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Research Paper

Painlevé analysis, Auto-Backlund transformation and new exact solutions for improved modified KdV equation

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Abstract

Improved modified Korteweg-de Vries (IMKdV) equation is shown to be non-integrable using Painlevé analysis. Exact travelling wave solutions are obtained using auto-Bäcklund transformation and Linearized transformation.

Keywords: *IMKdV equation; Painlevé analysis; extended homogeneous balance method, auto-Bäcklund transformation, Linearized transformation, and exact solutions.*

1. Introduction

Nonlinear evolution equations (NLEEs) are important mathematical models to describe physical phenomena. They are also an important field in the contemporary study of nonlinear physics, especially in soliton theory. The research on the explicit solution and integrability in helpful in clarifying the movement of matter under nonlinear interaction and plays an important role in scientifically explaining the physical phenomena see for example, fluid mechanics, plasma physics, quantum hydrodynamic model, optical fibers, solid state physics, chemical kinematic, chemical physics and geochemistry.
In this paper we will consider the following IMKdV equation as:

$$u_t + u^2 u_x + u_{xxx} - u_{xxt} = 0. \tag{1}$$

The investigation of exact solutions to nonlinear evolution has become an interesting subject in nonlinear science field. Many other methods have been developed, such as the inverse scattering transform [1] Bäcklund transformation method [2-6], Painlevé analysis [7-8], truncated Painlevé analysis [9], bilinear transformation [10], tanh method [11-12], extended homogeneous balance method [13-15], extended tanh function method [16-20] and linearized transformation [21-22]. The Bäcklund transformations (BT) of nonlinear partial differential equations (PDEs) play an important role in soliton theory, which is an efficient method to obtain exact solutions of nonlinear PDEs. In order to obtain the BT of the given nonlinear PDE, various methods, such as Painlevé method [7-8], homogenous balance (HB) method [13-15], have been presented. The paper is organized as follows : After this introduction Section 2, we will confirm whether or not (1) passes the Painlevé test by using WTC method [7]. In Section 3, auto-BTs of the IMKdV equation is obtained by using an extended homogeneous balance method. In Section 4, new exact solutions of (1) are given via linearized transformation. Finally, In section 5, the discussion and conclusion are illustrated [1].

2. Painlevé analysis

The Painlevé analysis for partial differential equations (PDEs) was suggested in Ref. [7] , which required that the solutions should be single valued around movable singularity manifolds. To be precise, if the singularity manifold is determined by

$$\phi(z_1, z_2, z_3, ...) = 0, \tag{2}$$

and $u = u(z_1, z_2, z_3, ..., z_n)$ is a solution of the PDE, then we assume that

$$u = \phi^{\alpha} \sum_{j=0}^{\infty} u_j \phi^j \tag{3}$$

where $\phi(z_1, z_2, z_3, ..., z_n)$, $u_j = u_j(z_1, z_2, z_3, ..., z_n)$, and $u_0 \neq 0$, are analytic functions of (z_j) in a neighborhood of the manifold [7] and a is an integer. Substitution of equation (3) into the PDE determines the allowed values of α and defines the recursion relations for u_j, $j = 0, 1, 2, ...$ When the anzatz equation (1) is correct, the PDE is said to possess the Painlevé analysis and is conjectured to be integrable.

There are essentially four steps involved in the Painlevé analysis of PDEs:
(i) Determination of the leading order behaviors.
(ii) Identification of the powers at which arbitrary functions can enter into the Laurent series called resonances.
(iii) Verifying that at the resonance values sufficient numbers of arbitrary functions exist without the introduction of movable critical manifolds.
Substituting (3) into (1), we can get the $(\alpha = -1)$. Thus, (3) becomes

$$u = \sum_{j=0}^{\infty} u_j \phi^{j-1}, \tag{4}$$

then we have

$$u_x = \sum_{j=0}^{\infty} \left[u_{j,x} \phi^{j-2} + (j-2) u_j \phi^{j-3} \phi_x \right],$$

