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Multiple Mellin Convolution and I-function Transformation involving r variables

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Abstract

An expression is obtained for an I-function of r variables, transform of the Multiple Mellin convolution of the two functions in terms of the Multiple Mellin convolution of I-function transforms of the function involving r variables, it generalize the result given earlier by Kumbhat and Khan [5].

Keywords: *Convolution , Multivariable I-Function, Multiple Mellin Convolution, Mellin Transformation.*

1 Introduction

Kumbhat and Khan [5] have obtained an interesting relation involving a convolution of the Mellin type in connection with the I-function transformation. The aim of the present paper is to generalize their result to r variables, I-function transformation.

Definition 1.1 : The multivariable I-function represented by Prasad [7] as

$$I[z_1,\ldots,z_r] = I^{\{0,n_i\}_{2,r}:\{(m^{(i)},n^{(i)})\}_{1,r}}_{\{p_i,q_i\}_{2,r}:\{(p^{(i)},q^{(i)})\}_{1,r}}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle|\begin{matrix} A:B \\ C:D \end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\ldots\int_{L_r}\psi(\zeta_1,\ldots,\zeta_r)\prod_{i=1}^{r}\left\{\phi_i(\zeta_i)z_i^{\zeta_i}\right\}d\zeta_1\ldots d\zeta_r \qquad (1.1)$$

where $\omega = \sqrt{-1}$,

$$\psi(\zeta_1,\ldots\zeta_r) = \frac{\prod_{k=2}^{r}\left[\prod_{j=1}^{n_k}\Gamma\left(1-a_{kj}+\sum_{i=1}^{k}\alpha_{kj}^{(i)}\zeta_i\right)\right]}{\prod_{k=2}^{r}\left[\prod_{j=n_k+1}^{p_k}\Gamma\left(a_{kj}-\sum_{i=1}^{k}\alpha_{kj}^{(i)}\zeta_i\right)\right]}$$

$$\times \frac{1}{\prod_{k=2}^{r}\left[\prod_{j=1}^{q_k}\Gamma\left(1-b_{kj}+\sum_{i=1}^{k}\beta_{kj}^{(i)}\zeta_i\right)\right]} \tag{1.2}$$

$$\phi_i(\zeta_i)=\frac{\left[\prod_{k=1}^{m^{(i)}}\Gamma\left(b_k^{(i)}-\beta_k^{(i)}\zeta_i\right)\right]\left[\prod_{j=1}^{n^{(i)}}\Gamma\left(1-a_j^{(i)}+\alpha_j^{(i)}\zeta_i\right)\right]}{\left[\prod_{j=n^{(i)}+1}^{p^{(i)}}\Gamma\left(a_j^{(i)}-\alpha_j^{(i)}\zeta_i\right)\right]\left[\prod_{k=m^{(i)}+1}^{q^{(i)}}\Gamma\left(1-b_k^{(i)}+\beta_k^{(i)}\zeta_i\right)\right]} \tag{1.3}$$

$\forall i\in\{1,...,r\}$. Also,

$\{0, n_i\}_{2,r}$ $:= 0, n_2 : ... : 0, n_r,$

$\{p_i, q_i\}_{2,r}$ $:= p_2, q_2 : ... : p_r, q_r,$

$\left\{\left(m^{(i)},n^{(i)}\right)\right\}_{1,r}$ $:= \left(m^{(1)},n^{(1)}\right);...;\left(m^{(r)},n^{(r)}\right),$

$\left\{\left(p^{(i)},q^{(i)}\right)\right\}_{1,r}$ $:= \left(p^{(1)},q^{(1)}\right);...;\left(p^{(r)},q^{(r)}\right),$

$$\left.\begin{aligned}
A &:= \left\{\left(a_{ij};\alpha_{ij}^{(1)},...,\alpha_{ij}^{(i)}\right)_{1,p_i}^{2,r}\right\} := \left(a_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)}\right)_{1,p_2};...;\left(a_{rj};\alpha_{rj}^{(1)},\alpha_{rj}^{(r)}\right)_{1,p_r}\\
B &:= \left\{\left(a_j^{(i)};\alpha_j^{(i)}\right)_{1,p^{(i)}}^{1,r}\right\} := \left(a_j^{(1)},\alpha_j^{(1)}\right)_{1,p^{(1)}};...;\left(a_j^{(r)},\alpha_j^{(r)}\right)_{1,p^{(r)}}\\
C &:= \left\{\left(b_{ij};\beta_{ij}^{(1)},...,\beta_{ij}^{(i)}\right)_{1,q_i}^{2,r}\right\} := \left(b_{2j};\beta_{2j}^{(1)},\beta_{2j}^{(2)}\right)_{1,q_2};...;\left(b_{rj};\beta_{rj}^{(1)},\beta_{rj}^{(r)}\right)_{1,q_r}\\
D &:= \left\{\left(b_j^{(i)};\beta_j^{(i)}\right)_{1,q^{(i)}}^{1,r}\right\} := \left(b_j^{(1)},\beta_j^{(1)}\right)_{1,q^{(1)}};...;\left(b_j^{(r)},\beta_j^{(r)}\right)_{1,q^{(r)}}
\end{aligned}\right\} \tag{1.4}$$

