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Coupled Coincidence and Common Fixed Point Theorems in Menger PM Spaces

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Abstract

Recently Many results on coupled fixed point theory exist in the literature, for more details, one can see in [2,11,22]. We established coupled coincidence and common fixed point theorems for two self mixed g-monotone mappings in the settings of Menger PM-spaces. Our result is also substantiated with the aid of an appropriate example and some open problems are also suggested for further studies.

Keywords: *Menger PM-spaces,Coupled fixed point, Coupled coincidence point,Partially ordered set and Mixed monotone mappings.*

1 Introduction

Menger [10] introduced the notion of a probabilistic metric space in 1942, since then the theory of probabilistic metric spaces is an important generalization of the metric space and appears to be of interest in the investigation of physical quantities and physiological thresholds (see [5,6,8,14,17]).It is also of fundamental importance in probabilistic functional analysis. In recent years there has been a growing interest in studying the existence of fixed points for contractive mappings satisfying monotone properties in ordered metric spaces. This trend was initiated by Ran and Reurings in [1] where they extended the Banach contraction principle in partially ordered sets with some applications to matrix equations. Recently, Ran and Reurings [1], Bhaskar and Lakshmikantham [20], Nieto and Lopez [7],Agarwal, El-Gebeily and O'Regan [15] and Lakshmikantham and Ciric [21] presented some new results for contractions in partially ordered metric spaces. The main idea in [1,7] involve combining the ideas of the iterative technique in the contraction mapping principle with those in the monotone technique. Bhaskar and Lakshmikantham [20] introduced the notion of a coupled fixed point and proved some coupled fixed point theorems for mixed monotone mappings in ordered metric spaces. Afterwards, Lakshmikantham and ciric [21] had established coupled coincidence and coupled fixed point theorems for two mappings F and g where F has the mixed g-monotone property. Many other results on coupled fixed point theory exist in the literature, for more details, we refer the reader to [2,11,12,22].

2 Preliminary Notes

Definition 2.1. [4] *A mapping* $\Delta : [0,1]\times[0,1] \to [0,1]$ *is called a triangular norm (shortly t-norm) if*

(i) $\Delta(a,1) = a, \Delta(a,0) = 0,$

$(ii)\Delta(a,b) = \Delta(b,a),$

$(iii)\Delta(a,b) \leq \Delta(c,d)$ *for* $a \leq c, b \leq d,$

$(iv)\Delta(\Delta(a,b)),c) = \Delta(a\Delta(b,c))$ *for all* $a,b,c \in [0,1]$.

Remark 2.1. *The following are the basic t-norms:*

(i) *The minimum t-norm:* $T_M(a,b) = \min\{a,b\}$.

(ii) *The weakest t-norm, the drastic product:*

$$H(x) = \begin{cases} \min(a,b) & if\ \max(a,b) = 1 \\ 0 & otherwise \end{cases}$$

Throughout this paper, Δ stands for an arbitrary continuous t-norm.

Definition 2.2. [4] *A mapping* $\mathcal{F} : \Re \to \Re^+$ *is called distribution function if it is non-decreasing, left continuous with*

$$\inf\{\mathcal{F}(t) : t \in \Re\} = 0, \qquad \sup\{\mathcal{F}(t) : t \in \Re\} = 1.$$

Let L be the set of all distribution functions whereas H stands for the specific distribution function (also known as Heaviside function) defined by

$$H(x) = \begin{cases} 0\ if\ x \leq 0 \\ 1\ if\ x > 0 \end{cases}$$

Definition 2.3. [4, 10] *A triplet* $(X, \mathcal{F}, \Delta)$ *is called a Menger probabilistic metric space (for short, a Menger PM-space) if* X *is a non empty set,* Δ *is a t-norm and* $\mathcal{F}$ *is a mapping from* $X \times X$ *to* $\mathfrak{D}$ *satisfying the following conditions: for all* $x,\ y,\ z \in X$ *we denote* $\mathcal{F}(x, y)$ *by* $F_{x,y}$.*:*
(MS-1)$F_{x,y}(t) = H(t)$ *for all* $t \in R$ *if and only if* $x = y$;
(MS-2)$F_{x,y}(t) = F_{y,x}(t)$; *for all* $x, y \in X$ *and* $t \in R$;
(MS-3)$F_{x,y}(s + t) \geq \Delta(F_{x,z}(s), F_{z,y}(t))$ *for all* $x, y, z \in X$ *and* $s, t \geq 0$.

