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***Research Paper***

# Effects of radiation on stability of triangular equilibrium points in elliptic restricted three body problem

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**Abstract**

This paper deals with the stability of triangular Lagrangian points in the elliptical restricted three body problem, under the effect of radiation pressure stemming from the more massive primary on the infinitesimal. We adopted a set of rotating pulsating axes centered at the centre of mass of the two primaries Sun and Jupiter. We have exploited method of averaging used by Grebenikov, throughout the analysis of stability of the system. The critical mass ratio depends on the radiation pressure, eccentricity and the range of stability decreases as the radiation parameter increases.

***Keywords***: *Dynamical system, elliptical restricted three body problems, lagrangian points, radiation pressure, and stability.*

## 1. Introduction

The restricted three body problem (R3BP) describes the motion of an infinitesimal mass moving under the gravitational effect of the two finite masses, called primaries, which move in circular orbits around their centre of masses on account of their mutual attraction and the infinitesimal mass not influencing the motion of primaries. The primaries move in circular keplarian orbit and hence the name circular restricted three body problems (CR3BP). Synodic reference system is used for the circular problem, in which the primaries are fixed with respect to uniformly rotating axes; hence the Hamiltonian does not depend on time explicitly. But (CR3BP) has a disadvantage that this formulation cannot treat the long-time behavior of practically important dynamical systems in celestial mechanics. The reason is that significant effects might be expected because of the eccentricity of the orbits of the primaries. On the other hand, in Elliptic restricted three body problems (ER3BP), the primaries move in an elliptical keplarian orbit. The introduction of non-uniformly rotating and pulsating co-ordinate system results again in fixed location of primaries. The elliptical restricted three body problem generalizes the original circular restricted three body problems, while some useful properties of circular model still can be satisfied to the elliptical case. The Hamiltonian however, in this case, does depend explicitly on time “as given by Szebehely [1]”. The ER3BP describes the dynamical system more accurately as the primaries move along the elliptical orbit.

In spite of the large amount of analytical and numerical work in CR3BP, there are relatively a few analytical results in ER3BP. The stability of the infinitesimal around the triangular equilibrium points in the elliptical restricted three body is “described in considerable details by Grebenikov [1], Danby [3], Bennet [4], Rabe [5], [6], Meire [7], Markeellos [8], Roberts [9], Zimvoschikov and Thakai [10], Ammar [11]”. The linear stability of elliptic Lagrange orbits in ER3BP by numerical integration technique has been “studied by Danby [3]”. The linear stability of the periodic orbits of Lagrange in the ER3BP using perturbation technique has been “investigated by Roberts [9]”; he proved that for some mass values; the elliptic orbits are linearly stable. The librational solutions to the photogravitational restricted three body problems by considering both primaries as radiating have been “studied by Khasan [12], [13]”. The effect of solar radiation pressure on the location and stability of lagrangian points in ER3BP has been “studied by Ammar [11]” and was seen that radiation pressure plays the role of reducing the effective mass and slightly changes the location of the Lagrangian points. It was found that the triangular equilibrium points are stable for $\mu \le 0.5$, satisfying the conditions $(27+6\beta)\mu(1-\mu) \le (1+4e\cos v)^2$, where $v$ is the true anomaly of the either primaries and $\beta$ the radiation pressure emanating from more massive primaries. The analytical investigation concerning the structure of asymptotic perturbative approximation for small amplitude motions has been "performed by Selaru and Cucu- Dumitrescu [14], [15]", provided the third point mass lies in the neighbourhood of a Lagrangian equilateral points in the planer , elliptical restricted three bodies.

Non-linear stability of the triangular equilibrium points of the elliptical restricted three body problem was "studied by Gyorgrey [16], Kumar and Choudhary [17], Erdi [18]". Furthermore, the nonlinear stability of the infinitesimal in the orbits or the size of the stable region around $L_4$ was "studied by Gyorgrey [16]" and the parametric resonance stability around $L_4$, in elliptical restricted three body problem was "studied by Erdi [18]". The influence of the eccentricity of the orbit of the primary bodies with or without radiation pressure on the existence of the equilibrium points and there stability was "discussed to some extent by Khasan [12], [13], Pinyol [19], Floria [20], Halan and Rana [21], Markeev [22], Selaru and Dumitrescu [14], [15], Nayayan and Ramesh [23], [24]". The stability of triangular points in the elliptical restricted three body problem under radiating and oblate primaries was "studied by Singh and Umar [25], [26]".

