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Type of equiangular tight frames with $n+1$ vectors in $\mathbb{R}^n$

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Abstract

An equiangular tight frame (ETF) is a $d \times n$ matrix that has orthogonal rows and unit-norm columns. ETFs have applications in communications, coding theory and quantum computing. In this paper we investigate type of ETFs that have $n+1$ vectors in $\mathbb{R}^n$. Also we state the connection between these frames with the complete graphs that containing $n+1$ vertices.

Keywords: *Equiangular tight frame, Complete graph, Mercedes-Benz frame, Siedel mateix, Adjacency matrix.*

1 Introduction

Equiangular tight frames are an important class of finite dimensional frames. These frames play an important role in several areas of mathematics, ranging from signal processing (see, e.g. [1,3,5,9,10], and references therein) to quantum computing (see, e.g. [2,4,12,13] and references therein). A detailed study of this class of frames was initiated by Strohmer and Heath [14], and Holmes and Paulsen [6]. The problem of the existence of equiangular tight frames is known to be equivalent to the existence of a certain type of matrix called a Seidel matrix [11] or signature matrix [6] with two eigenvalues. A matrix Q is a Seidel matrix provided that it is self-adjoint, its diagonal entries are 0, and its off diagonal entries are all of modulus one. In the real case, these

off-diagonal entries must all be ± 1; such matrices can then be interpreted as adjacency matrices of graphs. There is a well established correspondence between graph-theory and Seidel matrices of real equiangular tight frames as seen in [6], and recently in [16].Type genus of equiangular tight frames are Mercedes-Benz frames which containing $n+1$ vectors in $\mathbb{R}^n$.
A system of unit vectors $\{\varphi_1, \varphi_2, \cdots, \varphi_{n+1}\}$ in the space $\mathbb{R}^n$ is called a Mercedes-Benz system if $< \varphi_j, \varphi_k >= -\dfrac{1}{n}$ for $j \neq k$.
This paper is organized as follows. We start by giving definitions and preliminaries of frame theory in Section 2. In Section 3, we explore the construction of the Mercedes-Benz frames and thair properties and in section 4 we characterize equiangular tight frames with $n+1$ vectors in the space $\mathbb{R}^n$. The paper is concluded in section 5.

2 Definitions and preliminaries

Definition 2.1 *A family of vectors $\{f_j\}_{j=1}^m$ is a frame for $\mathbb{R}^n$, $m \geq n$, provided that there exist two constants $A, B > 0$ such that the equality*

$$A\|x\|^2 \leq \sum_{j=1}^{m} |\langle f, f_j \rangle|^2 \leq B\|x\|^2$$

satiesfies for all $x \in \mathbb{R}^n$. When $A = B = 1$, then the frame is called normalized frame or Parseval frame.

Definition 2.2 *Let $\{f_1, f_2, \cdots, f_m\}$ be a frame in $\mathbb{R}^n$, linear mapping*

$$V : \mathbb{R}^n \to \mathbb{R}^m, \quad (Vx_j) =< x, f_j > \quad \text{for all } j \in \{1, 2, \cdots, m\},$$

which is called the analysis operator of frame.
Because V is linear, we may identify V with an $m \times n$ matrix and the vectors $\{f_1, f_2, \cdots, f_m\}$ are columns of V^*. If V is the analysis operator of Parseval frame, then V is an isometry. We see that $V^*V = I_n$ and the $m \times m$ matrix VV^* is a self-adjoint projection of rank n. VV^* has entires $(VV^*)_{ij} = (< f_i, f_j >)$ and is Grammian matrix of frame.