$$u_t = \sum_{j=0}^{\infty} \left[u_{j,t} \phi^{j-2} + (j-2) u_j \phi^{j-3} \phi_t \right],$$

$$u_{xxx} = \sum_{j=0}^{\infty} [u_{j,xxx}\phi^{j-2} + 3(j-2)u_{j,xx}\phi^{j-3}\phi_x + 3(j-2)u_{j,x}\phi^{j-3}\phi_{xx} + (j-2)u_j\phi^{j-3}\phi_{xxx} + 3(j-2)(j-3)u_{j,x}\phi^{j-4}\phi_x^2 +$$
$$3(j-2)(j-3)u_j\phi^{j-4}\phi_x\phi_{xx} + (j-2)(j-3)(j-4)u_j\phi^{j-5}\phi_x^3],$$

$$u_{xxt} = \sum_{j=0}^{\infty} [u_{j,xxt}\phi^{j-1} + (j-1)u_{j,xx}\phi^{j-2}\phi_t + 2(j-1)u_{j,xt}\phi^{j-2}\phi_x + 2(j-1)(j-2)u_{j,x}\phi^{j-3}\phi_x\phi_t + 2(j-1)u_{j,x}\phi^{j-2}\phi_{xt} +$$
$$2(j-1)(j-2)u_j\phi^{j-3}\phi_x\phi_{xt} + (j-1)u_{j,t}\phi^{j-2}\phi_{xx} + (j-1)(j-2)u_{j,t}\phi^{j-3}\phi_x^2 + (j-1)(j-2)u_j\phi^{j-3}\phi_{xx}\phi_t +$$

$$(j-1)u_j\phi^{j-2}\phi_{xxt} + (j-1)(j-2)(j-3)u_j\phi^{j-4}\phi_t\phi_x^2]. \tag{5}$$

Substituting equations (5) into equation (1) we have the following recursion relation:

$$u_{j-3,t} + (j-3)u_{j-2}\phi_t + u_m u_n u_{j-m-n-1,x} + (j-m-n-1)u_m u_n u_{j-m-n}\phi_x + u_{j-3,3x} +$$

$$3(j-3)u_{j-2,xx}\phi_x + 3(j-3)u_{j-2,x}\phi_{xx} + (j-3)u_{j-2}\phi_{3x} + 3(j-2)(j-3)u_{j-1,x}\phi_x^2 + 3(j-2)(j-3)u_{j-1}\phi_x\phi_{xx} +$$
$$(j-1)(j-2)(j-3)u_j\phi_x^3 - (u_{j-3,xxt} + u_{j-2,xx}\phi_t + 2(j-3)u_{j-2,xt}\phi_x + 2(j-2)(j-3)u_{j-1,x}\phi_x\phi_t +$$
$$2(j-3)u_{j-2,x}\phi_{xt} + (j-2)(j-3)u_{j-1,t}\phi_x^2 + (j-1)(j-2)(j-3)u_j\phi_t\phi_x^2 + 2(j-2)(j-3)u_{j-1}\phi_x\phi_{xt} +$$

$$(j-3)u_{j-2,t}\phi_{xx} + (j-2)(j-3)u_{j-1}\phi_t\phi xx + (j-3)u_{j-2}\phi_{xxt}) = 0. \tag{6}$$