The present study aims to examine the combined effects of gravitational forces of the primaries rotating in an elliptic orbit around their centre of mass and radiation pressure emanating from the Sun on the infinitesimal particle analytically and numerically. An application of this problem can be seen in the motion of the Trojan asteroids around the triangular points $L_4$. The asteroids in this case are only influenced by the gravitational forces of the Sun and Jupiter and the orbit of Jupiter around the sun is assumed to be fixed ellipse. We have exploited the method of averaging"used by Grebenikov [2]", throughout the analysis for studying the existence and stability of infinitesimal. Attempt has been made to study the condition of convergence of the series representing the solution of problem by Fourier series expansion. Finally, transition Curves are plotted between critical mass ratio ( $\mu^*$ ) and eccentricity (e) for different values of radiation pressure using Matlab 7.1 Software.

## 2. Equation of motion

The ER3BP models the motion of a test particle having infinitesimal mass m, and moving under the influence of the gravitational field of two massive bodies of masses $m_1$ and $m_2$ that revolve in an elliptic orbit. Accordingly to the usual practice and without loss of generality we choose a system of units as the gravitational constants and the sum of the finite masses equals to the unity, i.e.

$$m_1 + m_2 = 1; \quad \mu = \frac{m_2}{m_1 + m_2}; 1 - \mu = \frac{m_1}{m_1 + m_2}; 0 < \mu < \frac{1}{2}. \tag{2.1}$$

If in addition to this the value of the orbital angular momentum of the relative motion of the primaries is unity, then the semi-latus rectum of the elliptical orbit will be equals to one and the polar equation of ellipse will take the form

$$r = \frac{1}{1 + e\cos v} \tag{2.2}$$

Where $v$ is the true anomaly of the either primaries.

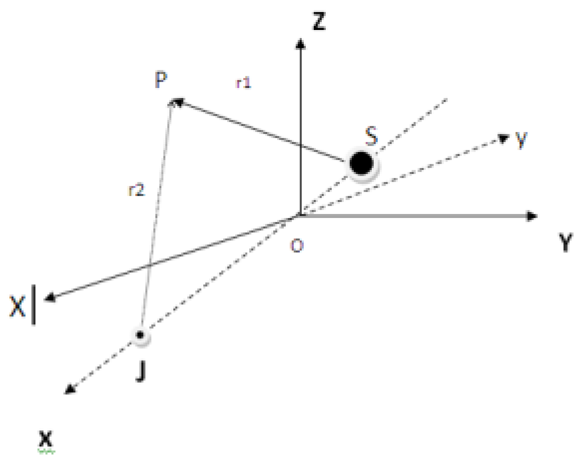


**Fig. 1:** The location of the sun (S), Jupiter (J) and the particle in the rotating frame.

The equations of motion of the infinitesimal mass in the elliptic restricted three body problem in rotating pulsating coordinate system are presented below "following Ammar [11]" by

$$\frac{d^2\bar{x}}{dv^2} - 2\frac{d\bar{y}}{dv} = \frac{1}{1 + e\cos v}\frac{\partial U}{\partial \bar{x}};$$

$$\frac{d^2\bar{y}}{dv^2} + 2\frac{d\bar{x}}{dv} = \frac{1}{1 + e\cos v}\frac{\partial U}{\partial \bar{y}};$$

$$\frac{d^2\bar{z}}{dv^2}+\frac{d\bar{z}}{dv}=\frac{1}{1+e\cos v}\frac{\partial U}{\partial \bar{z}}. \tag{2.3}$$

