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Research Paper

Solving the Kuramoto-Sivashinsky equation via Variational Iteration Method

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Abstract

In this study, the approximate solutions for the Kuramoto-Sivashinsky equation by using the Variational Iteration Method (VIM) are obtained. Comparisons with the exact solutions and the solutions obtained by the Homotopy Perturbation Method (HPM), the numerical example show that the Variational Iteration Method (VIM) is accurate and effective and suitable for this kind of problem.

Keywords: *Kuramoto-Sivashinsky equation, Variational Iteration Method.*

1. Introduction

It is well known that most of the phenomena that arise in mathematical physics and engineering fields can be described by partial differential equations (PDEs) [1], one of the recently method for solving (PDEs) is Variational Iteration Method, VIM was introduced by Ji-Huan He in 1997.[2] The method has been favorably applied to various kinds of problems; for example, this scheme is used for solving the fractional KdV–Burgers–Kuramoto equation.[3] This technique computes the exact solution of equations using the initial condition only. It is also important to note that the present method does not require discretization of the equation. Therefore, it is not affected by computation round-off errors and one is not faced with the necessity of large computer memory and time. Furthermore, using this idea we do not need to solve any linear or nonlinear system of equations. (VIM) is employed to solve fourth-order parabolic equations. [4] Also, this method is employed to solve delay differential equations. [5]

2. Mathematical model

The Kuramoto–Sivashinsky equation was derived by Kuramoto [6] as a model for phase turbulence in reaction diffusion systems and by Sivashinsky [7] as a model for plane flame propagation. This equation describes many kinds of physical phenomenons such as the flow of thin liquid films on inclined planes and dendritic fronts in dilute binary alloys. Boundary control of Kuramoto-Sivashinsky equation has practical applications in engineering. However, even after we linearize the Kuramoto-Sivashinsky equation by dropping the quadratic convective term, the problems of the boundary control of the linearized Kuramoto-Sivashinsky equation are still largely unexplored (see [8]), and there are few results obtained.[9]
Consider the Kuramoto-Sivashinsky equation [10]

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\alpha\frac{\partial^2 u}{\partial x^2}+\gamma\ \frac{\partial^3 u}{\partial x^3}+\beta\ \frac{\partial^4 u}{\partial x^4}=0 \qquad (1)$$

Where α , γ and β are arbitrary constants,
Subject to the initial condition

$$u(x,0)=f(x) \qquad a\le x\le b \qquad (2)$$

And boundary conditions

$$u(a,t)=g_1(t),\quad u(b,t)=g_2(t),\qquad t\ge 0\,. \qquad (3)$$

And Neumann boundary condition

$\frac{\partial^2 u}{\partial x^2} = h_1$, at $x = a$ and $x = b$ where $h_1 \geq 0$. (4)

3. Basic idea of Variational Iteration Method

To clarify the basic ideas of VIM, we consider the following differential equation

$$Lu + Nu = g(x,t) \tag{5}$$

Where L is a linear operator defined by $L = \frac{\partial^m}{\partial t^m}$, $m \in \mathbb{N}$, N is a nonlinear operator and $g(x,t)$ is a known analytic function. According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left[Lu_n(x,\tau) + N\tilde{u}_n(x,\tau) - g(x,\tau)\right] d\tau \tag{6}$$

Where λ is a general lagrangian multiplier defined as: [11]

$$\lambda(t,\tau) = \frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1} \quad , m \geq 1. \tag{7}$$

The subscript n indicates the nth approximation and $\tilde{u}_n$ is considered as a restricted variation. [2]

4. Derivative of (VIM) for Kuramoto-Sivashinsky equation

To solve Kuramoto-Sivashinsky equation (1), with initial condition (2) by means of VIM, we construct a correction functional:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left(\frac{\partial u_n(x,\tau)}{\partial \tau} + u_n(x,\tau)\frac{\partial \tilde{u}_n(x,\tau)}{\partial x} + \alpha \frac{\partial^2 \tilde{u}_n(x,\tau)}{\partial x^2} + \gamma \frac{\partial^3 \tilde{u}_n(x,\tau)}{\partial x^3} + \beta \frac{\partial^4 \tilde{u}_n(x,\tau)}{\partial x^4}\right) d\tau \tag{8}$$