Remark 2.2. [3, 5] *Point out that if the t-norm* Δ *of a Menger PM-space* $(X, \mathcal{F}, \triangle)$ *satisfies the condition* $\sup_{0<t<1} \Delta(t, t) = 1$, *then* $(X, \mathcal{F}, \triangle)$ *is a Hausdorff topologic space in the* (ε, λ)*-topology* $\mathcal{J}$ *i.e. the family of sets*

$$\{\cup_x(\varepsilon, \lambda) : \varepsilon > 0, \lambda \in (0, 1]\} \quad (x \in X)$$

is a basis of neighbourhood of point x *for* $\mathcal{J}$, *where*

$$\cup_x(\varepsilon, \lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\}.$$

By virtue of this topology $\mathcal{J}$, *a sequence* $\{x_n\}$ *in* $(X, \mathcal{F}, \triangle)$ *is said to be* $\mathcal{J}$*-convergent to* $x \in X$ *(we write* $x_n \overrightarrow{\mathcal{J}} x$ *if* $lim_{n\to\infty} Fx_n, x(t) = 1$ *for all* $t > 0; \{x_n\}$ *is called a* $\mathcal{J}$*-Cauchy sequence in* $(X, \mathcal{F}, \triangle)$ *if for any given* $\varepsilon > 0$ *and* $\lambda \in (0, 1]$, *there exists a positive integer* $N = N(\varepsilon, \lambda)$ *such that* $lim_{n\to\infty} Fx_n, x_m(\varepsilon) > 1 - \lambda$, *whenever* $n, m \geq N$; $(X, \mathcal{F}, \triangle)$ *is said to be* $\mathcal{J}$*-complete, if each* $\mathcal{J}$*-Cauchy sequence in* X *is* $\mathcal{J}$*-convergent to some point in* X.
in the sequel, we will always assume that $(X, \mathcal{F}, \triangle)$ *is a Menger space with the* (ε, λ)*-topology.*

Lemma 2.1. [9] *Let* $\{y_n\}$ *be a sequence in a Menger PM-space* $(X, \mathcal{F}, \triangle)$, *where the t-norm* $\Delta = \Delta_M$. *If there exists a function* $\phi \in \Phi$ *such that*

$$F_{y_n,y_{n+1}}(\phi(t)) \geq \min\{[F_{y_{n-1},y_n}(t), F_{y_n,y_{n+!}}(t)$$

for all $t > 0$, $n \in \mathbb{N}$. *The* $\{y_n\}$ *is a Cauchy sequence in* X.

Lemma 2.2. [18] *If* $(X, \mathcal{F}, \triangle)$ *is a Menger PM-space,* Δ *is continuous, then probabilistic distance function* F *is a low semi continuous function of points,i.e. for every fixed point* $t > 0$, *if* $x_n \to x, y_n \to y$, *then* $\lim_{n\to\infty} F_{x_n,y_n}(t) = F_{x,y}(t)$.

Lemma 2.3. [9] *Let* $(X, \mathcal{F}, \triangle)$ *is a Menger PM-space where the t-norm* $\Delta = \Delta_M$. *and* $x, y \in X$. *If there exists a function* $\phi \in \Phi$ *such that*

$$F_{x,y}(\phi(t) + 0) \geq F_{x,y}(t) \quad \textit{for all} \quad t > 0.$$

then $x = y$.

Definition 2.4. [20] *An element $(x, y) \in X \times X$ is called coupled fixed point of the mapping $G : X \times X \to X$ if*

$$G(x, y) = x, \quad G(y, x) = y.$$

Definition 2.5. [20] *Let $(X, \preceq)$ be a partially ordered set and $G : X \times X \to X$. The mapping G is said to has the mixed monotone property if G is monotone, non-decreasing in its first argument and is monotone non-increasing in its second argument, that is, for any $x, y \in X$,*

$$x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow G(x_1, y) \preceq G(x_2, y) \tag{1}$$

and

$$y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow G(x, y_1) \preceq G(x, y_2) \tag{2}$$

Definition 2.6. [19] *An element $(x, y) \in X \times X$ is called coupled coincidence point of the mapping $G : X \times X \to X$ and $g : X \times X$ if*

$$G(x, y) = g(x), \;\; G(y, x) = g(y).$$

Definition 2.7. [19] *Let X be a non-empty set, $G : X \times X \to X$ and $g : X \times X$, we say G and g are commutative if*

$$g(G(x, y)) = x, \quad G(g(x), g(y)) = y \quad \text{for all} \quad x, y \in X.$$

Definition 2.8. [19] *Let $(X, \preceq)$ be a partially ordered set and $G : X \times X \to X$ and The mapping $g : X \to X$. We say G has mixed g-monotone property if G is monotone, g non-decreasing in its first argument and is monotone g-non-increasing in its second argument that is, for any $x, y \in X$,*

$$x_1, x_2 \in X, g(x_1) \preceq g(x_2) \quad \Rightarrow G(x_1, y) \preceq G(x_2, y), \tag{3}$$

and

$$y_1, y_2 \in X, g(y_1) \preceq g(y_2) \Rightarrow G(x, y_1) \preceq G(x, y_2). \tag{4}$$

If we take g as identity mapping then this definition reduced in mixed monotone property.