Where,

$$U=(1-\mu)\left[\frac{1}{2}\bar{r}_1^2+\frac{(1-\beta)}{\bar{r}_1}\right]+\mu\left[\frac{1}{2}\bar{r}_2^2+\frac{1}{\bar{r}_2}\right]-\frac{1}{2}\frac{\bar{z}^2}{r}; \tag{2.4}$$

$$\bar{r}_1^2=(\bar{x}+\mu)^2+\bar{y}^2+\bar{z}^2; \tag{2.5}$$

$$\bar{r}_2^2=(\bar{x}+\mu-1)^2+\bar{y}^2+\bar{z}^2. \tag{2.6}$$

$$x=r\bar{x};y=r\bar{y};z=r\bar{z};r_1=r\bar{r}_1;r_2=r\bar{r}_2 \tag{2.7}$$

The above transformation is used to convert the motion of infinitesimal from rotating frame of reference to pulsating coordinate system,

Where $v$ is the true anomaly as the independent variable in elliptical orbit.

The total action from the Sun on the particle can be expressed by the acceleration "as given by Ammar [11]".

$$F_{total}=F_{gr}-F_{rad}=(1-\beta)F_{gr}=-\frac{(1-\beta)m_1}{\bar{r}_1^2}; \tag{2.8}$$

Equation (2.3) has the particular solution "as given by Ammar [11]".

$$\bar{x}_0=-\mu+\frac{1}{2}(1-\beta)^{2/3};\ \bar{y}_0=\pm(1-\beta)^{1/3}\sqrt{1-\frac{(1-\beta)^{2/3}}{4}};\ \bar{z}_0=0; \tag{2.9}$$

The three bodies nearly form an equilateral triangle in the coordinate system mentioned above. Since the equilateral points are symmetrical to each other, the nature of motion near the two triangular points is the same. Therefore, it is sufficient to analyze the motion of the triangular equilibrium points having the location ( $\bar{x}_0,\bar{y}_0,\bar{z}_0$ ).

In order to investigate the stability of the equilibrium points (2.9) in the first approximation, we derive the equation for variations in the coordinates. Let $\xi,\eta,\varsigma$ denotes small displacement in $\bar{x}_0,\bar{y}_0,\bar{z}_0$

Then $\bar{x}=\bar{x}_0+\xi;\bar{y}=\bar{y}_0+\eta;\bar{z}=\zeta.$ (2.10)

Differentiating with respect to $v$ , we get;

$$\bar{x}'=\xi'\ ;\ \bar{y}'=\eta' \qquad \bar{x}''=\xi''\ ;\ \bar{y}''=\eta'' \tag{2.11}$$

Assuming $\Omega(\bar{x},\bar{y},\bar{z})=U(1+e\cos v)^{-1}$.

Where $U$ defined by equation (2.4), is potential function. From the third equation of the system (2.3), the conditions $\Omega_{\bar{z}}=0$ ,implies that $\bar{z}=0$ .That is all the critical points are planar and no equilibrium points can be found outside the $\bar{x}\ \bar{y}$ plane "as explained by Szebehely [1]".Considering,

$$\Omega_{\bar{x}}=\Omega_{\bar{x}}(\bar{x},\bar{y})=\Omega_{\bar{x}}(\bar{x}_0+\xi,\bar{y}_0+\eta) \tag{2.12}$$

Applying Taylor's theorem in equation (2.11) and retaining first order terms in the infinitesimal $\xi$ and $\eta$ , we get:

$$\Omega_{\bar{x}}=\Omega_{\bar{x}}^0+\xi\Omega_{\bar{x}\bar{x}}^0+\eta\Omega_{\bar{x}\bar{y}}^0; \tag{2.13}$$

$$\Omega_{\bar{y}}=\Omega_{\bar{y}}^0+\xi\Omega_{\bar{y}\bar{x}}^0+\eta\Omega_{\bar{y}\bar{y}}^0. \tag{2.14}$$

Here, the subscript in $\Omega$ indicates the first and the second order partial derivative as the subscript appears once or twice, and the superscript '0' denotes partial derivatives evaluated at the equilibrium point ( $\bar{x}_0,\bar{y}_0$ ).Also at the equilibrium point, ( $\bar{x}_0,\bar{y}_0$ ), we have; $\Omega_{\bar{x}}^0=\Omega_{\bar{y}}^0=0.$