For $j = 0$, in (6), we obtain

$$u_0 = \pm\sqrt{6}\sqrt{\phi_x\phi_t - \phi_x^2}, \tag{7}$$

Substituting from equation (7) into (6), and collecting coefficients of u_j we obtain

$$(j+1)(j-3)(j-4)u_j\phi_x^2(\phi_x - \phi_t) = F_j\left(u_{j-1}, ..., u_0, \phi_t, \phi_x, \phi_{xx}, ...\right), \quad j = 1, 2, 3, ... \tag{8}$$

where F_j is a non-linear function. We can see that $j = -1, 3, 4,$ are resonances at which u_j becomes arbitrary. Resonance at -1 corresponds to the arbitrariness of ϕ. For $j = 1$, in (8) or (6), we obtain

$$u_1 = \frac{-1}{2\sqrt{6}(\phi_x\phi_t - \phi_x^2)^{3/2}}\left(\phi_x^2\phi_{tt} - 6\phi_x^2\phi_{xt} + 4\phi_x\phi_t\phi_{xt} + \phi_{xx}\phi_t^2 - 6\phi_x\phi_t\phi_{xx} + 6\phi_x^2\phi_{xx}\right), \tag{9}$$

There is incompatibility at $j = 2, 3$ and the recurrence relation is too lengthy and complicated at $j = 3$. From this analysis we see that IMKdV is non-Painlevè and because of Painlevé conjecture it is non-integrable.

3. Auto-Bäcklund transformations for IMKdV equation

According the idea of improved HB [23-24], we seek for ABT of Eq. (1), when balancing uu_x with u_{xxx} then gives $N = 1$. therefore, we may choose

$$u(x,t) = \frac{\partial f(w)}{\partial x} = f'(w)w_x + a, \tag{10}$$

where a is a constant, f, w are functions to be determined later,

$$u_t = f''w_tw_x + f'w_{xt},$$

$$u_x = f''w_x^2 + f'w_{xx},$$

$$u^2u_x = f''f^{(\prime)^2}w_x^4 + f^{(\prime)^3}w_x^2w_{xx} + 2af''f'w_x^3 + 2af^{(\prime)^2}w_xw_{xx} + a^2f''w_x^2 + a^2f'w_{xx},$$

$$u_{xxx} = f^{(4)}w_x^4 + 6f'''w_x^2w_{xx} + f''(3w_{xx}^2 + 4w_x^2w_{xxx}) + f'w_{xxxx},$$

$$u_{xxt} = f^{(4)}w_x^3w_t + 3f'''w_x^2w_{xt} + 3f'''w_tw_xw_{xx} + 3f''w_{xx}w_{xt} + 3f''w_xw_{xxt} + f''w_tw_{xxx} + f'w_{4x}. \tag{11}$$

Substituting (11), into equation (1), we have

$$u_t + u^2u_x + u_{xxx} - u_{xxt} = f^{(4)}(w_x^4 - w_tw_x^3) + f'''(6w_x^2w_{xx} + 3w_tw_xw_{xx} - 3w_x^2w_{xt}) +$$

$$f''(w_xw_t + a^2w_x^2 - 3w_{xt}w_{xx} + 3w_{xx}^2 - 3w_xw_{xxt} - w_tw_{3x} + 4w_xw_{3x}) + f'(w_{xt} + a^2w_{xx} - w_{3xt} + w_{4x}) +$$

$$2af'f''w_x^3 + f''f^{(\prime)^2}w_x^4 + 2af^{(\prime)^2}w_xw_{xx} + f^{(\prime)^3}w_x^2w_{xx} = 0. \tag{12}$$

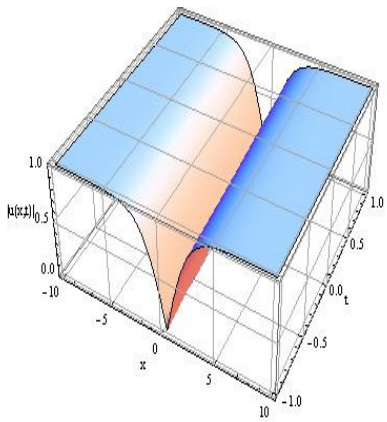
We assume the solution as the form

$$f(w) = cln(w), \tag{13}$$

substituting from equation (13) into equation (12), we obtain

$$f^{(4)}(w_x^4 - w_tw_x^3 + \frac{c^2}{6}) + f'''(-acw_x^3 + 6w_x^2w_{xx} - 3w_tw_xw_{xx} - 3w_x^2w_{xt} + \frac{c^2}{2}) +$$

$$f''(w_xw_t + a^2w_x^2 - 3w_{xt}w_{xx} + 3w_{xx}^2 - 3w_xw_{xxt} - w_tw_{3x} + 4w_xw_{3x} - 2acw_xw_{xx}) + f'(w_{xt} + a^2w_{xx} - w_{3xt} + w_{4x}) = 0. \tag{14}$$