In our equation, $m = 1$, then by formula (7), $\lambda = -1$ substituting in equation (8) we get:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\tau)}{\partial \tau} + u_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial x} + \alpha \frac{\partial^2 u_n(x,\tau)}{\partial x^2} + \gamma \frac{\partial^3 u_n(x,\tau)}{\partial x^3} + \beta \frac{\partial^4 u_n(x,\tau)}{\partial x^4}\right) d\tau,\ n = 0,1,... \tag{9}$$

We start with the initial approximation of $u(x,0)$ given by equation (2). Using the iteration formula (9), we can obtain the other components as follows:

$$u_0(x,t) = u(x,0) = f(x). \tag{10}$$

For $n = 0$;

$$u_1(x,t) = u_0(x,t) - \int_0^t \left(\frac{\partial u_0(x,\tau)}{\partial \tau} + u_0(x,\tau)\frac{\partial u_0(x,\tau)}{\partial x} + \alpha \frac{\partial^2 u_0(x,\tau)}{\partial x^2} + \gamma \frac{\partial^3 u_0(x,\tau)}{\partial x^3} + \beta \frac{\partial^4 u_0(x,\tau)}{\partial x^4}\right) d\tau \tag{11}$$

For $n = 1$;

$$u_2(x,t) = u_1(x,t) - \int_0^t \left(\frac{\partial u_1(x,\tau)}{\partial \tau} + u_1(x,\tau)\frac{\partial u_1(x,\tau)}{\partial x} + \alpha \frac{\partial^2 u_1(x,\tau)}{\partial x^2} + \gamma \frac{\partial^3 u_1(x,\tau)}{\partial x^3} + \beta \frac{\partial^4 u_1(x,\tau)}{\partial x^4}\right) d\tau \tag{12}$$

And by the same way for $n = 2,3,...$

5. Applications

In this section, we have solved the Kuramoto-Sivashinsy equation numerically by using Variational Iteration Method (VIM). For clarifying, we take the following example:

Example:

Consider Kuramoto-Sivashinsky equation by [12]

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0\,, \qquad x \in [0,32\pi], \qquad t \in [0,0.001]\,; \tag{13}$$

With the initial condition

$$u(x,0) = cos\left(\frac{x}{16}\right)\left(1 + sin\left(\frac{x}{16}\right)\right), \tag{14}$$

And the exact solution of the problem is given by

$$u(x,t) = cos\left(\frac{x}{16} - t\right)\left(1 + sin\left(\frac{x}{16} - t\right)\right). \tag{15}$$

For solving by (VIM) we obtain the recurrence relation

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\tau)}{\partial \tau} + u_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial x} + \frac{\partial^2 u_n(x,\tau)}{\partial x^2} + \frac{\partial^4 u_n(x,\tau)}{\partial x^4}\right) d\tau, \quad n = 0,1,... \tag{16}$$

Starting with the initial approximation

$$u_0(x,t) = \cos\left(\frac{x}{16}\right)\left(1 + \sin\left(\frac{x}{16}\right)\right) \tag{17}$$

$$u_1(x,t) = \cos\left(\frac{x}{16}\right)\left(1+\sin\left(\frac{x}{16}\right)\right) + t\left(\frac{8447\cos\left(\frac{x}{16}\right)}{65536} - \frac{\cos^3\left(\frac{x}{16}\right)\sin\left(\frac{x}{16}\right)}{8} - \frac{3\cos^3\left(\frac{x}{16}\right)}{16} + \frac{575\sin\left(\frac{x}{16}\right)\cos\left(\frac{x}{16}\right)}{4096}\right) \quad (18)$$