such that n_i, p_i, q_i, $m^{(i)}$, $n^{(i)}$, $p^{(i)}$ $q^{(i)}$ are non-negative integers and all a_{ij}, b_{ij}, α_{ij}, β_{ij}, $a_j^{(i)}$, $b_j^{(i)}$, $\alpha_j^{(i)}$, $\beta_j^{(i)}$ are complex numbers and the empty product denotes unity.
The contour integral (1.1) converges, if

$$|\arg z_i|<\frac{1}{2}U_i\pi, U_i>0, i=1,...,r \tag{1.5}$$

where

$$U_i=\sum_{j=1}^{n^{(i)}}\alpha_j^{(i)}-\sum_{j=n^{(i)}+1}^{p^{(i)}}\alpha_j^{(i)}+\sum_{j=1}^{m^{(i)}}\beta_j^{(i)}-\sum_{j=m^{(i)}+1}^{q^{(i)}}\beta_j^{(i)}+\left(\sum_{j=1}^{n_2}\alpha_{2j}^{(i)}-\sum_{j=n_2+1}^{p_2}\alpha_{2j}^{(i)}\right)$$
$$+...+\left(\sum_{j=1}^{n_r}\alpha_{rj}^{(i)}-\sum_{j=n_r+1}^{p_r}\alpha_{rj}^{(i)}\right)-\left(\sum_{j=1}^{q_2}\beta_{2j}^{(i)}+...+\sum_{j=1}^{q_r}\beta_{rj}^{(i)}\right) \tag{1.6}$$

and $I[z_1, ..., z_r] = O\left(|z_1|^{\alpha_1},...,|z_r|^{\alpha_r}\right)$, $\max\{|z_1|,...,|z_r|\}\rightarrow 0$,
where $\alpha_i=\min_{1\le j\le m^{(i)}}\Re\left(b_j^{(i)}/\beta_j^{(i)}\right)$, and $\beta_i=\max_{1\le j\le n^{(i)}}\Re\left(a_j^{(i)}-1/\alpha_j^{(i)}\right)$, $i = 1, ..., r$.
For the condition of convergence and analyticity of multivariable I-function we refer [7, 8]

Definition 1.2 : Generalized Mellin convolution can be taken in the form, given by Datta & Arora [1]

$$(k*f)(y_1,y_2,....,y_r)=\int_0^\infty ...\int_0^\infty \prod_{i=1}^{r} x_i^{-1} k\left(\frac{y_1}{x_1},...\frac{y_r}{x_r}\right) f(x_1,...x_r).dx_1,dx_2...dx_r \qquad (1.7)$$

Further the Mellin convolution of two I functions involving r-variables given by Datta & Arora [1] as

Theorem : Suppose the condition (9) to be satisfied and $\mu_i > 0, v_i > 0, n_i = 0, n_i^{'} = 0, i \in \{1,...,r\}$. Then the Multiple Mellin transform of the product of two I-function of r variable.

$$(\Re I \cdot I)(s)=\int_0^\infty ...\int_0^\infty \prod_{i=1}^{r} t_i^{s_i-1} I[z_1 t_1^{-1},...z_r t_r^{-1}] I'[\eta_1 t_1,...,\eta_r t_r] dt_1...dt_r$$

$$= \prod_{i=1}^{r}\{\eta_i^{-s_i}\} I^{\{0,0\}_{2,r}:\{(m^{(i)}+m'^{(i)},n^{(i)}+n'^{(i)})\}_{1,r}}_{\{p_i+p_i^{'},q_i+q_i^{'}\}_{2,r}\{(p^{(i)}+p'^{(i)},q^{(i)}+q'^{(i)})\}_{1,r}}\left[\begin{matrix} z_1\eta_1^{s_1} \\ M \\ z_r\eta_r^{s_r}\end{matrix}\middle| \begin{matrix} A:B \\ C:D\end{matrix}\right] \qquad (1.8)$$

where

$$-\min_{1\le j\le n^{(i)}} \Re\left(\frac{1-a_j^{(i)}}{\alpha_j^{(i)}}\right) - \min_{1\le j\le m'^{(i)}} \Re\left(\frac{b'^{(i)}_j}{\beta'^{(i)}_j}\right) < \Re(s_i) <$$