3 Main Results

Theorem 3.1. *Let $(X, \mathcal{F}, \Delta)$ be a complete Menger PM space under continuous t-norm $\Delta = \Delta_M$. and $(X, \preceq)$ be a partially ordered set and Let G :*

$X \times X \to X$, $g : X \times X$ be two self mappings such that G has a mixed g-monotone property and there exists $\phi \in \Phi$ such that

$$F_{G(x,y),G(u,v)}(\phi(t)) \geq \min\Big\{F_{g(x),G(x,y)}(t), F_{g(u),G(u,v)}(t), \frac{1}{F_{g(x),G(u,v)}(2t)F_{g(u),G(x,y)}(2t)}, F_{g(x),g(u)}(t), \frac{F_{g(u),G(x,y)}(2t)}{F_{g(x),G(u,v)}(2t)}\Big\} \tag{5}$$

for all $x, y, u, v \in X, t > 0$ for which $g(x) \succeq g(u)$ and $g(y) \preceq g(v)$ or $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$.
Suppose $G(X \times X) \subseteq g(X)$ is continuous and commutes with G and also suppose either
(i) G is continuous or
(ii) X has the following properties:
(a) If a non decreasing sequence $x_n \to X$ then $x_n \preceq x$ for all n, and
(b) If a non increasing sequence $y_n \to y$ then $y_n \succeq y$ for all n.
if there exists an $x_0, y_0 \in X$ such that

$$g(x_0) \preceq G(x_0, y_0) \quad g(y_0) \succeq G(y_0, x_0)$$

then there exists $x, y \in X$ such that
$g(x) = G(x, y)$ and $g(y) = G(y, x)$.
then G and g have a coupled coincidence point.

Proof : Let $x_0, y_0 \in X$ such that X. By $g(x_0) \preceq G(x_0, y_0)$ and $g(y_0) \succeq G(y_0, x_0)$. Since $G(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $g(x_1) = G(x_0, y_0)$ and $g(y_1) = G(y_0, x_0)$. Again from $G(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $g(x_2) = G(x_1, y_1)$ and $g(y_2) = G(y_1, x_1)$. Repeating this process we get the sequences x_{n+1} and y_{n+1} in X such that

$$g(x_{n+1}) = G(x_n, y_n) \text{ and } g(y_{n+1}) = G(y_n, x_n). \text{ for all } n \geq 0. \tag{6}$$

we have to show that

$$g(x_n) \preceq g(x_{n+1}) \text{ for all } n \geq 0, \tag{7}$$

and

$$g(y_n) \preceq g(y_{n+1}) \text{ for all } n \geq 0. \tag{8}$$

Now by the mathematical induction.
Let $n = 0$. $g(x_0) \preceq G(x_0, y_0)$, $g(y_0) \succeq G(y_0, x_0)$
and $g(x_1) = G(x_0, y_0)$, $g(y_1) = G(y_0, x_0)$, we have $g(x_0) \preceq g(x_1)$ and $g(y_0) \succeq g(y_1)$. Thus eq.(7) and eq.(8) holds for $n = 0$. suppose now that eq.(7) and

eq.(8)holds for $n \geq 0$. since $g(x_n) \preceq g(x_{n+1})$ and $g(y_n) \succeq g(y_{n+1})$, and as G has the mixed $g-$monotone property, from eq.(3),we get

$$g(x_{n+1}) = G(x_n, y_n) \preceq G(x_{n+1}, y_n) \text{ and } G(y_{n+1}, x_n) \preceq G(y_n, x_n) = g(y_{n+1}) \tag{9}$$

and from eq.(3) we get

$$\begin{aligned} g(x_{n+2}) &= G(x_{n+1}, y_{n+1}) \succeq G(x_{n+1}, y_n), \\ G(y_{n+1}, x_n) &\succeq G(y_{n+1}, x_{n+1}) = g(y_{n+2}). \end{aligned} \tag{10}$$

now from eq. (9) and eq.(10) we get $g(x_{n+1}) \preceq g(x_{n+2})$ and $g(y_{n+1}) \succeq g(y_{n+2})$. thus by the mathematical induction we conclude that eq.(7) and eq.(8) holds for all $n \geq 0$. therefore,

$$g(x_0) \preceq g(x_1) \preceq g(x_2) \preceq \cdots \preceq g(x_n) \preceq g(x_{n+1}) \preceq \cdots \tag{11}$$

and

$$g(y_0) \succeq g(y_1) \succeq g(y_2) \cdots \succeq g(y_n) \succeq g(y_{n+1}) \succeq \cdots \tag{12}$$

Now putting $x = x_{n-1}$, $y = y_{n-1}$ and $u = x_n$, $v = y_n$ in eq.(5)

$$\begin{aligned} F_{G(x_n,y_n),G(x_{n-1},y_{n-1})}(\phi(t)) \geq \min\{&F_{g(x_{n-1}),G(x_{n-1},{n-1}y)}(t), F_{g(x_n),G(x_n,y_n)}(t), \\ &\frac{1}{F_{g(x_{n-1}),G(x_n,y_n)}(2t)F_{g(x_n),G(x_{n-1},y_{n-1})}(2t)}, \\ &F_{g(x_n),g(x_{n-1})}(t), \frac{F_{g(x_n),G(x_{n-1},y_{n-1})}(2t)}{F_{g(x_{n-1}),G(x_n,y_n)}(2t)}\} \end{aligned}$$

$$\begin{aligned} F_{g(x_{n+1},g(x_n)}(\phi(t)) \geq &\min\{F_{g(x_{n-1}),g(x_n)}(t), F_{g(x_n),g(x_{n+1})}(t), \\ &\frac{1}{1.\Delta[F_{g(x_{n-1}),g(x_n)}(t), F_{g(x_n),g(x_{n+1})}(t)]}, \\ &F_{g(x_n),g(x_{n-1})}(t), \frac{1}{\Delta[F_{g(x_{n-1}),g(x_n)}(t), F_{g(x_n),g(x_{n+1})}(t)]}\} \end{aligned}$$