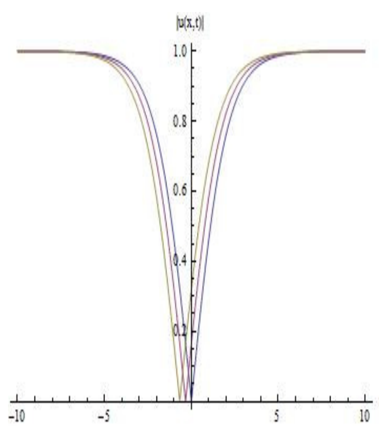


Fig. 1: The solitary solution $u(x,t)$ defined in equation (17).

To obtain the solution, we set the coefficients of $f^{(4)}, f''', f''$ and f' equal zero and we assumed $w(x,t)$ as the form:

$$w(x,t) = 1 + e^{\theta}, \quad where \quad \theta = \lambda(t) + kx. \tag{15}$$

Substituting from (15) into (14) we get

$$c = \pm\sqrt{-2(3+a^2)}, \quad k = \mp\frac{\sqrt{2}a}{\sqrt{-(3+a^2)}}, \quad \lambda(t) = \pm\frac{\sqrt{2}a^3}{\sqrt{-3(3+a^2)}}. \tag{16}$$

Substituting form (16) in the auto-Backlund transformation (13) gives the solution of (1) provided that $a = \sqrt{-1}$

$$u(x,t) = -itanh\left(\frac{1}{6}(3x+t)\right). \tag{17}$$

We have represented this solution (17) for a set of parameter values in Fig. 1

4. Linearized transformation for IMKdV equation

By using the linearized transformation [21], we find the solution for the IMKdV equation (1) by substitution of the following:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{in(k_1 x - \omega t)}. \tag{18}$$

To deal with the nonlinear terms of equation (1) we need to employ the extension of Cauchy's product rule for multiple series:

$$\prod_{i=1}^{p} F^i = \sum_{n=p}^{\infty}\sum_{m=p-1}^{n-1} \cdots \sum_{r=2}^{k-1}\sum_{s=1}^{r-1} f_s^1 f_{r-s}^2 \cdots f_{n-m}^p, \tag{19}$$

where

$$F_i \sum_{k=1}^{\infty} f_k^i \qquad (i = 1,2,3,...,p), \tag{20}$$

If we substitute the solution u(x,t) into equation (1) and apply Cauchy's rule for the double product appearing in the nonlinear term, then we obtain

$$\sum_{n=1}^{\infty}[-in\omega - in^3k^3 - in^3k^2\omega]A_n e^{in(kx-\omega t)} + ik\sum_{n=3}^{\infty}\sum_{m=2}^{n-1}\sum_{\ell=1}^{m-1}(\ell)A_\ell A_{m-\ell}A_{n-m}e^{in(k-\omega t)} \tag{21}$$

Now, we deriving a recursion relation and we determine the coefficients A_n. Firstly, we put $n = 1$, we obtain the dispersion relation $\omega = \frac{-k^3}{1+k^2}$ and $A_1 \neq 0$ is arbitrary. Secondly, we put $n = 2$, we see that the coefficients $A_2 = A_4 = A_6 = ... = A_{2n} = 0$. Then, we can determine the expansion coefficients by the following recursion relation:

$$n(n^2-1)A_n e^{in(kx-\omega t)} = \frac{1+k^2}{k^2} \sum_{m=2}^{n-1} \sum_{\ell=1}^{m-1} \ell A_\ell A_{m-\ell} A_{n-m} e^{in(kx-\omega t)} \tag{22}$$