Definition 2.3 *A frame $\{f_1, f_2, \cdots, f_m\}$ in $\mathbb{R}^n$ is called equal norm if there is $b > 0$ such that $\|f_j\| = b$.*

Definition 2.4 *A finite family $\{f_1, f_2, \cdots, f_m\}$ in $\mathbb{R}^n$ is called an equiangular tight frame if it is equal norm and if there is $b \geq 0$, $| < f_i, f_j > | = b$ for all $i, j \in \{1, 2, \cdots, m\}$ with $i \neq j$.*

3 The Mercedes-Benz frames in $\mathbb{R}^n$

Consider three vectors in $\mathbb{R}^2$:

$$f_1^2 = (-\frac{\sqrt{3}}{2}, -\frac{1}{2})^T \quad , \quad f_2^2 = (\frac{\sqrt{3}}{2}, -\frac{1}{2})^T \quad , \quad f_3^2 = (0,1)^T,$$

where the superscript indicates the dimension of vectors. Compose the matrix with columns f_1^2, f_2^2 and f_3^2:

$$A_2 = \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}.$$

It is easily shown that $A_2A_2^T = \frac{3}{2}$.
Hence the system $\{f_j^2\}_{j=1}^3$ is a tight frame , known as the Mercedes-Benz frame.Note

$$\sum_{j=1}^{3} f_j^2 = 0, \; < f_j^2, f_k^2 >= -\frac{1}{2} \quad \text{for} \quad k \neq j. \tag{1}$$

In $\mathbb{R}$ there are only two unit vectors $f_1^1 = -1$ and $f_2^1 = 1$. These vectors have a property similar to (1):

$$f_1^1 + f_2^1 = 0, \qquad < f_1^1, f_2^1 >= -1.$$

It is natural to call the system $\{f_1^1, f_2^1\}$ a Mercedes-Benz frame in $\mathbb{R}$. Figure 1 shows that the Mercedes-Benz frames in $\mathbb{R}$ and $\mathbb{R}^2$.

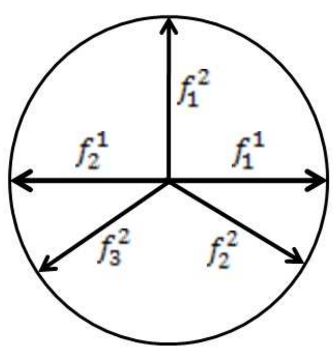

Figure 1: Mercedes-Benz frames in $\mathbb{R}$ and $\mathbb{R}^2$.

We see that the system $\{f_1^2, f_2^2, f_3^2\}$ is obtained from the system $\{f_1^1, f_2^1\}$ in

the following way.The vectors f_1^1 and f_2^1 rotate downward by the system angle until are formed vectors f_1^2 and f_2^2. Then we add $f_3^2 = (0,1)^T$ to f_1^2 and f_2^2. This observation is influence for Mercedes-Benz constructingin the space $\mathbb{R}^n$ by induction.
Let the system of unit vectors $\{f_1^{n-1}, f_2^{n-1}, \cdots, f_n^{n-1}\}$ has been constructed in $\mathbb{R}^{n-1}$ and

$$\sum_{j=1}^{n} f_j^{n-1} = 0, \; < f_j^{n-1}, f_k^{n-1} >= -\frac{1}{n-1} \quad \text{for } k \neq j.$$

We set $f_{n+1}^n = (0,0,\cdots,1)^T$ and for $j \in \{1,2,\cdots,n\}$

$$f_j^n = c_n (f_j^{n-1}, -h_n)^T$$

Since the vectors are unit, we have

$$1 = \|f_j^n\|^2 = c_n^2(1+h_n^2)$$

Since the vectors are unit, we have

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Hence

$$c_n = \frac{1}{\sqrt{1+h_n^2}}$$

The equality $\sum_{j=1}^{n+1} f_j^n = 0$ results $c_n h_n = \frac{1}{n}$. For the constants c_n and h_n, we have

$$c_n = \frac{\sqrt{n^2-1}}{n} \qquad , \qquad h_n = \frac{1}{\sqrt{n^2-1}}$$

With the right choice of c_n and h_n for $j \neq k$, we have

$$< f_j^n, f_k^n >= c_n^2(< f_j^{n-1}, f_k^{n-1} > + h_n^2) = -\frac{n+1}{n^2} + \frac{1}{n} = -\frac{1}{n}.$$

For $j \in \{1,2,\cdots,n\}$ and $k = n+1$, we have

$$< f_j^n, f_{n+1}^n >= -c_n h_n = -\frac{1}{n}.$$

Thus for all natural numbers n, we can construct a system of unit vectors $\{f_1^n, f_2^n, \cdots, f_{n+1}^n\}$ in $\mathbb{R}^n$ such that

$$\sum_{j=1}^{n+1} f_j^n = 0, \; < f_j^n, f_k^n >= -\frac{1}{n} \text{ for } k \neq j.$$

The construction of the system $\{f_j^n\}_{j=1}^{n+1}$ was first described in [7,8].