By using eq.(6)

$$\begin{aligned} = &\min\{F_{g(x_{n-1}),g(x_n)}(t), F_{g(x_n),g(x_{n+1})}(t), \\ &\frac{1}{\min[F_{g(x_{n-1}),g(x_n)}(t), F_{g(x_n),g(x_{n+1})}(t)]}, F_{g(x_{n-1}),g(x_n)}(t), \\ &\frac{1}{\min[F_{g(x_{n-1}),g(x_n)}(t), F_{g(x_n),g(x_{n+1})}(t)]}\} \end{aligned}$$

Case-I

If $F_{g(x_{n-1}),g(x_n)}(t) < F_{g(x_n),g(x_{n+1})}(t)$ then by above

$$F_{g(x_{n+1},g(x_n)}(\phi(t)) \geq \min\{F_{g(x_n),g(x_{n-1})}(t), \frac{1}{F_{g(x_n)}(t), g(x_{n-1})}, \frac{1}{F_{g(x_{n-1}),g(x_n)}(t)}\}$$

Now by Lemma (2.1) $\{g(x_n)\}$ is Cauchy sequence.
Case-II
If $F_{g(x_n),g(x_{n+1})}(t) < F_{g(x_{n-1}),g(x_n)}(t)$ then

$$F_{g(x_{n+1},g(x_n)}(\phi(t)) \geq \min\{F_{g(x_n),g(x_{n+1})}(t), \frac{1}{F_{g(x_n),g(x_{n+1})}(t)}, \frac{1}{F_{g(x_n),g(x_{n+1})}(t)}\}$$

Now by Lemma (2.1) $\{g(x_n)\}$ is Cauchy sequence.
Again putting $x = y_n$, $y = x_n$ and $u = y_{n-1}$ $v = x_{n-1}$ in eq.(5) we have

$$\begin{aligned} F_{G(y_n,x_n),G(y_{n-1},x_{n-1})}(\phi(t)) \geq \min\{ & F_{g(y_n),G(y_n,x_n)}(t), F_{g(y_{n-1}),G(y_{n-1},x_{n-1})}(t), \\ & \frac{1}{F_{g(y_n),G(y_{n-1},x_{n-1})}(2t)}, F_{g(x),g(y_{n-1})}(t), \\ & \frac{F_{F_{g(y_{n-1}),G(y_n,x_n)}}(2t)}{F_{g(y_n),G(y_{n-1},x_{n-1})}(2t)}\} \end{aligned}$$

$$\begin{aligned} F_{g(y_{n+1},g(y_n))}(\phi(t)) \geq \min\{ & F_{g(y_n),g(y_{n+1})}(t), F_{g(y_{n-1}),g(y_n)}(t), \\ & \frac{1}{F_{g(y_n),g(y_n)}(2t).F_{g(y_{n-1}),g(y_{n+1})}(2t)}, F_{g(x),g(y_{n-1})}(t), \\ & \frac{F_{g(y_{n-1}),g(y_{n+1})}(2t)}{F_{g(y_n),g(y_n)}(2t)}\} \\ = \min\{ & F_{g(y_n),g(y_{n+1})}(t), F_{g(y_{n-1}),g(y_n)}(t), \\ & \frac{1}{1.\Delta[F_{g(y_{n-1}),g(y_n)}(t), F_{g(y_n),g(y_{n+1})}(t)]}, F_{g(x),g(y_{n-1})}(t), \\ & \frac{\Delta[F_{g(y_{n-1}),g(y_n)}(t), F_{g(y_n),g(y_{n+1})}(t)]}{1}\} \\ = \min\{ & F_{g(y_n),g(y_{n+1})}(t), F_{g(y_{n-1}),g(y_n)}(t), \\ & \frac{1}{\min[F_{g(y_{n-1}),g(y_n)}(t), F_{g(y_n),g(y_{n+1})}(t)]}, \\ & F_{g(y_n),g(y_{n+1})}(t), \min[F_{g(y_{n-1}),g(y_n)}(t), F_{g(y_n),g(y_{n+1})}(t)]\} \end{aligned}$$

Here arises two cases as follows
Case-I
If $F_{g(y_{n-1}),g(y_n)}(t) < F_{g(y_n),g(y_{n+1})}(t)$ then

$$F_{g(y_{n+1},g(y_n)}(\phi(t)) \geq \min\{F_{g(y_n),g(y_{n-1})}(t)$$

Now by Lemma (2.1) $\{g(y_n)\}$ is also a Cauchy sequence.
Case-II
If $F_{g(y_{n-1}),g(y_n)}(t) > F_{g(y_n),g(x_{n+1})}(t)$ then

$$F_{g(y_{n+1},g(y_n)}(\phi(t)) \geq \min\{F_{g(y_n),g(y_{n+1})}(t)\}$$

Now by Lemma (2.1) $\{g(y_n)\}$ is also a Cauchy sequence.
Since X is complete, there exists $x, y \in X$ such that

$$lim_{n\to\infty} g(x_n) = x \text{ and } lim_{n\to\infty} g(y_n) = y \tag{13}$$

