Projective Synchronization of Chaotic Systems Via Backstepping Design

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Abstract

Chaos synchronization of discrete dynamical systems are investigated. An algorithm is proposed for projective synchronization of chaotic 2D Duffing map and chaotic Tinkerbell map. The control law was derived from the Lyapunov stability theory. The numerical simulation results are presented to verify the effectiveness of the proposed algorithm

Keywords: *Lyapunov function, Projective synchronization, Backstepping Design, Duffing map and Tinkerbell map.*

1 Introduction

Adjacent chaotic trajectories governed by the same nonlinear systems may evolve into a state utterly uncorrelated, but in 1990 Pecora and Carrol [1] shown that it could be synchronized through a proper coupling. Since their seminal paper in 1990, chaos synchronization is an interesting research topic of great attention. Hayes et.al. (1993)[2] have studied to some potential applications in secure communication. Blasius et.al. [3] have observed complex dynamics and phase synchronization in spatially extended ecological systems in 1999 and system identification was investigated by Kocarev et.al. in 1996 [4]. Different forms of synchronization phenomena have been observed in a variety of chaotic systems, such as identical synchronization [1]. In 1996 Rosenblum et.al. [5] have studied phase synchronization of chaotic oscillators. A generalized synchronization for unidirectionally coupled dynamical systems was proposed by Rulkov et.al. [6] in

1995, where two systems are called synchronized if a static functional relation exists between the states of the systems. The linear generalized chaos synchronization and predictability have investigated by S.Poria in 2007 [7]. Recently, in 2009 Khan et.al. [8] have studied generalized chaos synchronization of coupled Rossler system. A generalized anti-synchronization of different chaotic systems have investigated by Khan et.al. in 2012 [9]. In 2009 Tarai et.al. [10] have studied synchronization of bi-directionally coupled chaotic Chen's system with delay and they also studied generalized synchronization of linearly bi-directionally coupled unified chaotic system in 2009 [11].

Along with the development of control technology [12], the requirement for control precision gets higher and higher. In order to achieve the satisfied control performance, nonlinear control techniques are used. Backstepping is one of these nonlinear control techniques that has attracted a great deal of research interest in last few years. In 2003, Yongguang Y et. al. [13] have studied controlling uncertain Lu system using backstepping design. In 2001, adaptive synchronization of uncertain chaotic systems via backstepping design was investigated by Wang et. al [14]. Tan et.al. [15] applied backstepping design for synchronizing of chaotic systems. Chaos control of 4D chaotic systems using recursive backstepping nonlinear controller have investigated by Laoye et.al. in 2007 [16].
The key idea of the backstepping approach is to recursively design controllers for subsystems in the structure and "step back" the feedback signals towards the control input. This differs from the conventional feedback linearization in that it offers a more flexible way of dealing with uncertainties. Using the Lyapunov functions, the system can be kept into account and harmful nonlinearities can be cancelled or dominated by the control signal.
In 1998, J.M. Gonzalez-Miranda [17] derived a general condition for projective synchronization (PS) in arbitrary dimensional systems. In projective synchronization the drive and response vectors synchronize up to a scaling factor. PS is characterized by a scale factor that defines a proportional relation between the synchronized systems. PS results from the partial linearity of coupled chaotic systems, and become the unique feature of partially linear systems, studied by Rehacek et.al.[18] in 1999. The proportionality allows us to duplicate a chaotic system with different scales, while the topological characteristic of the two synchronized systems remain unchanged.

2 Backstepping Design for Duffing Map

We consider the drive Duffing Map as
$$x_1(k+1) = x_2(k)$$
$$x_2(k+1) = -bx_1(k) + ax_2(k) - x_2^3(k) \qquad (1)$$
and driven Duffing map as

$$y_1(k+1) = y_2(k)$$
$$y_2(k+1) = -by_1(k) + ay_2(k) - y_2^3(k) + u(k) \tag{2}$$

Here a and b are the parameters of the Duffing map. Uncontrolled Duffing map displays chaos for $a = 2.75$ and $b = 0.2$. Now our aim is to find $u(k)$ such that the drive and response vectors become proportional, i.e., $\lim_{n\to\infty} \dfrac{y_i(n)}{x_i(n)} = \alpha$.

Defining the errors as $e_1(k) = \alpha y_1(k) - x_1(k), e_2(k) = \alpha y_2(k) - x_2(k)$ we obtain the dynamical system for the error as

$$e_1(k+1) = e_2(k)$$
$$e_2(k+1) = -be_1(k) + ae_2(k) - \alpha y_2^3(k) + x_2^3(k) + \alpha u(k) \tag{3}$$

Now the systems (1) and (2) will synchronize if for suitable choice of $u(k)$ the error system stabilizes at the origin.

Following backstepping technique we assume $z_1(k) = e_1(k)$ and choose the Lyapunov function $V_1(k) = |e_1(k)|$. Then the variation of $V_1(k)$ with time is

$$\begin{aligned}
\Delta V_1(k) &= V_1(k+1) - V_1(k) \\
&= |e_1(k+1)| - |e_1(k)| \\
&= |e_2(k)| - |z_1(k)| \\
&= |e_2(k) - z_2(k) + z_2(k)| - |z_1(k)| \\
&= |\beta(k) + z_2(k)| - |z_1(k)| \quad introducing\ z_2(k) = e_2(k) - \beta(k) \\
&= |c_1 z_1(k) + z_2(k)| - |z_1(k)| \quad choosing\ \beta(k) = c_1 z_1(k) \\
&\leq |z_2(k)| + (|c_1| - 1)|z_1(k)| \tag{4}
\end{aligned}$$

where c_1 is a design constant to be chosen later.

   Now

$$z_1(k+1) = e_1(k+1) = e_2(k) = z_2(k) + \beta(k) = c_1 z_1(k) + z_2(k) \tag{5}$$

Again,

$$\begin{aligned}
z_2(k+1) &= e_2(k+1) - \beta(k+1) \\
&= -be_1(k) + ae_2(k) - \alpha y_2^3(k) + x_2^3(k) + \alpha u(k) - c_1(c_1 z_1(k) + z_2(k)) \tag{6}
\end{aligned}$$

At this step we can determine the control law $u(k)$ as

$$u(k) = \left[c_2 z_2(k) + b e_1(k) - a e_2(k) + \alpha y_2^3(k) - x_2^3(k) + c_1(c_1 z_1(k) + z_2(k)) \right] / \alpha \qquad (7)$$

where c_2 is another design constant to be choosen later. Now with this $u(k)$ equation (7) becomes

$$z_2(k+1) = c_2 z_2(k) \qquad (8)$$

Choosing the second Lyapunov function as $V_2(k) = V_1(k) + d \mid z_2(k) \mid$, where d is a positive constant, then the variation of $V_2(k)$ can be computed as

$$\begin{aligned}
\Delta V_2(k) &= V_2(k+1) - V_2(k) \\
&= V_1(k+1) + d \mid z_2(k+1) \mid - V_1(k) - d \mid z_2(k) \mid \\
&= \Delta V_1(k) + d \mid z_2(k+1) \mid - d \mid z_2(k) \mid \\
&\leq \mid z_2(k) \mid + (\mid c_1 \mid -1) \mid z_1(k) \mid + d \mid c_2 \parallel z_2 \mid - d \mid z_2(k) \mid \\
&= (\mid c_1 \mid -1) \mid z_1(k) \mid + (d \mid c_2 \mid - d + 1) \mid z_2(k) \mid \qquad (9)
\end{aligned}$$

We choose c_1 and c_2 such that the right hand side of (9) become negative definite. Clearly the right hand side of (9) is negative definite if we choose $\mid c_1 \mid < 1$ and $\mid c_2 \mid < \dfrac{d-1}{d}$ where $d > 1$. We can choose the parameter d to be large enough. Hence the close loop system (5) and (8)

$$\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} = \begin{pmatrix} c_1 & 1 \\ 0 & c_2 \end{pmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} = A \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} \qquad (10)$$

is globally stable about the origin. Therefore system (1) is synchronized with system (2) up to a constant scaling factor α. Notice that if we choose any one of the design constants c_1, c_2 to be zero, the dead-beat response will happen since $A^2 = 0$, which implies that the error state will be bought to zero in atmost 2 time steps. Therefore projective synchronization in finite time can be achieved.
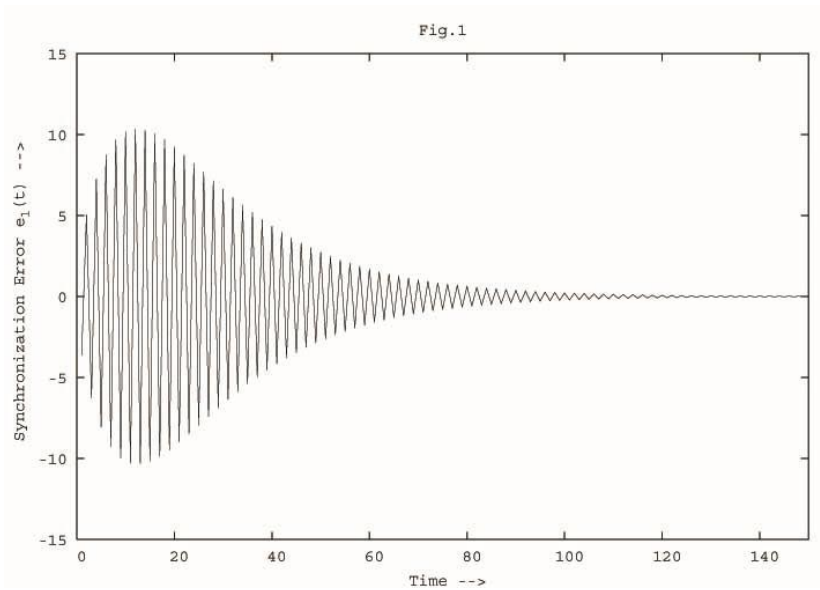
Figure 1: Time evolution of the synchronization error $e_1(t)$ for $c_1=-0.9$, $c_2=-0.95$ are drawn for Duffing map.
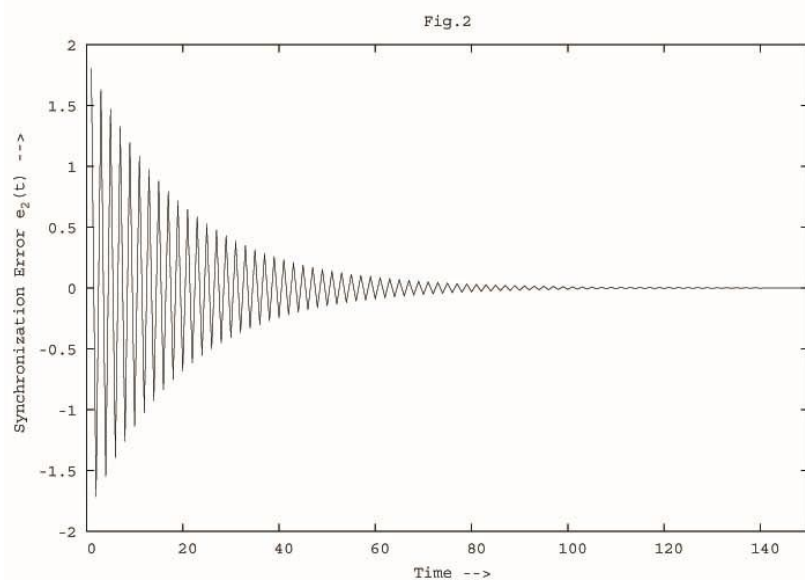

Figure 2: Time evolution of the synchronization error $e_2(t)$ for $c_1=-0.9$, $c_2=-0.95$ are drawn for Duffing map.

3 Backstepping Design for Tinkerbell Map

We consider the driving Tinkerbell's map as

$$x_1(k+1) = x_1^2(k) - x_2^2(k) + ax_1(k) + bx_2(k)$$

$$x_2(k+1) = 2x_1(k)x_2(k) + cx_1(k) + dx_2(k) \tag{11}$$

and driven Tinkerbell's map as

$$y_1(k+1) = y_1^2(k) - y_2^2(k) + ay_1(k) + by_2(k)$$
$$y_2(k+1) = 2y_1(k)y_2(k) + cy_1(k) + dy_2(k) + u(k) \tag{12}$$

Here $a, b, c\, d$ are parameters of the map. Uncontrolled map displays chaos for $a = 0.9, b = 0.6013, c = 2$ and $d = 0.5$. Now our aim is to find an controller $u(k)$ such that the drive and response system synchronize projectively. Defining the errors as $e_1(k) = \alpha y_1(k) - x_1(k), e_2(k) = \alpha y_2(k) - x_2(k)$ we obtain the dynamical system for the error as

$$e_1(k+1) = ae_1(k) + be_2(k) + \alpha(y_1^2(k) - y_2^2(k)) - (x_1^2(k) - x_2^2(k))$$
$$e_2(k+1) = ce_1(k) + d_2(k) + 2\alpha(y_1(k)y_2(k) - x_1(k)x_2(k)) + \alpha u(k) \tag{13}$$