$$u_2(x,t) = \cos\left(\frac{x}{16}\right)\left(1+\sin\left(\frac{x}{16}\right)\right) + t\left(\frac{3839\cos\left(\frac{x}{16}\right)}{32768} - \frac{\cos^3\left(\frac{x}{16}\right)\sin\left(\frac{x}{16}\right)}{8} - \frac{3\cos^3\left(\frac{x}{16}\right)}{16} + \frac{575\sin\left(\frac{x}{16}\right)\cos\left(\frac{x}{16}\right)}{4096} - \frac{\sin\left(\frac{x}{4}\right)}{64} - \frac{3\cos\left(\frac{3x}{16}\right)}{64} + \frac{319\sin\left(\frac{x}{8}\right)}{8192319}\right) + t^2\left(\frac{25\cos\left(\frac{5x}{16}\right)}{4096} - \frac{7780863\cos\left(\frac{x}{16}\right)}{4294967296} - \frac{6765\cos\left(\frac{3x}{16}\right)}{2097152} - \frac{1207\sin\left(\frac{x}{4}\right)}{131072} - \frac{119199\sin\left(\frac{x}{8}\right)}{33554432} + \frac{3\sin\left(\frac{3x}{8}\right)}{2048}\right) + t^3\left(\frac{440637\cos\left(\frac{3x}{16}\right)}{17179869184} + \frac{34435\cos\left(\frac{5x}{16}\right)}{134217728} - \frac{1127873\cos\left(\frac{x}{16}\right)}{17179869184} - \frac{21\cos\left(\frac{7x}{16}\right)}{131072} - \frac{\sin\left(\frac{x}{2}\right)}{32768} - \frac{27937\sin\left(\frac{x}{4}\right)}{1073741824} + \frac{89601\sin\left(\frac{x}{8}\right)}{137438953472} + \frac{2685\sin\left(\frac{3x}{8}\right)}{8388608}\right) \quad (19)$$

Then by the same way for $u_3(x,t), u_4(x,t), \ldots$

Table 1: Absolute Error of (HPM) and (VIM) At 3rd Order, When $t = 0.0004$

$x \times \pi$	$\lvert u_{exact} - u_{HPM} \rvert$	$\lvert u_{exact} - u_{VIM} \rvert$
0	$3.766364391323274 \times 10^{-4}$	$3.297496586894821 \times 10^{-4}$
6.4	$6.750910825057410 \times 10^{-4}$	$6.174481700532697 \times 10^{-4}$
12.8	$1.248599280267992 \times 10^{-4}$	$1.511274889944847 \times 10^{-4}$
19.2	$3.679513787964717 \times 10^{-4}$	$3.862367434608327 \times 10^{-4}$
25.6	$5.536563414442614 \times 10^{-5}$	$5.260390638544416 \times 10^{-5}$
32	$3.766364391323274 \times 10^{-4}$	$3.297496586894821 \times 10^{-4}$

Table 2: Absolute Error of (HPM) and (VIM) At 3rd Order, when $t = 0.0008$

$x \times \pi$	$\lvert u_{exact} - u_{HPM} \rvert$	$\lvert u_{exact} - u_{VIM} \rvert$
0	$7.534327929941131 \times 10^{-4}$	$6.596600858967960 \times 10^{-4}$
6.4	$1.349941768170049 \times 10^{-3}$	$1.234642967767430 \times 10^{-3}$
12.8	$2.501510819914454 \times 10^{-4}$	$3.026732222584094 \times 10^{-4}$
19.2	$7.360770433769148 \times 10^{-4}$	$7.726453090434737 \times 10^{-4}$
25.6	$1.105927804877852 \times 10^{-4}$	$1.050693972385001 \times 10^{-4}$
32	$7.534327929941131 \times 10^{-4}$	$6.596600858967960 \times 10^{-4}$

A **B**

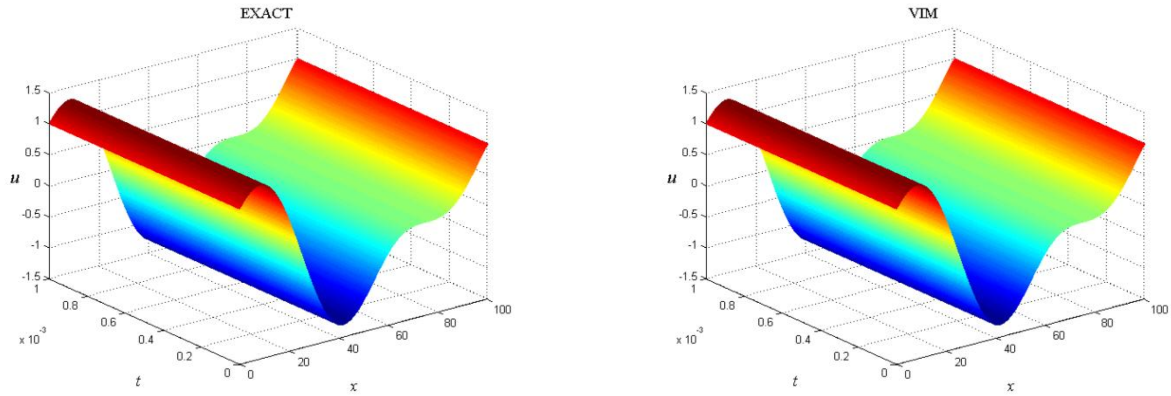

Fig. 1: The Surface shows the solution $u(x,t)$, when $x \in [0,32\pi], t \in [0,0.001]$: (**A**) Exact solution, (**B**) 3rd Order of approximate solution (VIM)