Since g is continuous therefore

$$lim_{n\to\infty} g(g(x_n)) = g(x) \text{ and } lim_{n\to\infty} g(g(y_n)) = g(y) \tag{14}$$

Using the commutativity of g and G and eq.(6) we have

$$g(g(x_{n+1})) = g(G(x_n, y_n) = G(g(x_n), g(y_n)), \tag{15}$$

$$g(g(y_{n+1})) = g(G(y_n, x_n) = G(g(y_n), g(x_n)). \tag{16}$$

Now we show that $g(x) = G(x, y)$ and $g(y) = G(y, x)$. Suppose that the assumption (i) holds. On taking $n \to \infty$ in eq.(15) and eq.(16) by eq.(13)eq.(14) and continuity of G we get.

$$g(x) = \lim_{n\to\infty} g(g(x_{n+1})) = \lim_{n\to\infty} G(g(x_n), g(y_n)) = G(\lim_{n\to\infty} g(x_n), \lim_{n\to\infty} g(y_n) = G(x, y).$$

$$g(y) = \lim_{n\to\infty} g(g(y_{n+1})) = \lim_{n\to\infty} G(g(y_n), g(x_n)) = G(\lim_{n\to\infty} g(y_n), \lim_{n\to\infty} g(x_n) = G(y, x).$$

i.e.$g(x) = G(x, y)$ and $g(y) = G(y, x)$.
Suppose that the assumption (ii) holds. Since $\{g(x_n)\}$ is non-decreasing and $g(x_n) \to x$, and $\{g(y_n)\}$ is non-increasing and $g(y_n) \to y$, we have by $g(x_n) \preceq x$ and $g(y_n) \succeq y$ for all n. Then by (MS-3), eq. (5), eq.(15) and eq.(16) we get

$$\begin{aligned}
F_{g(x),G(x,y)}(\phi(t)) \geq & \min\{F_{g(x),g(g(x_{n+1}))}(\phi(t) - \phi(qt)), F_{g(g(x_{n+1})),G(x,y)}(\phi(qt))\} \\
= & \min\{F_{g(x),g(g(x_{n+1}))}(\phi(t) - \phi(qt)), F_{G(g(x_n),g(y_n),G(x,y))}(\phi(qt))\} \\
\geq & \{F_{g(x),g(g(x_{n+1}))}(\phi(t) - \phi(qt)), F_{g(g(x_n)),G(g(x_n),g(y_n))}(qt), \\
& F_{g(x),G(x,y)}(qt), \frac{1}{F_{g(g(x_n)),G(x,y)}(2qt).F_{g(x),G(g(x_n),g(y_n))}(2qt)}, \\
& F_{g(g(x_n)),g(x)}(qt), \frac{F_{g(x),G(g(x_n),g(y_n))}(2qt)}{F_{g(g(x_n)),G(x,y)}(2qt)}\}
\end{aligned}$$

For all $t > 0$, $q \in (0, 1)$.

$$\begin{aligned}
& F_g(x), G(x, y)(\phi(t)) \geq \\
& \quad \min\{F_{g(x),g(g(x_{n+1}))}(\phi(t) - \phi(qt)), F_{g(g(x_n)),G(g(x_n),g(y_n))}(qt), F_{g(x),G(x,y)}(qt), \\
& \frac{1}{\min[F_{g(g(x_n)),g(g(x_{n-1}))}(qt), F_{g(g(x_n)),g(x)}(qt)]. \min[F_{g(x),g(g(x_{n-1}))}(qt), F_{g(g(x_{n-1})),g(g(x_n))}(qt)]} \\
& F_{g(g(x_n)),g(x)}(qt), \frac{\min[F_{g(x),g(g(x_{n-1}))}(qt), F_{g(g(x_{n-1})),g(g(x_n))}(qt)]}{\min[F_{g(g(x_n)),g(g(x_{n-1}))}(qt), F_{g(g(x_n)),g(x)}(qt)]}\}
\end{aligned}$$

$$F_{g(x),G(x,y)}(\phi(t)) \geq \min\{F_{g(x),g(x)}(\phi(t)-\phi(qt)), F_{g(x),G(g(x,y))}(qt), F_{g(x),G(x,y)}(qt), \frac{1}{F_{g(g(x_n)),G(x,y)}(2qt).F_{g(x),G(g(x_n),g(y_n))}(2qt)}, F_{g(x),g(x)}(qt), \frac{F_{g(x),G(g(x_n),g(y_n))}(2qt)}{F_{g(g(x_n)),G(x,y)}(2qt)}\}$$

which on letting $n \to \infty$ and taking lower limit,by Lemma(2.2) we get

$$F_{g(x),G(x,y)}(\phi(t)) \geq \min\{1, F_{g(x),G(g(x,y))}(qt), F_{g(x),G(x,y)}(qt), 1, 1, 1\} = F_{g(x),G(g(x,y))}(qt)$$

Taking $q \to 1$ with the left continuity of F, gives

$$F_{g(x),G(x,y)}(\phi(t)) \geq F_{g(x),G(g(x,y))}(t)$$

by Lemma(2.3) we get $g(x) = G(x,y)$.
Similarly we can show that $g(y) = G(y,x)$. Thus G and g have a coupled coincidence point.
We are giving two corollaries, in corollary first taking identity mapping in place of g and in corrollary second replacing $\phi(t)$ to qt in Theorem (3.1) as :