Definition 3.1 *A finite family of unit vectors $\{\varphi_j\}_{j=1}^{n+1}$ in $\mathbb{R}^n$ is called a Mercedes-Benz system if*

$$<\varphi_j, \varphi_k> = -\frac{1}{n} \text{ for } k \neq j.$$

Theorem 3.2 *A Mercedes-Benz system $\{f_j^n\}_{j=1}^{n+1}$ in $\mathbb{R}^n$ is a tight frame.*

Proof. We apply the induction on n.
For $n = 1$ and $n = 2$, since $A_1 A_1^T = 2\ I_1$ and $A_2 A_2^T = \frac{3}{2}\ I_2$, then $\{f_j^1\}_{j=1}^2$ and $\{f_j^2\}_{j=1}^3$ are tight frames.
By induction assume $\{f_j^{n-1}\}_{j=1}^n$ in $\mathbb{R}^{n-1}$ be a tight frame such that

$$\sum_{j=1}^{n} | < x, f_j^{n-1} > |^2 = \frac{n}{n-1}\|x\|^2 \quad \forall x \in \mathbb{R}^{n-1}.$$

Consider $x \in \mathbb{R}^n$ and set $x = (x^{n-1}, x_n)^T$, we find

$$\begin{aligned}
\sum_{j=1}^{n+1} | < x, f_j^n > |^2 &= \sum_{j=1}^{n} | < x, f_j^n > |^2 + | < x, f_{n+1}^n > |^2 \\
&= c_n^2 \sum_{j=1}^{n+1} | < x^{n-1}, f_j^{n-1} > |^2 + (n\ c_n^2\ h_n^2 + 1)\ x_n^2 \\
&= c_n^2 \frac{n}{n-1} \|x^{n-1}\|^2 + (n\ c_n^2\ h_n^2 + 1)\ x_n^2 \\
&= \frac{n+1}{n} \|x^{n-1}\|^2 + \frac{n+1}{n} x_n^2 \\
&= \frac{n+1}{n} (\|x^{n-1}\|^2 + x_n^2) \\
&= \frac{n+1}{n} \|x\|^2.
\end{aligned}$$

Theorem 3.3 *A Mercedes-Benz system $\{f_j^n\}_{j=1}^{n+1}$ in $\mathbb{R}^n$ is an equiangular tight frame.*

Proof. By Theorem (3.2), $\{f_j^n\}_{j=1}^{n+1}$ is tight frame.
Because $\{f_j^n\}_{j=1}^{n+1}$ is a Mercedes-Benz system, then by definition (3.1), for $j \neq k$ we have

$$< f_j^n, f_k^n > = -\frac{1}{n}$$

Thus since the angle between any pair of frame vectors is a constant, therefore a Mercedes-Benz system $\{f_j^n\}_{j=1}^{n+1}$ is equiangular tight frame.

4 Classification equiangular tight frames

In this section the first we define the adjacency matrix of a graph. Then show that there exists a one-to-one correspondence between real equiangular tight frames and graphs. This one-to-one correspondence has recently been studied in the case of equiangular tight frames (see, e.g., [2,14,15]). At last by induction on n (space dimensional), we get the number of equiangular tight frames with $n+1$ vectors.

Definition 4.1 *The Seidel matrix or adjacency matrix Q of a graph G with n vertices is the $n \times n$ matrix with a -1 in the (j,k)-entry if the j and k vertices are adjacent (connected by an edge), a 1 if they are nonadjacent, and 0 diagonal entries.*

Since frames are determined to unitary equivalence by their Gramian matrices, the Gramian of an equiangular frame that $< f_j, f_k >= C > 0$ and $\|f_j\|^2 = r$ has the form

$$G = \begin{bmatrix} r & cf_{12} & cf_{13} & \cdots & cf_{1n} \\ c\overline{f_{21}} & r & cf_{23} & \cdots & cf_{2n} \\ c\overline{f_{31}} & c\overline{f_{31}} & r & \cdots & cf_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c\overline{f_{n1}} & c\overline{f_{n2}} & c\overline{f_{n2}} & \cdots & r \end{bmatrix} = rI + c\,Q$$