If we put $n = 3, 5, 7, ...$ in equation (22), we find the coefficients $A_3, A_5, A_7, ...$ respectively as the following:

$$A_3 = \frac{1+k^2}{24k^2} A_1^3, \quad A_5 = \left(\frac{1+k^2}{24k^2}\right)^2 A_1^5, A_7 = \left(\tfrac{1+k^2}{24k^2}\right)^3 A_1^7, \quad A_9 = \left(\tfrac{1+k^2}{24k^2}\right)^4 A_1^9. \tag{23}$$

Substituting from equation (23) into equation (18), we obtain

$$u(x,t) = A_1 e^{i(kx-\omega t)} + A_3 e^{3i(kx-\omega t)} + A_5 e^{5i(kx-\omega t)} + A_7 e^{7i(kx-\omega t)} + A_9 e^{9i(kx-\omega t)} + ... =$$

$$A_1 e^{i(kx-\omega t)} + \left(\frac{1+k^2}{24k^2}\right) A_1^3 e^{3i(kx-\omega t)} + \left(\frac{1+k^2}{24k^2}\right)^2 A_1^5 e^{5i(kx-\omega t)} + \left(\frac{1+k^2}{24k^2}\right)^3 A_1^7 e^{7i(kx-\omega t)} + ... \tag{24}$$

If we take $A_1 = 2k\sqrt{6/(1+k^2)}$, then equation (24) takes the form:

$$u(x,t) = 2k\sqrt{\frac{6}{1+k^2}} e^{i(kx-\omega t)} \left(1 + e^{2i(kx-\omega t)} + e^{4i(kx-\omega t)} + e^{6i(kx-\omega t)}\right) ....= ik\sqrt{\tfrac{6}{1+k^2}} cosec \left(kx + \tfrac{k^3}{1+k^2} t\right). \tag{25}$$

If we take $k = a$ in (25), then we can obtain the solitary wave solution of (1) as:

$$u(x,t) = a\sqrt{\frac{6}{1+a^2}} csc \left(a\left(x + \frac{a^2}{1+a^2} t\right)\right). \tag{26}$$

Also if we take $k = ia$ in (25), then we can obtain the solitary wave solution of (1) as:

$$u(x,t) = ia\sqrt{\frac{6}{1-a^2}} cosech \left(a\left(x - \frac{a^2}{1-a^2} t\right)\right). \tag{27}$$

In (24) if we take $k = ia$ and $A_1 = 2a\sqrt{6/(1-a^2)}$, we obtain a new solitary solution of (1)

$$u(x,t) = a\sqrt{\frac{6}{1-a^2}} sech \left(a\left(x - \frac{a^2}{a^2-1} t\right)\right). \tag{28}$$

We have represented these solutions (25)-(27) for a set of parameter values in Figs. (2)-(4) respectively.

5. Conclusion

In this paper, the Bäcklund transformations and a series of new exact explicit solutions of the IMKdV equation have been established. An extension of the homogeneous balance method was successfully used to develop these solutions. The solutions include, the algebraic solitary wave solution of rational function, single-soliton solutions, singular traveling solutions, and the periodic wave solutions of trigonometric function type. Linearized transformation method was described to find exact solutions of the Improved Modified KdV (IMkdV) equation. Consequently, three exact soliton solutions were obtained to the IMkdV equation. In spite of the fact that these new soliton solutions may be important for physical problems, this study also suggests that one may find different solutions by choosing different methods. Therefore, this method can be utilized to solve many equations of nonlinear partial differential equation arising in the theory of soliton and other related areas of research.