Corollary 3.1. *Let $(X, \mathcal{F}, \Delta)$ be a complete Menger PM space under continuous t-norm $\Delta = \Delta_M$. and $(X, \preceq)$ be a partially ordered set and Let $G : X \times X \to X$, be a self mapping of X such that G has a mixed g-monotone property and there exists $\phi \in \Phi$ such that*

$$F_{G(x,y),G(u,v)}(\phi(t)) \geq \min\{F_{x,G(x,y)}(t), F_{u,G(u,v)}(t), \frac{1}{F_{x,G(u,v)}(2t)F_{u,G(x,y)}(2t)}, F_{x,u}(t), \frac{F_{u,G(x,y)}(2t)}{F_{x,G(u,v)}(2t)}\}$$

for all $x, y, u, v \in X, t > 0$ for which $x \preceq u$ and $y \succeq v$.
Suppose either
(i) G is continuous or
(ii) X has the following properties:
(a) If a non decreasing sequence $x_n \to X$ then $x_n \preceq x$ for all n, and
(b) If a non increasing sequence $y_n \to y$ then $y_n \succeq y$ for all n.
if there exists an $x_0, y_0 \in X$ such that

$$x_0 \preceq G(x_0, y_0) \quad and \quad y_0 \succeq G(y_0, x_0)$$

then there exists $x, y \in X$ such that
$x = G(x,y)$ and $y = G(y,x)$.
then G has a coupled fixed point.

Corollary 3.2. *Let $(X, \mathcal{F}, \Delta)$ be a complete Menger PM space under continuous t-norm $\Delta = \Delta_M$. and $(X, \preceq)$ be a partially ordered set and Let $G : X \times X \to X$, $g : X \times X$ be two self mappings such that G has a mixed g-monotone property and there exists $q \in (0, 1)$ such that*

$$F_{G(x,y),G(u,v)}(qt) \geq \min\{F_{g(x),G(x,y)}(t), F_{g(u),G(u,v)}(t), \frac{1}{F_{g(x),G(u,v)}(2t)F_{g(u),G(x,y)}(2t)},$$
$$F_{g(x),g(u)}(t), \frac{F_{g(u),G(x,y)}(2t)}{F_{g(x),G(u,v)}(2t)}\}$$

for all $x, y, u, v \in X, t > 0$ for which $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$.
Suppose $G(X \times X) \subseteq g(X)$ is continuous and commutes with G and also suppose either
(i) G is continuous or
(ii) X has the following properties:
(a) If a non decreasing sequence $x_n \to X$ then $x_n \preceq x$ for all n, and
(b) If a non increasing sequence $y_n \to y$ then $y_n \succeq y$ for all n.
if there exists an $x_0, y_0 \in X$ such that

$$g(x_0) \preceq G(x_0, y_0) \quad \textit{and} \quad g(y_0) \succeq G(y_0, x_0)$$

then there exists $x, y \in X$ such that
$g(x) = G(x, y)$ and $g(y) = G(y, x)$.
then G and g have a coupled coincidence point.

Theorem 3.2. *In addition to the hypotheses of Theorem (3.1), suppose that for every $(x, y), (x^*, y^*) \in X \times X$ there exists a $(u, v) \in X \times X$ satisfying $g(u) \preceq g(v)$ or $g(v) \preceq g(u)$ such that $G(u, v), G(v, u) \in X \times X$ is comparable to $(G(x, y), G(y, x))$ and $(G(x^*, y^*), G(y^*, x^*))$. Then G and g have a unique coupled fixed point, that is , there exists a unuque $(x, y) \in X \times X$ such that*
$x = g(x) = G(x, y)$ and $y = g(y) = G(y, x)$.

Proof : we have to show that if $(x, y), (x^*, y^*)$ are coupled coincidence points, that is if $g(x) = G(x, y)$ and $g(y) = G(y, x)$. and $g(x^*) = G(x^*, y^*)$ and $g(y^*) = G(y^*, x^*)$. then

$$g(x) = g(x^*) \text{ and } g(y) = g(y^*). \tag{17}$$

By assumption there is $(u, v) \in X \times X$ such that $G(u, v), G(v, u) \in X \times X$ is comparable to $(G(x, y), G(y, x))$ and $(G(x^*, y^*), G(y^*, x^*))$. Putting $u_0 = u, v_0 = v$ and choose $u_1, v_1 \in X$ so that $g(u_1) = G(u_0, v_0)$ and $g(v_1) = G(v_0, u_0)$. We define the sequence $\{g(u_n)\}$ and $\{g(v_n)\}$ such that $g(u_{n+1}) = G(u_n, v_n)$ and $g(v_{n+1}) = G(v_n, u_n)$. These two sequences $\{g(u_n)\}$ and $\{g(v_n)\}$ exists as similar proof given in Theorem (3.1). In addition put $x_0 =$