The perturbations in $\xi,\eta,\zeta$ are given by the equations:

$$\xi''-2\eta'=\frac{1}{1+e\cos v}\left[\left(\Omega_{\bar{x}\bar{x}}^0\xi+\Omega_{\bar{x}\bar{y}}^0\eta\right)\right];\quad \eta''+2\xi'=\frac{1}{1+e\cos v}\left[\left(\Omega_{\bar{x}\bar{y}}^0\xi+\Omega_{\bar{y}\bar{y}}^0\eta\right)\right]; \qquad \xi''=-\zeta. \tag{2.15}$$

Now, we have:

$$\Omega^0_{\bar{x}\bar{x}}=\frac{3}{4(1+e\cos v)}\left[Q+\mu(4-Q)(1-Q)\right];$$

$$\Omega^0_{\bar{x}\bar{y}}=\pm\frac{3}{4(1+e\cos v)}\sqrt{Q(4-Q)}\left[1+\mu Q-3\mu\right]; \tag{2.16}$$

$$\Omega^0_{\bar{y}\bar{y}}=\frac{3}{4(1+e\cos v)}\left[4-Q-\mu(4-Q)(1-Q)\right].$$

Where $Q=(1-\beta)^{2/3}$.

From equations (2.15) and (2.16), the perturbations in $\xi,\eta,\zeta$ are given by the following equations:

$$\xi'' - 2\eta' = \frac{3}{4(1+e\cos v)}\left[Q + \mu(4-Q)(1-Q)\right]\xi + \frac{3}{4(1+e\cos v)}\left[\sqrt{Q(4-Q)}(1+\mu Q - 3\mu)\right]\eta;$$
$$\eta'' + 2\xi' = \frac{3}{4(1+e\cos v)}\left[\sqrt{Q(4-Q)}(1+\mu Q - 3\mu)\right]\xi + \frac{3}{41+e\cos v)}\left[(4-Q-\mu(4-Q)(1-Q)\right]\eta;$$
$$\xi'' = -\zeta. \tag{2.17}$$

The last equation of (2.17) yields:

$$\zeta = C_1\cos v + C_2 \sin v. \tag{2.18}$$

Where $C_1$ and $C_2$ are arbitrary constants. Hence in order to investigate the stability of equilibrium points, we consider first two equations of (2.17).The stability of the equilibrium points has been investigated by introducing a new variable given by:

$$\bar{x}_1 = \xi;\ \bar{x}_2 = \eta;\ \bar{x}_3 = \frac{d\xi}{dv};\ \bar{x}_4 = \frac{d\eta}{dv}. \tag{2.19}$$

The equations of motion (2.17) in the variables $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4$ can be written as:

$$\frac{d\bar{x}_i}{dv} = P_{i1}\bar{x}_1 + P_{i2}\bar{x}_2 + P_{i3}\bar{x}_3 + P_{i4}\bar{x}_4; \quad i = 1,2,3,4 \tag{2.20}$$

Where, $P_{11} = P_{12} = P_{14} = P_{21} = P_{22} = P_{23} = P_{33} = P_{44} = 0;\ P_{13} = 1,\ P_{24} = 1,\ \ P_{34} = 2,\ \ P_{43} = -2;$

$$P_{31} = \frac{1}{(1+e\cos v)}\Omega^0_{\bar{x}\bar{x}} = \frac{3}{4(1+e\cos v)}\left[Q + \mu(4-Q)(1-Q)\right];$$
$$P_{32} = \frac{1}{(1+e\cos v)}\Omega^0_{\bar{x}\bar{y}} = \pm\left[\frac{3}{4(1+e\cos v)}\sqrt{Q(4-Q)}(1+\mu Q - 3\mu)\right] = P_{41};$$
$$P_{42} = \frac{1}{(1+e\cos v)}\Omega^0_{\bar{y}\bar{y}} = \frac{3}{4(1+e\cos v)}\left[4-Q-\mu(4-Q)(1-Q)\right]. \tag{2.21}$$

The coefficients in the system of equation (2.20) are periodic function of '$v$ ' with period $2\pi$. