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Research Paper

Oscillatory behavior of a class of fractional difference equations with damping

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Abstract

This paper investigates the oscillation of a class of fractional difference equations with damping term of the form

$$\Delta(c(t)(\Delta^{\alpha} x(t))^{\gamma}) + p(t)(\Delta^{\alpha} x(t))^{\gamma} + q(t)\left(\sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s)\right)^{\gamma} = 0,\ t \in N_{t_0+1-\alpha}$$

where Δ^{α} denotes the Riemann-Liouville difference operator of order $0<\alpha\le 1$ and $\gamma>0$ is a quotient of odd positive integers. Based on a generalized Riccati transformation and some inequalities, we establish some sufficient conditions of oscillation criteria for it. Some applications are also presented for the established results.

Keywords: *Difference Equations, Oscillation, Fractional Order, Damping*.

1. Introduction

Recent years have witnessed the study of qualitative properties, especially oscillation of solutions, of fractional difference equations, [3], and [7]. In this paper, we investigate the oscillatory properties a class of fractional difference equations with damping term of the form

$$\Delta(c(t)(\Delta^{\alpha} x(t))^{\gamma}) + p(t)(\Delta^{\alpha} x(t))^{\gamma} + q(t)\left(\sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s)\right)^{\gamma} = 0,\ t \in N_{t_0+1-\alpha} \tag{1}$$

where Δ^{α} denotes the Riemann-Liouville difference operator of order $0<\alpha\le 1$ and $\gamma>0$ is a quotient of odd positive integers.

(H). $p(t)$ and $q(t)$ are positive sequences on $t_0>0$ and $f: R \to R$ is a continuous function and $\sum_{s=t_0}^{\infty} \frac{1}{\mu(s)c(s)} = \infty.$

A solution x (t) of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

2. Preliminaries and basic lemmas

In this section, we introduce some preliminary results of discrete fractional calculus, which will be used throughout this paper.

Definition 2.1: *(See [8]) Let* $\nu>0$. *The* ν *-th fractional sum* f *is defined by*

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} f(s),$$

Where f is defined for $s \equiv a$ mod (1) and $\Delta^{-\nu} f$ is defined for $t \equiv (a+\nu) \bmod (1)$ and $t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t-\nu+1)}$. The fractional sum $\Delta^{-\nu} f$ maps functions defined on N_a to functions defined on $N_{a+\nu}$.

Definition 2.2: *(see [8]) let $\mu > 0$ and $m-1 < \mu < m$ where m denotes a positive integer $m = \lceil \mu \rceil$. Set $\nu = m - \mu$. The μ-th fractional difference is defined as*

$$\Delta^{\mu} f(t) = \Delta^{m-\nu} f(t) = \Delta^{m} \Delta^{-\nu} f(t).$$

Lemma 2.3: Let $a \geq 0, b, X \in R$, then $b\sqrt{a}X - aX^2 \leq \frac{b^2}{4}$

Lemma 2.4: *Let x (t) be a solution of (1) and let*

$$G(t) = \sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s) \tag{2}$$

Then

$$\Delta(G(t)) = \Gamma(1-\alpha)\Delta^{\alpha}(x(t)). \tag{3}$$

Proof:

$$G(t) = \sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s) = \sum_{s=t_0}^{t-(1-\alpha)} (t-s-1)^{(1-\alpha)-1} x(s)$$

$$= \Gamma(1-\alpha)\Delta^{-(1-\alpha)} x(t),$$

which implies?

$$\Delta(G(t)) = \Gamma(1-\alpha)\Delta\Delta^{-(1-\alpha)} x(t) = \Gamma(1-\alpha)\Delta^{\alpha} x(t).$$

Now, we assume that $c(t) > p(t)$ and define a sequence

$$\mu(t) = \prod_{s=t_0}^{t-1} \frac{c(s)}{c(s)-p(s)} \quad \forall t \geq t_0. \tag{4}$$

and

$$\Delta\mu(t) = \frac{p(t)}{c(t)-p(t)}\mu(t) \text{ and } \mu(t+1) = \frac{c(t)}{c(t)-p(t)}\mu(t) \tag{5}$$

3. Main results

Theorem 3.1: *Suppose that (H) holds and* $\sum_{s=t_0}^{\infty} \frac{1}{\mu(s)c(s)} = \infty$ *, there exists a positive sequence b (t) such that*

$$\limsup_{t\to\infty} \sum_{s=t_1}^{t-1} \left(b(s)q(s)\mu(s+1) - \frac{(\Delta b_+(s))^2}{4b^2(s+1)R(s)} \right) = \infty, \tag{6}$$

where $\Delta b_+(s) = \max\{\Delta b(s), 0\}$, $R(t) = \frac{b(t)\Gamma(1-\alpha)^{\gamma}}{b^2(t+1)\mu(t)c(t)}$. Then every solution of (1) is oscillatory.