$x, y_0 = y, x_0^* = x^*, y_0^* = y^*$ and on the same way define the sequence $\{g(x_n)\}$, $\{g(y_n)\}$ and $\{g(x_n^*)\}$ and $\{g(y_n^*)\}$ such that

$$g(x_{n+1}) = G(x_n, y_n) \text{ and } g(y_{n+1}) = G(y_n, x_n).$$

$$g(x_{n+1}) = G(x_n, y_n) \text{ and } g(y_{n+1}) = G(y_n, x_n).$$

Since

$$(G(x,y), G(y,x)) = (g(x_1), g(y_1)) = (g(x), g(y))$$

and $(G(u,v), G(v,u)) = (g(u_1), g(v_1))$ are comparable, now we suppose that $g(x) \preceq g(u_1)$ and $g(y) \preceq g(v_1)$. It is easy to show that $(g(x), g(y))$ and $G(u_n, v_n)$ are comparable, that is $g(x) \preceq g(u_n)$ $g(y) \preceq g(v_n)$ for all $n \geq 1$ Thus from eq.(5) we get

$$\begin{aligned} F_{g(x),g(u_{n+1})}(\phi(t)) =& F_{G(x,y),G(u_n,v_n)}(\phi(t) \\ \geq & \min\{F_{g(x),G(x,y)}(t), F_{g(u_n),G(u_n,v_n)}(t), \\ & \frac{1}{F_{g(x),G(u_n,v_n)}(2t).F_{g(u_n),G(x,y)}(2t)}, \\ & F_{g(x),g(u_n)}(t), \frac{F_{g(u_n),G(x,y)}(2t)}{F_{g(x),G(u_n,v_n)}(2t)}\} \end{aligned}$$

$$\begin{aligned} =& \min\{F_{g(x),g(x)}(t), F_{g(u_n),g(u_{n+1})}(t), \frac{1}{F_{g(x),g(u_{n+1})}(2t)F_{g(u_{n+1}),g(x)}(2t)}, \\ & F_{g(x),g(u_n)}(t), \frac{F_{g(u_n),g(x)}(2t)}{F_{g(x),G(u_n)}(2t)}\} \\ =& \min\{1, F_{g(u_n),g(u_{n+1})}(t), \frac{1}{g(u_n), F_{g(x)}(2t).F_{g(x),g(u_{n+1})}(2t)}, \\ & F_{g(x),g(u_n)}(t), \frac{F_{g(u_n),g(x)}(2t)}{F_{g(x),G(u_n)}(2t)}\} \end{aligned}$$

$$\begin{aligned} F_{g(y),g(v_{n+1})}(\phi(t)) =& F_{G(y,x),G(v_n,u_n)}(\phi(t) \\ \geq & \min\{F_{g(x),G(y,x)}(t), F_{g(v_n),G(v_n,u_n)}(t), \\ & \frac{1}{F_{g(y),G(v_n,u_n)}(2t)F_{g(v_n),G(y,x)}(2t)}, \\ & F_{g(y),g(v_n)}(t), \frac{F_{g(v_n),G(y,x)}(2t)}{F_{g(y),G(v_n,u_n)}(2t)}\} \end{aligned}$$

$$= \min\{1, F_{g(v_n),g(v_{n+1})}(t), \frac{1}{F_{g(y),g(v_{n+1})}(2t)F_{g(v_{n+1}),g(y)}(2t)},$$
$$F_{g(y),g(v_n)}(t), \frac{F_{g(v_n),g(y)}(2t)}{F_{g(y),G(v_n)}(2t)}\}$$
$$= \min\{1, F_{g(v_n),g(v_{n+1})}(t), \frac{1}{g(v_n), F_{g(y)}(2t).F_{g(y),g(v_{n+1})}(2t)},$$
$$F_{g(y),g(v_n)}(t), \frac{F_{g(v_n),g(y)}(2t)}{F_{g(y),G(v_n)}(2t)}\}$$

for each $n \geq 1$ on taking $n \to \infty$ and lower limit, by Lemma (2.2) and Lemma (2.3) we get

$$lim_{n\to\infty} g(u_{n+1}) = g(x) \quad \text{and} \quad lim_{n\to\infty} g(v_{n+1}) = g(y) \tag{18}$$

similarly we can prove that

$$lim_{n\to\infty} g(u_{n+1}) = g(x^*) \quad \text{and} \quad lim_{n\to\infty} g(v_{n+1}) = g(y^*) \tag{19}$$

By (MS-3), eq. (18) and eq. (19).we have

$$F_{g(x),g(x^*)}(t) \geq \min F_{g(x),g(u_{n+1})}(t/2), F_{g(u_{n+1}),g(x^*)}(t/2) \to 1 \text{ as } n \to \infty$$

which shows that $g(x) = g(x^*)$. similarly we can prove that $g(y) = g(y^*)$
Since $g(x) = G(x, y)$ and $g(y) = G(y, x)$, be commutativity of G and g, we have

$$g(g(x)) = g(G(x, y)) = G(g(x), g(y)) \tag{20}$$

and

$$g(g(y)) = g(G(y, x)) = G(g(y), g(x)) \tag{21}$$

say $g(x) = Z, g(y) = w$ Then from eq.(20) and eq.(21)

$$g(z) = G(z, w) \text{ and } g(w) = G(w, z) \tag{22}$$

Thus (z, w) is a coupled common fixed point. Then from eq. (17) with $x^* = z$ and $y^* = w$, it follows $g(z) = g(x)$ and $g(w) = g(y)$, i.e.

$$g(z) = z \text{ and } g(w) = w. \tag{23}$$

now from (22)and (23),

$$z = g(z) = G(z, w) \text{ and } w = g(w) = G(w, z).$$

Therefore (z, w) is a coupled common fixed point of G and g. To prove the uniqueness, assume that (p, q) is another couples common fixed point. Then by eq.(17) we have

$$p = g(p) = g(z) = z \text{ and } q = g(q) = g(w) = w.$$

Now we substiontiate our theorem with the aid of following example.

Example 3.1. *Consider $X = [0,6]$ with $d(x,y) = |x-y|$, $F_{x,y}(t) = H(t - d(x,y)), \Delta = \Delta_M$, then $(X, \mathcal{F}, \Delta)$ is complete Menger PM space. let $g : X \times X$ and $G : X \times X$ be defined as $g(x) = \frac{x}{2}$, for all $x \in X$*

$$G(x,y) = \begin{cases} \frac{x-y}{6} & ; x, y \in [0,1], x \geq y \\ \\ 0 & ; x < y \end{cases}$$

G enjoys the mixed g-monotone property. $G(X \times X) \subset g(X)$, g is continuous and commute with G. G is also continuous.

Let $\phi(t) = (2/3)t$, for all $f \in [0, \infty)$. Let $x_0 = 0$ and $y_0 = c$ are two points in X then

$$g(x_0) = g(0) = G(0,c) \leq G(x_0, y_0)$$

and

$$g(y_0) = g(c) = \Big\{ \frac{c}{2} \geq \frac{c}{6} = G(c,0) = G(y_0, x_0) \Big\}.$$

Next we verify the inequality in Theorem (3.1). We take $x, y, u, v \in X$, such that $g(x) \geq g(u)$ and $g(y) \geq g(v)$, that is, $x \geq u, y \leq v$.
Thus the inequality in Theorem(3.1)takes the following form:

$$H\Big(\frac{2t}{3} - |G(x,y) - G(u,v)|\Big) \geq \min \Big\{ H\big(t - \frac{x}{2} - G(x,y)\big), H\big(t - |\frac{u}{2} - G(u,v)|\big), \frac{1}{H\big(2t - |\frac{x}{2} - G(u,v)|\big).H\big(2t - |\frac{u}{2} - G(x,y)|\big)}, H\big(t - \frac{x-u}{2}\big), \frac{H\big(2t - |\frac{u}{2} - G(x,y)|\big)}{H\big(2t - |\frac{x}{2} - G(u,v)|\big)} \Big\}.$$

By the definition of H,we only need to verify that

$$\frac{2t}{3} > |G(x,y) - G(u,v)| \tag{24}$$

$$\text{If} \quad t > |\frac{x}{2} - G(x,y)|, \quad t > |\frac{u}{2} - G(u,v)|, \quad t > \frac{x-u}{2}, \quad 2t > |\frac{u}{2} - G(x,y)|, \quad 2t > |\frac{x}{2} - G(u,v)|. \tag{25}$$

We consider the following cases
Case I

If $x \geq y$ and $u \geq v$, i.e. $y \leq v \leq u \leq x$, then eq.(25)implies

$$t > \frac{x-u}{2} \quad t > \frac{x}{2} - \frac{x-y}{6} = \frac{2x+y}{6}, \quad t > \frac{u}{2} - \frac{u-v}{6} = \frac{2u+v}{6}$$

we get $t > \frac{2x+y}{6} \geq \frac{x}{4}$, *and then eq.(24) holds.*

$$t > \frac{x}{4} \geq \frac{x}{4} - \frac{u-v}{4} = \frac{x-u}{4} + \frac{v}{4} \geq \frac{x-u}{4} + \frac{v-y}{4},$$

which means that $\frac{2}{3}t \geq \frac{x-y}{6} - \frac{u-v}{6}$, *eq.(24) satisfied.*
Case II

If $x \geq y$ *and* $u < v$, *eq.(25)implies*

$$t > \frac{x}{2} - \frac{x-y}{6} = \frac{2x+y}{6} \geq \frac{x-y}{4}$$

i.e. $\frac{2}{3}t \geq \frac{x-y}{6} = |G(x,y) - G(u,v)|$, *eq.(24) satisfied.*
Case III

If $x < y$ *and* $u \geq v$. *it can not happen since* $u \leq x$ *and* $y \leq v$.
Case IV

If $x < y$ *and* $u < v$. *Since* $t > 0$. *obviously eq.(24) holds. Hence all the conditions of Theorem (3.1) are satisfied and the coupled coincidence point is* $(0,0)$ *of* G *and* g *in* X.

Open Problem: Recently Luong and Thuan[13] presented some coupled fixed point theorems for a mixed monotone mappings in a partially ordered metric space. Further a result on trippled coincidence points for monotone operators in partially ordered metric spaces is obtained by Alsulami and Alotaibi[16]. Whether our result can be obtained for trippled coincidence point theorems in the settings of Menger PM-Spaces.

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