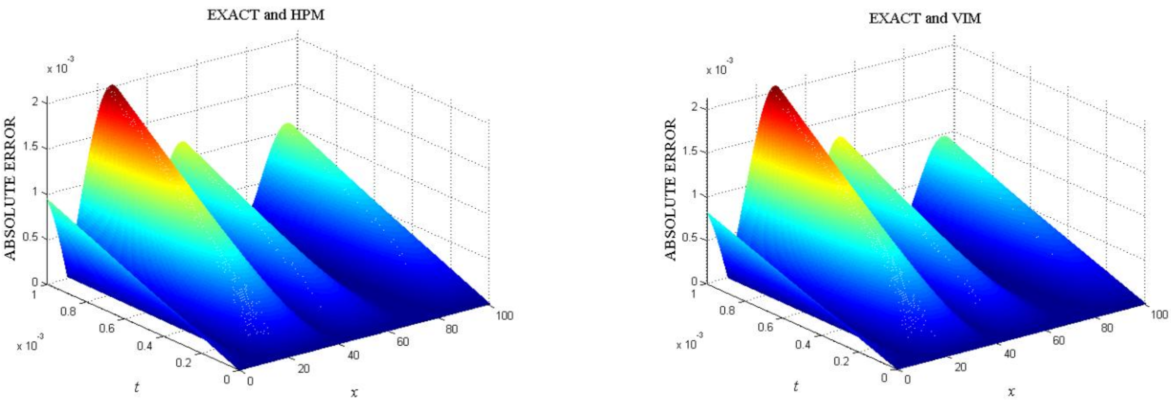

Fig. 2: Absolute error of exact solution, when $x \in [0,32\pi], t \in [0,0.001]$ and: (**A**) 3rd Order of (HPM), (**B**) 3rd Order of (VIM) approximate solution

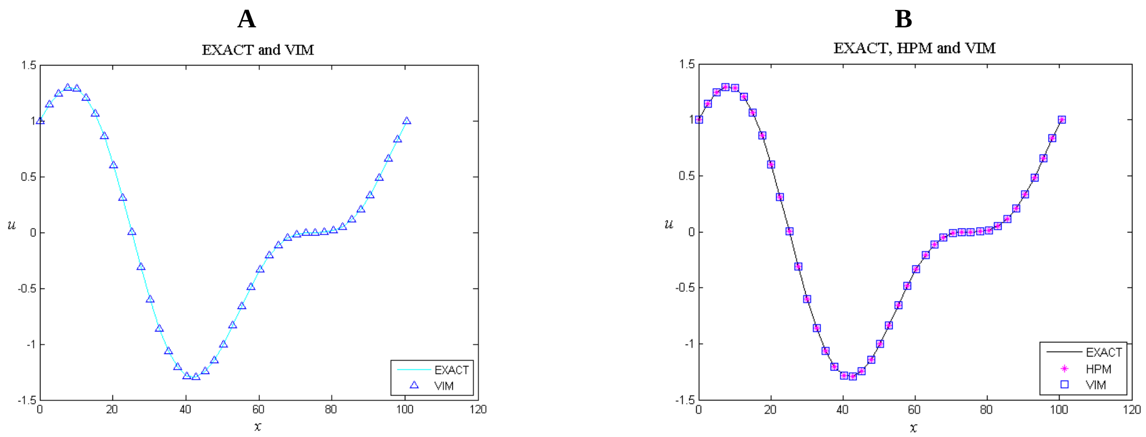

Fig. 3: The curve shows the solution $u(x,t)$, when $x \in [0,32\pi], t = 0.0004$: (**A**) Exact solution and 3rd Order of (VIM), (**B**) Exact Solution, and 3rd Order of (HPM) and 3rd Order of (VIM).