Proof: Suppose that x (t) is a nonoscillatory solution of (1). Without loss of generality, we may assume that x(t) is an eventually positive of (1). Then there exists $t_1 \geq t_0$ such that

$$x(t) > 0 \text{ and } G(t) > 0 \text{ for } t \geq t_1, \tag{7}$$

where G is defined as in (2). Therefore, it follows from (1) that

$$\Delta(\mu(t)c(t)(\Delta^{\alpha} x(t))^{\gamma}) = \mu(t+1)\Delta(c(t)(\Delta^{\alpha} x(t))^{\gamma}) + (c(t)(\Delta^{\alpha} x(t))^{\gamma})\Delta\mu(t)$$

$$= \mu(t+1)\Delta(c(t)(\Delta^{\alpha} x(t))^{\gamma}) + (c(t)(\Delta^{\alpha} x(t))^{\gamma}) \left(\frac{p(t)}{c(t)-p(t)} \right) \left(\frac{c(t)-p(t)}{c(t)} \right) \mu(t+1)$$

$$= \mu(t+1)(\Delta(c(t)(\Delta^{\alpha} x(t))^{\gamma}) + p(t)(\Delta^{\alpha} x(t))^{\gamma})$$

$$\Delta(\mu(t)c(t)(\Delta^{\alpha} x(t))^{\gamma}) = -\mu(t+1)q(t)(G(t))^{\gamma} < 0 \text{ for } t \geq t_1 \tag{8}$$

Thus $\mu(t)c(t)(\Delta^{\alpha}x(t))^{\gamma})$ is a strictly non-increasing sequence and is eventually of one sign on $t \geq t_1$. First we show that $\mu(t)c(t)(\Delta^{\alpha}x(t))^{\gamma}$ is eventually positive. Suppose there is an integer $t_1 > t_0$ such that $\mu(t)c(t)(\Delta^{\alpha}x(t))^{\gamma} = \delta < 0$ for $t \geq t_1$, so that

$$\mu(t)c(t)(\Delta^{\alpha}x(t))^{\gamma} \leq \mu(t_1)c(t_1)(\Delta^{\alpha}x(t_1))^{\gamma} = \delta < 0$$

$$(\Delta^{\alpha}x(t))^{\gamma} \leq \frac{\delta}{\mu(t)c(t)} < 0$$

which implies that

$$\frac{\Delta G(t)}{\Gamma(1-\alpha)} = \Delta^{\alpha}x(t) \leq \delta^{1/\gamma}\mu^{-1/\gamma}(t)c^{-1/\gamma}(t) \quad \text{for} \quad t \geq t_1. \tag{9}$$

Summing both sides of the inequality (9) from t_1 to $t-1$ yields

$$G(t) \leq G(t_1) + \Gamma(1-\alpha)\delta^{1/\gamma}\sum_{s=t_1}^{t-1}\mu^{-1/\gamma}(s)c^{-1/\gamma}(s) \to -\infty \quad \text{as} \quad t \to \infty, \tag{10}$$

This contradicts the fact that G (t) > 0. Hence $\mu(t)c(t)(\Delta^{\alpha}x(t))^{\gamma} > 0$ is eventually positive. Define the function ω*(t)* by the Riccati substitution

$$\omega(t) = b(t)\frac{\mu(t)c(t)(\Delta^{\alpha}x(t))^{\gamma}}{G^{\gamma}(t)} \quad \text{for} \quad t \geq t_1. \tag{11}$$

Then we have $\omega(t) > 0$ for $t \geq t_1$. It follows that

$$\Delta\omega(t) \leq \Delta b_{+}(t)\frac{\omega(t+1)}{b(t+1)} - b(t)q(t)\mu(t+1) - \frac{b(t)\mu(t+1)c(t+1)(\Delta^{\alpha}x(t+1))^{\gamma}\Delta G^{\gamma}(t)}{G^{2\gamma}(t+1)}.$$

Now using the following inequality (see [1]), we obtain

$$x^{\beta} - y^{\beta} \geq (x-y)^{\beta},$$

We have

$$\mu(t)c(t)(\Delta^{\alpha}x(t))^{\gamma} \geq \mu(t+1)c(t+1)(\Delta^{\alpha}x(t+1))^{\gamma}$$

$$(\Delta^{\alpha}x(t))^{\gamma} \geq \frac{\mu(t+1)c(t+1)}{\mu(t)c(t)}(\Delta^{\alpha}x(t+1))^{\gamma}.$$

Using the above inequality, we obtain

$$\Delta\omega(t) \leq \Delta b_{+}(t)\frac{\omega(t+1)}{b(t+1)} - b(t)q(t)\mu(t+1) - \frac{b(t)\mu(t+1)c(t+1)(\Delta^{\alpha}x(t+1))^{\gamma}(\Delta G(t))^{\gamma}}{G^{2\gamma}(t+1)}$$

$$\leq \Delta b_{+}(t)\frac{\omega(t+1)}{b(t+1)} - b(t)q(t)\mu(t+1) - \frac{b(t)\Gamma(1-\alpha)^{\gamma}}{b^{2}(t+1)\mu(t)c(t)}\omega(t+1)^{2}$$

$$\Delta\omega(t) \leq \Delta b_{+}(t)\frac{\omega(t+1)}{b(t+1)} - b(t)q(t)\mu(t+1) - R(t)\omega(t+1)^{2} \tag{12}$$

where $R(t) = \dfrac{b(t)\Gamma(1-\alpha)^{\gamma}}{b^{2}(t+1)\mu(t)c(t)}$. Take $a = R(t)$, $b = \dfrac{\Delta b_{+}(t)}{b(t+1)\sqrt{R(t)}}$ and X=ω(t+1).

Using Lemma 2.3, we get

$$\frac{\Delta b_{+}(t)}{b(t+1)\sqrt{R(t)}}\sqrt{R(t)}\omega(t+1) - R(t)\omega(t+1)^{2} \leq \frac{\left(\dfrac{\Delta b_{+}(t)}{b(t+1)\sqrt{R(t)}}\right)^{2}}{4}$$

From (12), we conclude that

$$\Delta\omega(t) \leq -b(t)q(t)\mu(t+1) + \frac{(\Delta b_{+}(t))^{2}}{4b^{2}(t+1)R(t)}.$$

Summing the above inequality from t_1 to $t-1$, we have

$$\sum_{s=t_1}^{t-1}\left(b(s)q(s)\mu(s+1) - \frac{(\Delta b_{+}(s))^{2}}{4b^{2}(s+1)R(s)}\right) \leq \omega(t_1) - \omega(t) \leq \omega(t_1) < \infty, \text{for } t \geq t_1$$

Letting $t \to \infty$, we get

$$\limsup_{t\to\infty}\sum_{s=t_1}^{t-1}\left(b(s)q(s)\mu(s+1)-\frac{(\Delta b_+(s))^2}{4b^2(s+1)R(s)}\right)\le\omega(t_1)<\infty.$$

This contradicts (6). The proof is complete.

Theorem 3.2: *Suppose that (H) holds. Furthermore, assume that there exists a positive sequence b (t) such that* $H(t,t)=0$ for $t\ge t_0$ $H(t,s)>0$ $t>s\ge t_0$

$\Delta_2 H(t,s)=H(t,s+1)-H(t,s)\le 0$ for $t\ge s\ge t_0$.

If

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\sum_{s=t_0}^{t-1}\left(b(s)q(s)\mu(s+1)H(t,s)-\frac{h_+^2(t,s)}{4H(t,s)R(s)}\right)=\infty \tag{13}$$

where $h_+(t,s)=\Delta_2 H(t,s)+\dfrac{H(t,s)\Delta b_+(s)}{b(s+1)}$ and $\Delta b_+(s)=\max\left[\Delta b(s),0\right]$. Then every solution of (1) is oscillatory.

Proof: Suppose the contrary that *x (t)* is a nonoscillatory solution of (1). Without loss of generality, we may assume that *x (t)* is an eventually positive solution of (1). We proceed as in the proof of Theorem (3.1) to get (12) hold.

Multiplying (12) by $H(t,s)$ and summing from t_1 to $t-1$, we obtain

$$\begin{aligned}\sum_{s=t_1}^{t-1}b(s)q(s)\mu(s+1)H(t,s)\le -\sum_{s=t_1}^{t-1}H(t,s)\Delta\omega(s)+\sum_{s=t_1}^{t-1}H(t,s)\Delta b_+(s)\frac{\omega(s+1)}{b(s+1)}\\ -\sum_{s=t_1}^{t-1}H(t,s)R(s)\omega^2(s+1)\end{aligned} \tag{14}$$

Using summation by parts formula, we get

$$\begin{aligned}-\sum_{s=t_1}^{t-1}H(t,s)\Delta\omega(s)&=-[H(t,s)\omega(s)]_{s=t_1}^{t}+\sum_{s=t_1}^{t-1}\omega(s+1)\Delta_2 H(t,s)\\ &=H(t,t_1)\omega(t_1)+\sum_{s=t_1}^{t-1}\omega(s+1)\Delta_2 H(t,s)\end{aligned} \tag{15}$$

Now, we have

$$\sum_{s=t_1}^{t-1}b(s)q(s)\mu(s+1)H(t,s)\le H(t,t_1)\omega(t_1)+\sum_{s=t_1}^{t-1}(h_+(t,s)\omega(s+1)-H(t,s)R(s)\omega^2(s+1)) \tag{16}$$

where $h_+(t,s)=\Delta_2 H(t,s)+\dfrac{H(t,s)\Delta b_+(s)}{b(s+1)}$.