For synchronization of (11) and (12) the error system should be stabilized at origin. According to backstepping design we assume $z_1(k) = e_1(k)$. Choosing the Lyapunov function as $V_1(k) = |e_1(k)|,$ we get the time variation of $V_1(k)$ as

$$\Delta V_1(k) = |e_1(k+1)| - |e_1(k)|$$
$$= |\alpha(y_1^2(k) - y_2^2(k)) - (x_1^2(k) - x_2^2(k) + ae_1(k) + be_2(k)| - |z_1(k)| \tag{14}$$

We choose

$$\beta(k) = c_1 z_1(k) - \alpha(y_1^2(k) - y_2^2(k)) - (x_1^2(k) - x_2^2(k) = c_1 z_1(k) + \phi(k)(say) \tag{15}$$

here c_1 is a constant to be chosen later and introduce the variable $z_2(k) = e_2(k) - \beta(k)$. Now,

$$\Delta V_1(k) = |\alpha(y_1^2(k) - y_2^2(k)) - (x_1^2(k) - x_2^2(k)) + az_1(k) + be_2(k) + z_2(k) - z_2(k)| - |z_1(k)|$$

$$\leq |\alpha(y_1^2(k) - y_2^2(k)) - (x_1^2(k) - x_2^2(k)) + be_2(k) + z_2(k) - z_2(k)| + (a-1)|z_1(k)|$$

$$= |\beta(k) + z_2(k) + \alpha(y_1^2(k) - y_2^2(k)) - (x_1^2(k) - x_2^2(k))| + (a-1)|z_1(k)|$$
$$= |c_1 z_1(k) + z_2(k)| + (a-1)|z_1(k)|$$
$$\leq z_2(k)| + (|c_1| + a - 1)|z_1(k)| \tag{16}$$

Therefore

$$z_1(k+1) = e_1(k+1) = \alpha(y_1^2(k) - y_2^2(k)) - (x_1^2(k) - x_2^2(k)) + ae_1(k) + be_2(k)$$
$$= \alpha(y_1^2(k) - y_2^2(k)) - (x_1^2(k) - x_2^2(k)) + az_1(k) + z_2(k) + \beta(k)$$
$$= c_1 z_1(k) + az_1(k) + z_2(k)$$
$$= (c_1 + a)z_1(k) + z_2(k) \tag{17}$$

Now we have
$$z_2(k+1) = be_2(k+1) - \beta(k+1)$$
$$= b[2\alpha(y_1(k)y_2(k) - x_1(k)x_2(k)) + ce_1(k) + de_2(k) + \alpha u(k)] \tag{18}$$
$$- c_1[(c_1 + a)z_1(k) + z_2(k)] - \phi(k+1)$$

At this step we choose the control lae $u(k)$ as

$$u(k) = \begin{bmatrix} c_2 z_2(k) - 2b\alpha(y_1(k)y_2(k) - x_1(k)x_2(k)) \\ -cbe_1(k) - bde_2(k) + c_1\{(c_1 + a)z_1(k) + z_2(k)\} \end{bmatrix} + \phi(k+1)/b\alpha \tag{19}$$

With this choice we get

$$z_2(k+1) = c_2 z_2(k) \tag{20}$$

Let us take second Lyapunov function as $V_2(k) = V_1(k) + d\,|z_2(k)|$ where d is a positive constant. Then the time variation of $V_2(k)$ is given by

$$\Delta V_2(k) = V_2(k+1) - V_2(k)$$
$$= \Delta V_1(k) + d\,|z_2(k+1)| + d\,|z_2(k)|$$
$$\leqq |z_2(k)| + (|c_1| + a - 1)\,|z_1(k)| + d\,|c_2\,\|z_2(k)| - d\,|z_2(k)|$$
$$= (|c_1| + a - 1)\,|z_1(k)| + (d\,|c_2| - d + 1)\,|z_2(k)| \tag{21}$$