$$\min_{1\le j\le m^{(i)}} \Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right) + \min_{1\le j\le n'^{(i)}} \Re\left(\frac{1-a'^{(i)}_j}{\alpha'^{(i)}_j}\right)$$

i = 1, ..., r and

$$A \equiv \left\{ \left(a_{ij};\alpha_{ij}^{(1)},...,\alpha_{ij}^{(i)}\right)_{1,p_i}^{2,r}; \left(a_{ij}^{'}+\sum_{k=1}^{'} s_k \alpha_{ij}^{'(k)} \alpha_{ij}^{'(1)},...,\alpha_{ij}^{'(i)}\right)_{1,p_i^{'}}^{2,r} \right\},$$

$$B \equiv \left\{ \left(a_j^{(i)},\alpha_j^{(i)}\right)_{1,p^{(i)}}^{1,r}; \left(a_j^{'(i)}+s_i\alpha_j^{'(i)},\alpha_j^{'(i)}\right)_{1,p'^{(i)}}^{1,r} \right\},$$

$$C \equiv \left\{ \left(b_{ij};\beta_{ij}^{(1)},...,\beta_{ij}^{(i)}\right)_{1,q_i}^{2,r}; \left(b_{ij}^{'}+\sum_{k=1}^{'} s_k \beta_{ij}^{'(k)} \beta_{ij}^{'(1)},...,\beta_{ij}^{'(i)}\right)_{1,q_i^{'}}^{2,r} \right\},$$

$$D \equiv \left\{ \left(b_j^{(i)},\beta_j^{(i)}\right)_{1,q^{(i)}}^{1,r}; \left(b_j^{'(i)}+s_i\beta_j^{'(i)},\beta_j^{'(i)}\right)_{1,q'^{(i)}}^{1,r} \right\}$$

2 The I-Function Transformation Involving R-Variables

The I-function Transformation of function f involving r-variables is given as

$$\hat{f}(y_1,y_2,...y_r) = I[f(x_1,x_2,...x_r); y_1,y_2,...y_r]$$

=

$$\int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{r} x_i^{-1} I[y_1x_1, y_2x_2, \ldots y_rx_r] f(x_1, \ldots, x_r).dx_1 \ldots dx_r$$

provided that the integral exists. (2.1)

3 The Convolution Property

$$\left.\begin{aligned} \text{Let} \quad & \hat{k}(s) = I[k(x_1, \ldots x_r); s_1, \ldots, s_r] \\ \text{and} \quad & \hat{f}(s) = I[f(x_1, \ldots x_r); s_1, \ldots, s_r] \end{aligned}\right\} \tag{3.1}$$

Now we obtain an interesting result which expresses the Mellin convolution of I function transforms of two functions of r variables as the I function transform of multiple Mellin convolution of the functions.

$$\left(\hat{k} * \hat{f}\right)(y_1, y_2, \ldots, y_r) = \int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{r} z_i^{-1} I[y_1z_1, y_2z_2, \ldots y_rz_r]$$

$$\cdot \left\{ \int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{r} x_i^{-1} k\left(\frac{z_1}{x_1}, \ldots \frac{z_r}{x_r}\right) f(x_1, \ldots, x_r) dx_1 \ldots dx_r \right\} \cdot dz_1 \ldots dz_r \tag{3.2}$$

provided that each side exists.

Proof : The Multiple Mellin transform of $\phi(x_1 \ldots x_r)$ is given as follows

$$M\{\phi(x_1, x_2, \ldots, x_r); s_i\} = \int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{r} x_i^{s_i - 1} \phi(x_1, \ldots, x_r) dx_1 \ldots dx_r \tag{3.3}$$

with the assumptions of absolute convergence of the integrals exist and

let $\quad K(s) = M\{k(x_1 \ldots x_r) : s_1 \ldots s_r), F(s) = M\{f(x_1 \ldots x_r): s_1 \ldots s_r\}$ (3.4)

Take Mellin transform of the left hand side of (3.2)

$$\Delta(s) = M\left\{\left(\hat{k} * \hat{f}\right) : s_1 \ldots s_r\right\}$$

=

$$\int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{r} y_i^{s_i - 1} \left\{ \int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{r} x_i^{-1} k\left(\frac{y_1}{x_1} \ldots \frac{y_r}{x_r}\right) f(x_1, \ldots, x_r) dx_1 \ldots dx_r \right\} dy_1 \ldots dy_r$$

(3.5)