Taking $a=H(t,s)R(s),\ b=\dfrac{h_+(t,s)}{\sqrt{H(t,s)R(s)}}$ and X=ω(t+1) and using the Lemma 2.3, we get

$$\frac{h_+(t,s)}{\sqrt{H(t,s)R(s)}}\sqrt{H(t,s)R(s)}\omega(t+1)-H(t,s)R(s)\omega(t+1)^2\le\frac{\left(\dfrac{h_+(t,s)}{\sqrt{H(t,s)R(s)}}\right)^2}{4}$$

We have $0<H(t,t_1)\le H(t,t_0)$ for $t>t_1\ge t_0$, from equation (16),

$$\sum_{s=t_1}^{t-1}b(s)q(s)\mu(s+1)H(t,s)\le H(t,t_1)\omega(t_1)+\sum_{s=t_1}^{t-1}\frac{h_+^2(t,s)}{4H(t,s)R(s)}$$

$$\begin{aligned}\sum_{s=t_1}^{t-1}\left(b(s)q(s)\mu(s+1)H(t,s)-\frac{h_+^2(t,s)}{4H(t,s)R(s)}\right)&\le H(t,t_1)\omega(t_1)\\ &\le H(t,t_0)\omega(t_1).\end{aligned}$$

Since $0<H(t,s)\le H(t,t_0)$ for $t>s\ge t_0$, we have $0<\dfrac{H(t,s)}{H(t,t_0)}\le 1$ for $t>s\ge t_0$. Hence it follows that

$$\frac{1}{H(t,t_0)}\sum_{s=t_0}^{t-1}\left(b(s)q(s)\mu(s+1)H(t,s)-\frac{h_+^2(t,s)}{4H(t,s)R(s)}\right)=$$

$$\frac{1}{H(t,t_0)}\sum_{s=t_0}^{t_1-1}\left(b(s)q(s)\mu(s+1)H(t,s)-\frac{h_+^2(t,s)}{4H(t,s)R(s)}\right)$$

$$+\frac{1}{H(t,t_0)}\sum_{s=t_1}^{t-1}\left(b(s)q(s)\mu(s+1)H(t,s)-\frac{h_+^2(t,s)}{4H(t,s)R(s)}\right)$$

$$\leq\frac{1}{H(t,t_0)}\sum_{s=t_0}^{t_1-1}b(s)q(s)\mu(s+1)H(t,s)+\omega(t_1)$$

$$\leq\sum_{s=t_0}^{t_1-1}b(s)q(s)\mu(s+1)+\omega(t_1)$$

Letting $t\to\infty$, we have

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\sum_{s=t_0}^{t-1}\left(b(s)q(s)\mu(s+1)H(t,s)-\frac{h_+^2(t,s)}{4H(t,s)R(s)}\right)$$

$$\leq\sum_{s=t_0}^{t_1-1}b(s)q(s)\mu(s+1)+\omega(t_1)<\infty,$$

This is a contradiction to (13). The proof is complete.

Example 3.3: Consider the fractional difference equation

$$\Delta(\Delta^{\alpha}x(t))+\frac{1}{t+1}\Delta^{\alpha}x(t)+\frac{1}{t+1}\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)=0, \tag{17}$$

where $\alpha=0.5,\gamma=1,c(t)=1,p(t)=\dfrac{1}{t+1}$ and $q(t)=\dfrac{1}{t+1}$. Since

$$\mu(\mathrm{t})=\prod_{s=1}^{t-1}\frac{1}{1-\frac{1}{s+1}}=t.$$

we find that (H) holds. We will apply Theorem (3.1) and it remains to show condition (6) is satisfied. Taking *b(s) = s*, we obtain

$$\limsup_{t\to\infty}\sum_{s=t_0}^{t-1}\left(b(s)q(s)\mu(s+1)-\frac{(\Delta b_+(s))^2}{4b^2(s+1)R(s)}\right)=\limsup_{t\to\infty}\sum_{s=t_0}^{t-1}\left(s-\frac{1}{4\sqrt{\pi}}\right)=\infty$$

which implies that (6) holds. Therefore, by Theorem (3.1) every solution of (17) is oscillatory.

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