In order to make the right hand side of (21) negative definite we have to choose $a - 1 < c_1 < 1 - a, |c_2| < \frac{d-1}{d}$ where $d > 1$. Now the close-loop system (17) and (20) become

$$\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} = \begin{pmatrix} c_1 + a & 1 \\ 0 & c_2 \end{pmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} = A \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} \tag{22}$$

is globally stable about the origin. Therefore system (1) is synchronized with system (2) up to a constant scaling factor α. Notice that if we choose any one of the design constants $c_1 = -a$, or c_2 to be zero, the dead-beat response will happen since $A^2 = 0$, which implies that the error state will be bought to zero in atmost 2 time steps. Therefore projective synchronization in finite time can be achieved.
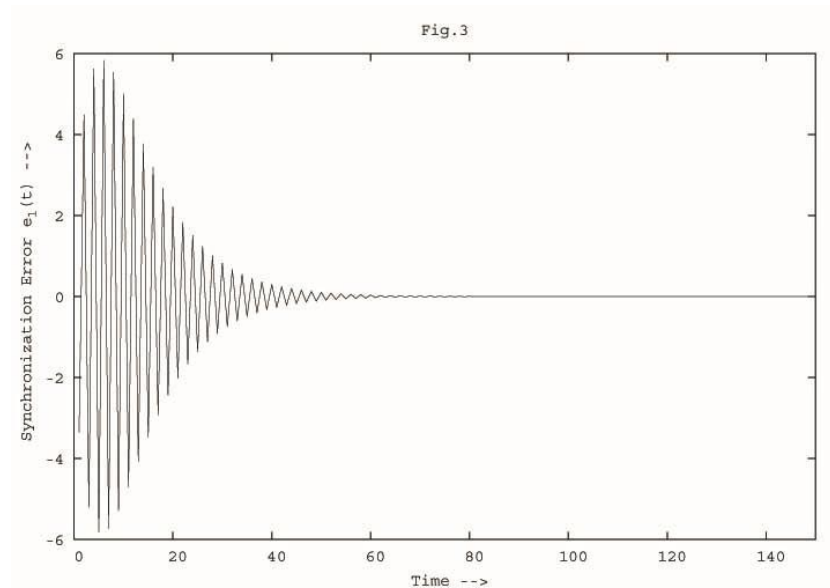
Figure 3: Time evolution of the synchronization error $e_1(t)$ for $c_1=-1.7$, $c_2=-0.9$ are drawn for Tinkerbell map.
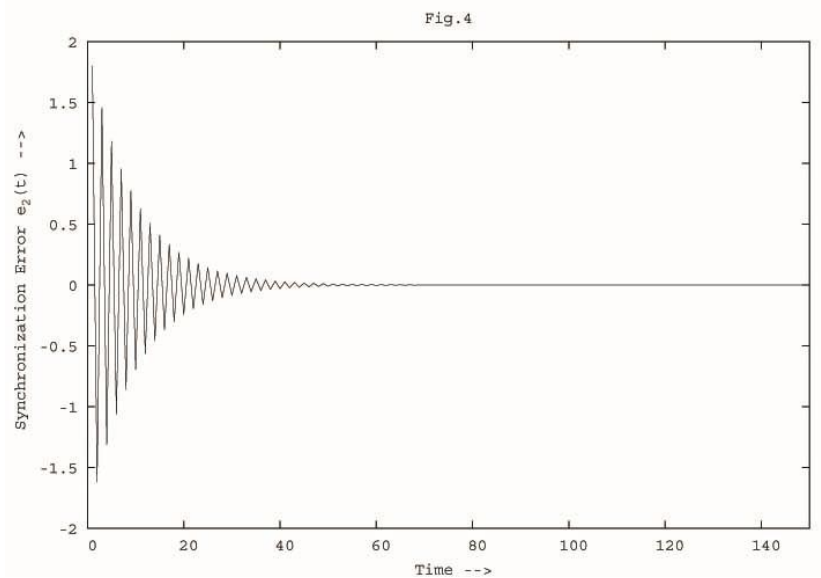

Figure 4: Time evolution of the synchronization error $e_2(t)$ for $c_1=-1.7$, $c_2=-0.9$ are drawn for Tinkerbell map.

4 Results and Discussions

We have successfully applied the backstepping method for projective synchronization of Duffing map and Tinkerbell map. The numerical simulation

results for the case of Duffing map are shown in Fig.1 and Fig.2 taking the value of $c_1 = -0.9$ and $c_2 = -0.95$. In both the cases the error goes to zero through oscillation. We assume the synchronization errors was initially $e_1(t) = 1.9$ and $e_2(t) = -1.9$. Simulation results for the case of Tinkerbell map are shown in Fig.4 and Fig.5 taking the values of $c_1 = -1.7$ and $c_2 = -0.9$. Here also the error goes to zero in a oscillatory way. The numerical simulation results indicate that this approach works very well.

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