, where Q is Seidel matrix corresponding to equiangular frame.
Consider the vectors $\{f_j^n\}_{j=1}^{n+1}$ in $\mathbb{R}^n$ that defined in section 3. Now by induction on n (space dimension) we have:

If **n=1**, there are only two vectors $f_1^1 = 1$, $f_2^1 = -1$ in $\mathbb{R}$ that form an equiangular tight frame. Let A_1 and A_1^T be the synthesis and analysis operator associated to $\{f_j^1\}_{j=1}^2$.The Gramian matrix of frame has the form

$$G = A_1^T A_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = I + Q.$$

which $Q = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, is the Siedel matrix correspondence to equiangular tight frame. The corresponding graph to Q is Complete graph K_2 and is as follows:

Figure 2: The complete graph K_2 corresponding equiangular tight frame of two vectors in $\mathbb{R}$.

If **n=2**, we investigate two cases.
Case 1: The vectors $f_1^2 = (0,1)^T$, $f_2^2 = (\frac{\sqrt{3}}{2}, -\frac{1}{2})^T$, $f_3^2 = (-\frac{\sqrt{3}}{2}, -\frac{1}{2})^T$ in $\mathbb{R}^2$ that form an equiangular tight frame. Consider A_2 and A_2^T be the synthesis and analysis operator associated to $\{f_j^2\}_{j=1}^3$.The Gramian matrix of frame has the form

$$G = A_2^T A_2 = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} = I + \frac{1}{2}\, Q.$$

which $Q = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$, is the Siedel matrix correspondence to

equiangular tight frame. The corresponding graph to Q is complete graph K_3. (Figure3 (a))

Figure 3: The complete graph K_3 and complete bigraph $\{K_1, K_2\}$ corresponding equiangular tight frames of three vectors in $\mathbb{R}^2$.

Case 2: If the one of the frame vectors change in the opposite direction, then the vectors $f_1^2 = (0,-1)^T$, $f_2^2 = (\frac{\sqrt{3}}{2}, -\frac{1}{2})^T$, $f_3^2 = (-\frac{\sqrt{3}}{2}, -\frac{1}{2})^T$ in $\mathbb{R}^2$ that form an equiangular tight frame. Consider A_2 and A_2^T be the synthesis and analysis operator associated to $\{f_j^2\}_{j=1}^3$.The Gramian matrix of frame has the form

$$G = A_2^T A_2 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} = I + \frac{1}{2}\, Q.$$

which $Q = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$, is the Siedel matrix correspondence to equiangular tight frame. The corresponding graph to Q is bigraph complete graph $\{K_1, K_2\}$. (Figure3 (b))

If **n=3**, we investigate three cases.
Case 1: The vectors $f_1^3 = (0, 0, 1)^T$, $f_2^3 = (0, \frac{2\sqrt{2}}{3}, -\frac{1}{3})^T$, $f_3^3 = (\frac{\sqrt{6}}{3}, -\frac{\sqrt{2}}{3}, -\frac{1}{3})^T$ and $f_4^3 = (-\frac{\sqrt{6}}{3}, -\frac{\sqrt{2}}{3}, -\frac{1}{3})^T$ in $\mathbb{R}^3$ that form an equiangular tight frame. Consider A_3 and A_3^T be the synthesis and analysis operator associated to $\{f_j^3\}_{j=1}^4$.The Gramian matrix of frame has the form

$$G = A_3^T A_3 = \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 1 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{bmatrix} = I + \frac{1}{3}\, Q.$$

which $Q = \begin{bmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$, is the Seidel matrix correspondence to equiangular tight frame. The corresponding graph to Q is complete graph K_4. (Figure4 (a))