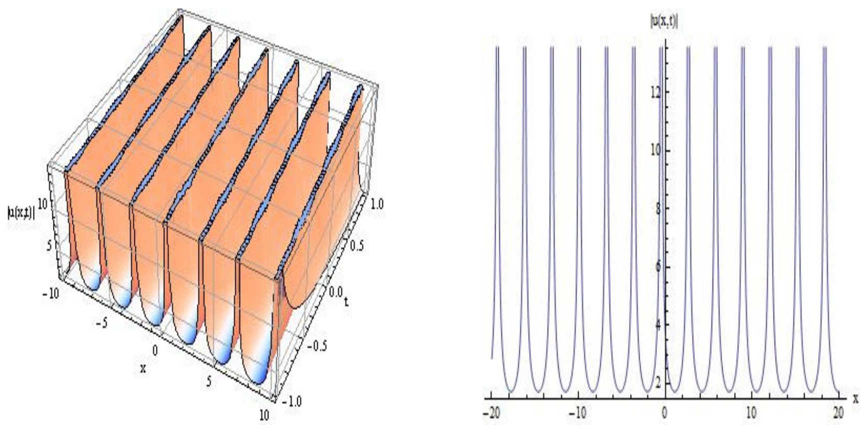


Fig. 2: The solitary solution $u(x,t)$ defined in equation (26).

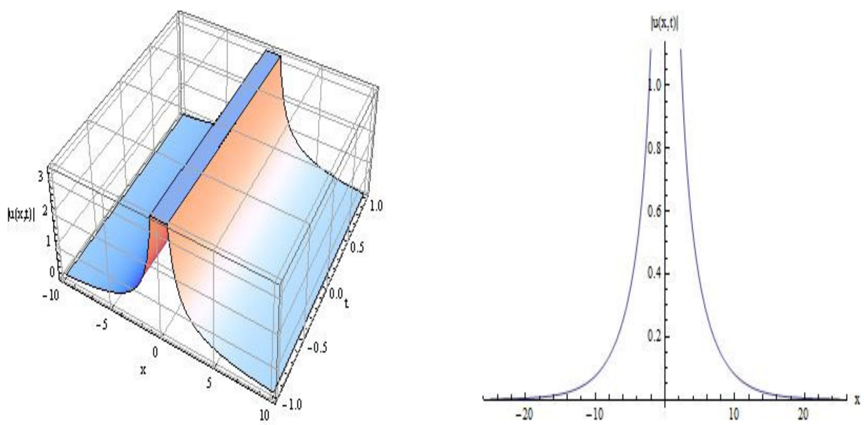


Fig. 3: The solitary solution $u(x,t)$ defined in equation (27).

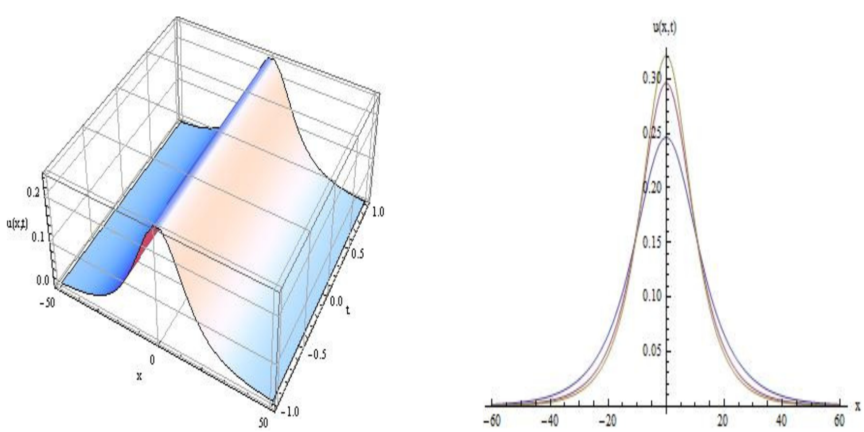


Fig. 4: The solitary solution $u(x,t)$ defined in equation (28).

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