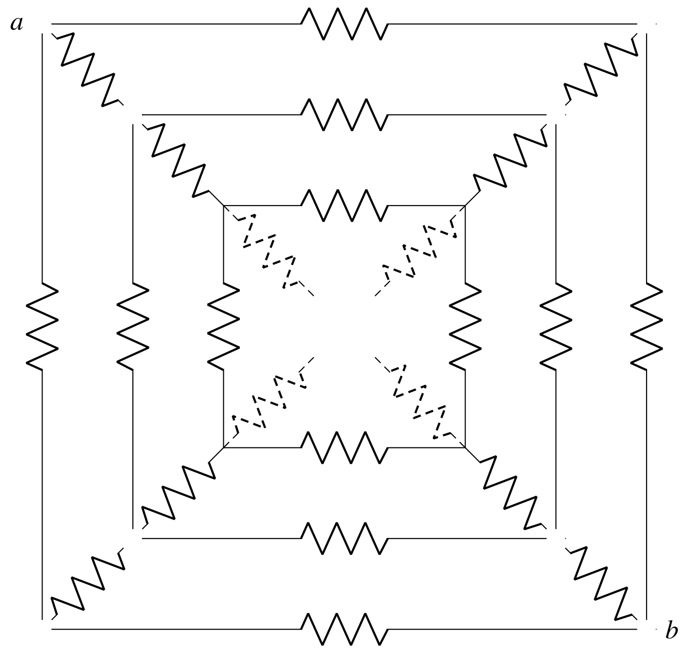


Figure 3: The planar structure of the infinite network

Theorem 1. Let R_n denote the resistance of the n^{th} subnetwork of Figure 1. Then the resistance of the $(n+1)^{th}$ subnetwork, denoted by R_{n+1}, is

$$R_{n+1} = \frac{1}{\frac{1}{R} + \frac{1}{2R+R_n}} = \frac{2R^2 + RR_n}{3R + R_n}.$$

Proof. Suppose that the network depicted in Figure 4 having n connected squares in total represents the n^{th} subnetwork of Figure 1. Then, R_n is the resistance between the nodes d and f in the network.

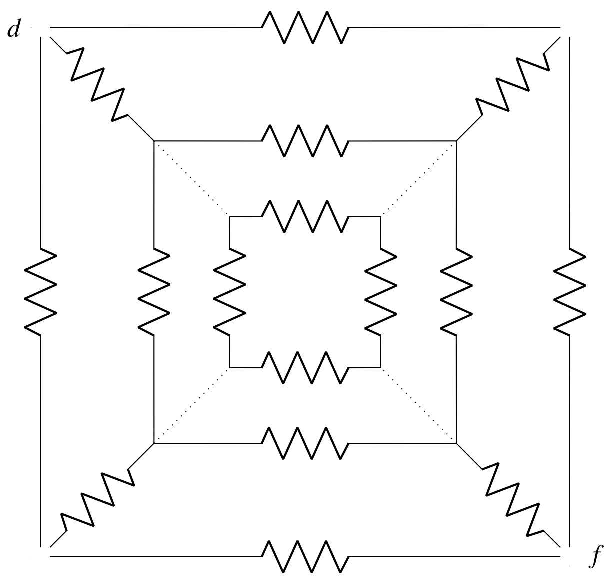


Figure 4: The n^{th} network

By adding an additional square with resistors of resistance R for each edge, and connecting the vertices of that square, via a resistor of resistance R as well, to the corresponding vertices of the n^{th} network, we have constructed the $(n+1)^{th}$ network as depicted in Figure 5 and R_{n+1} is the resistance between the nodes a and b in the $(n+1)^{th}$ network.

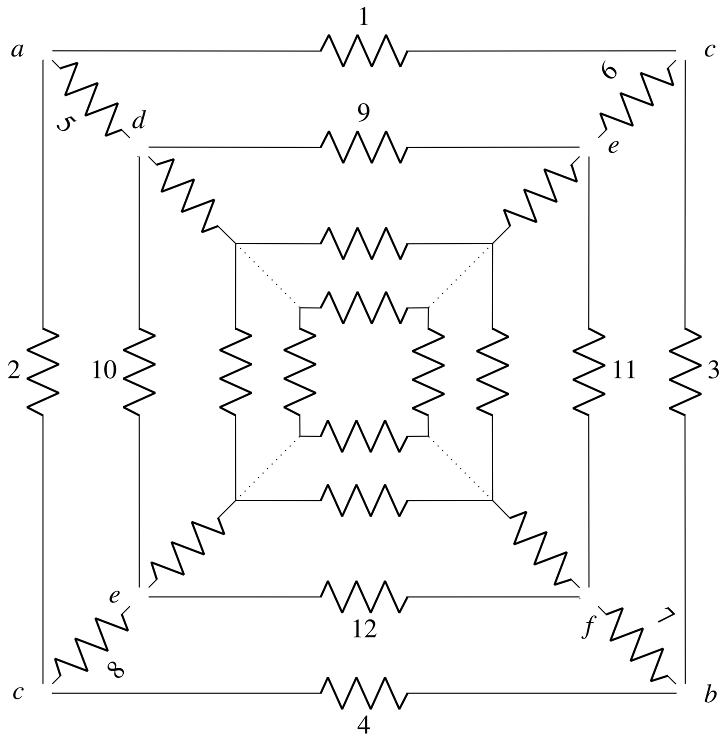


Figure 5: The $(n+1)^{th}$ network

By symmetry, the nodes marked c have the same voltage. Thus, as described in balanced bridge in the previous sections, they can be shorted to produce the network depicted in Figure 6. Note that for brevity, only the perimeter of the $(n+1)^{th}$ network is depicted in Figure 6. We label the resistors in Figure 5 and Figure 6 to help readers understand the transition from Figure 5 to Figure 6.

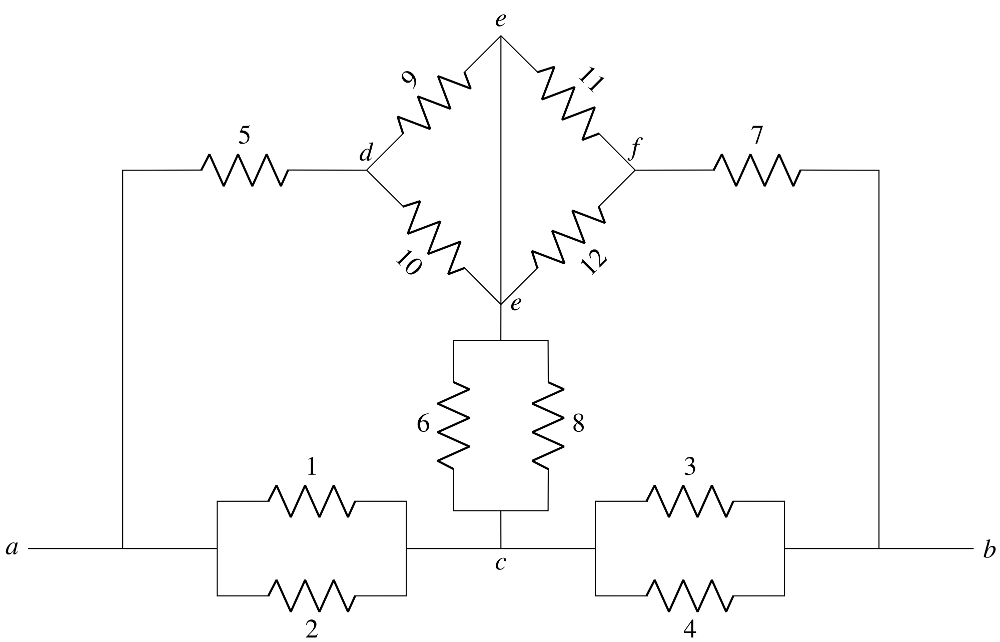


Figure 6: The simplified $(n+1)^{th}$ network using symmetry and shorts

Since the resistor pair joining the nodes marked e and c is a bridge, the network as a whole is a bridge as well. The question remains as to whether it is a balanced bridge. Note that the n^{th} network is symmetric across the e axis, the network is balanced since the remainder of the network is balanced. Thus, the resistors joining e and c are removable. After further simplification, we arrive at the simplified network depicted in Figure 7.

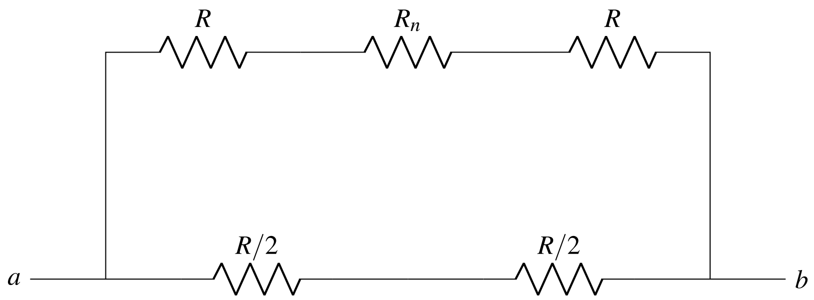


Figure 7: The further simplified $(n+1)^{th}$ network using balanced bridge

Applying the resistance formulas for parallel and series networks we conclude that

$$R_{n+1} = \frac{1}{\frac{1}{R/2+R/2} + \frac{1}{2R+R_n}} = \frac{2R^2+RR_n}{3R+R_n}.$$

Theorem 2. The sequence $\{R_n\}$ defined by

$$R_{n+1} = \frac{2R^2+RR_n}{3R+R_n}$$

is contractive, and, consequently, convergent. Furthermore, the limit is $(\sqrt{3}-1)R$.
Proof. Note that we have

$$\begin{aligned} &\left|R_{n+2}-R_{n+1}\right| \\ &= \left|\frac{2R^2+RR_{n+1}}{3R+R_{n+1}} - \frac{2R^2+RR_n}{3R+R_n}\right| \\ &= \left|\frac{R^2R_{n+1}-R^2R_n}{9R^2+3RR_n+3RR_{n+1}+R_{n+1}R_n}\right| \\ &\leq \left|\frac{R^2R_{n+1}-R^2R_n}{9R^2}\right| = \frac{1}{9}\left|R_{n+1}-R_n\right|. \end{aligned}$$

This shows that the sequence $\{R_n\}$ is contractive. By the well known contraction principle stated earlier, the sequence is convergent. Let $\lim_{n\to\infty} R_n = A$. Then $\lim_{n\to\infty} R_{n+1} = A$ as well. Taking limits on both sides of

$$R_{n+1} = \frac{2R^2+RR_n}{3R+R_n}$$

implies that

$$A = \frac{2R^2+RA}{3R+A}.$$

Solving the equation above gives $A = -R \pm R\sqrt{3}$. Because the resistance is nonnegative, we claim $A = (\sqrt{3}-1)R$.

3. Conclusion

The equivalent resistance between two nodes of an infinite electrical network is studied and derived in this paper. First, with the clever use of balanced Wheatstone bridges, we are able to convert the original infinite network into a much simple planar infinite network. Then, from the simplified infinite network, the resistance of the $(n+1)^{th}$ network is obtained in terms of the resistance of the n^{th} network which establish a recursive sequence of resistances. After that, with the help of real analysis, we prove that the resistance sequence is contractive, hence, it is convergent by the famous contraction principle. Finally, we claim that the limit of the resistance sequence is the equivalent resistance of the 3-dimensional infinite network.

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