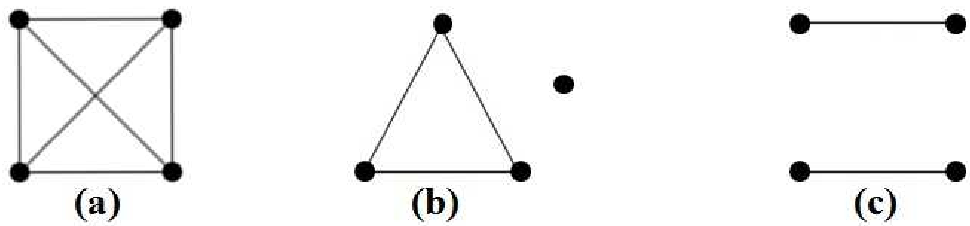

Figure 4: The complete graph K_4, complete bigraph $\{K_1, K_3\}$ and complete bigraph $\{K_2, K_2\}$ corresponding equiangular tight frames of four vectors in $\mathbb{R}^3$.

Case 2: If the one of the frame vectors change in the opposite direction, then the vectors $f_1^3 = (0, 0, -1)^T$, $f_2^3 = (0, \frac{2\sqrt{2}}{3}, -\frac{1}{3})^T$, $f_3^3 = (\frac{\sqrt{6}}{3}, -\frac{\sqrt{2}}{3}, -\frac{1}{3})^T$ and $f_4^3 = (-\frac{\sqrt{6}}{3}, -\frac{\sqrt{2}}{3}, -\frac{1}{3})^T$ in $\mathbb{R}^3$ that form an equiangular tight frame. Consider A_3 and A_3^T be the synthesis and analysis operator associated to $\{f_j^3\}_{j=1}^4$.The Gramian matrix of frame has the form

$$G = A_3^T A_3 = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 1 & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & 1 & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{bmatrix} = I + \frac{1}{3}\, Q.$$

which $Q = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 0 \end{bmatrix}$, is the seidel matrix correspondence to equiangular tight frame. The corresponding graph to Q is complete bigraph $\{K_1, K_3\}$. (Figure4 (b))

Case 3: If the two of the frame vectors change in the opposite direction, then the vectors $f_1^3 = (0, 0, -1)^T$, $f_2^3 = (0, \frac{-2\sqrt{2}}{3}, \frac{1}{3})^T$, $f_3^3 = (\frac{\sqrt{6}}{3}, -\frac{\sqrt{2}}{3}, -\frac{1}{3})^T$ and $f_4^3 = (-\frac{\sqrt{6}}{3}, -\frac{\sqrt{2}}{3}, -\frac{1}{3})^T$ in $\mathbb{R}^3$ that form an equiangular tight frame. Consider A_3 and A_3^T be the synthesis and analysis operator associated to $\{f_j^3\}_{j=1}^4$.The

Gramian matrix of frame has the form

$$G = A_3^T A_3 = \begin{bmatrix} 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & 1 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 1 & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 1 \end{bmatrix} = I + \frac{1}{3}\,Q.$$

which $Q = \begin{bmatrix} 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$, is the seidel matrix correspondence to equiangular tight frame. The corresponding graph to Q is complete bigraph $\{K_2, K_2\}$. (Figure4 (c))

With continued this process, we can obtain the number of equiangular tight frames with $n+1$ vectors in $\mathbb{R}^n$.
If n be an odd number, the number of equiangular tight frames with $n+1$ vectors in $\mathbb{R}^n$ is $\frac{n+3}{2}$. The set $\{\{K_0, K_{n+1}\}, \{K_1, K_n\}, ..., \{K_{n+1-\frac{n+1}{2}}, K_{\frac{n+1}{2}}\}\}$ is consisting of complete bigraph that $\{K_0, K_{n+1}\}$ is complete graph K_{n+1}. Also if n be an even number, the number of equiangular tight frames with $n+1$ vectors in $\mathbb{R}^n$ is $\frac{n+2}{2}$. The set $\{\{K_0, K_{n+1}\}, \{K_1, K_n\}, ..., \{K_{n+1-\frac{n}{2}}, K_{\frac{n}{2}}\}\}$ is consisting of complete bigraph that $\{K_0, K_{n+1}\}$ is complete graph K_{n+1}.

5 Conclusion

In this paper, the Mersedes-Benz frames have been investigated. By using of the correspondence one-to-one between equiangular tight frames, Seidel matrix and graph theory, we obtained the number of these frames with $n+1$ vectors in $\mathbb{R}^n$.

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