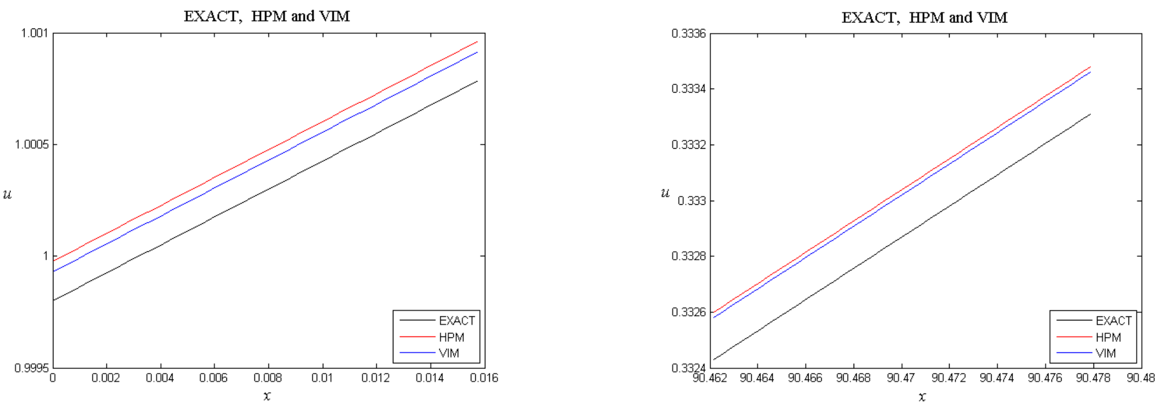

Fig. 4: The zoom curve shows the solution $u(x,t)$ of exact solution, 3rd Order of (HPM) and 3rd Order of (VIM): (**A**) when $x \in [0,0.005\pi]$, $t = 0.0004$, (**B**) when $x \in [28.795\pi, 28.8\pi], t = 0.0004$:.

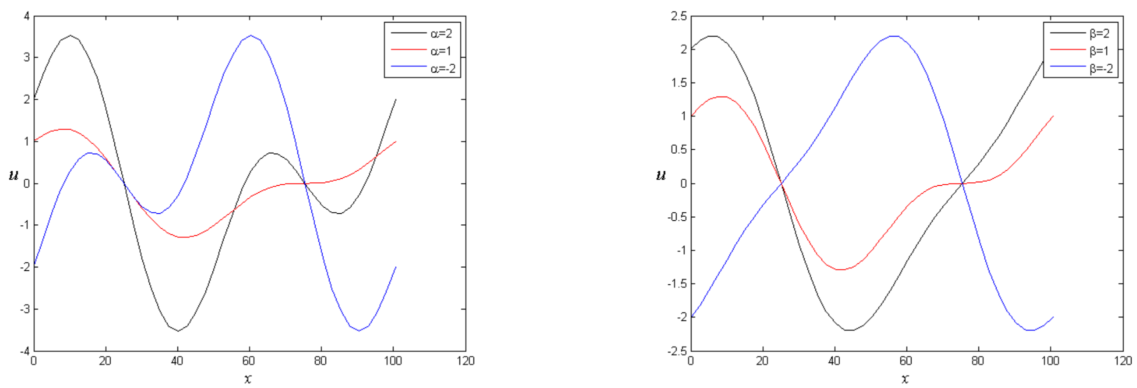

Fig. 5: The curve shows the solution $u(x,t)$ By 3rd Order of (VIM), when $\in [0,32\pi], t = 0.0004$: (**A**) when $\alpha = -2$, $\alpha = 1$, $\alpha = 2$, and $\beta = 1$ (**B**) when $\beta = -2$, $\beta = 1$, $\beta = 2$, and $\alpha = 1$.

6. Conclusion

The Variational Iteration Method (VIM) applied to Kuramoto-Sivashinsky equation and comparing 3rd order of VIM with the exact solution and 3rd order of Homotopy Perturbation Method (HPM) those obtained by Fadhil [13], by Fig.1, Fig.2, Fig.3 and Fig.4, and the absolute error between them in Table1 and Table2. Show that the Variational Iteration Method (VIM) is more accurate and the absolute error is so small and the approximate solution is so closed to the exact solution, Fig.5 show that α , β are so effective in this model.

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