On suitable changing the order of two sets of integrations on the right hand side of (3.5) we get

$$\Delta(s) = M\{(\hat{k} * \hat{f}) : s_1 \ldots s_r\}$$

$$= \int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{r} x_i^{-1} f(x_1, \ldots, x_r) \left\{ \int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{r} y_i^{s_i - 1} k\left(\frac{y_1}{x_1} \ldots \frac{y_r}{x_r} \right) dy_1 \ldots dy_r \right\} dx_1 \ldots dx_r \tag{3.6}$$

$$= M\{f(x_1 \ldots x_r) : s_1 \ldots s_r\} \quad M\{k(y_1 \ldots y_r) : s_1 \ldots s_r\}$$

$$= F(s) \cdot K(s) \tag{3.7}$$

Now using the following result,

$$M\{\hat{k}(y_1 \ldots y_r) : s_1 \ldots s_r\} = M\{I(y_1 \ldots y_r) : s_1 \ldots s_r\} K(-s)$$

$$M\{\hat{f}(y_1 \ldots y_r) : s_1 \ldots s_r\} = M\{I(y_1 \ldots y_r) : s_1 \ldots s_r\} F(-s) \text{ in (3.7)}$$

which can easily proved as

$$M\{\hat{k}(y_1 \ldots y_r)\} = \int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{r} y_i^{s_i - 1} \left\{ \int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{r} x_i^{-1} I(y_1 x_1, \ldots, y_r x_r) k(x_1, \ldots, x_r) dx_1 \ldots dx_r \right\} dy_1 \ldots dy_r$$

$$= \int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{r} x_i^{-1} k(x_1, \ldots, x_r) dx_1 \ldots dx_r \int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{r} y_i^{s_i - 1} I(y_1 x_1, \ldots, y_r x_r) dy_1 \ldots dy_r$$

Let $\prod_{i=1}^{r} y_i x_i = \prod_{i=1}^{r} z_i$

$$\prod_{i=1}^{r} dy_i = \frac{\prod_{i=1}^{r} dz_i}{\prod_{i=1}^{r} x_i}$$

$$= \int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{r} x_i^{-s_i - 1} k(x_1, \ldots, x_r) dx_1 \ldots dx_r \int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{r} z_i^{s_i - 1} I(z_1, \ldots, z_r) dz_1 \ldots dz_r$$

$$M\{\hat{k}(y_1 \ldots y_r)\} = M\{I(y_1 \ldots y_r)\} K(-s) \tag{3.8}$$

Similarly

$$M\{\hat{f}(y_1 \ldots y_r)\} = M\{I(y_1 \ldots y_r)\} F(-s) \tag{3.9}$$

Hence from (3.7) we get

$$\Delta(s) = M\left\{\left(\hat{k} * \hat{f}\right)(y_1 \ldots y_r)\right\} = M\left\{\hat{k}(y_1 \ldots y_r)\right\} M\left\{\hat{f}(y_1 \ldots y_r)\right\}$$

By using (3.8) & (3.9) we get

$$\Delta(s) = M\left\{I^{m,n}_{p,q:r}(y_1 \ldots y_r)\right\} K(-s) \cdot M\left\{I^{m',n'}_{p',q':r'}(y_1 \ldots y_r)\right\} F(-s) \qquad (3.10)$$

On the other hand Mellin convolution of the right hand side of (3.2) is

$$\Delta'(s) = M\left\{\int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{r} z_i^{-1} = I^{m+m',n+n'}_{p+p',q+q'}\left[y_1 z_1, \ldots, y_r z_r\right] \cdot \left(k * f\right)(z) dz\right\}$$

$$\Delta'(s) = \int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{r} y_i^{s_i - 1} \left\{\int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{r} z_i^{-1} I\left[y_1 z_1, \ldots, y_r z_r\right]\left(k * f\right)(z) dz_1 \ldots dz_r\right\} dy_1 \ldots dy_r$$

(3.11)

Changing the order of two sets of integrations in (3.11), we get

$$\Delta'(s) = M\left\{I\left(y_1 z_1 \ldots y_r z_r\right)\right\} \int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{r} z_i^{-s_i - 1} (k * f)(z) dz_1 \ldots dz_r$$

Now applying (1.8) and (3.7)

$$\Delta' s) = M\left\{I^{m,n}_{p,q:r}\left[\xi\right]; s_1 \ldots s_r\right\} \cdot M\left\{I^{m',n'}_{p',q':r'}\left[\eta\right]; s_1', s_2' \ldots s_r'\right\} K(-s) F(-s) \qquad (3.12)$$

Equation (3.10) and (3.12) together exhibit the fact that the Mellin transforms of the two sides of (3.2) are equal, and so the two sides of (3.2) must also be equal.

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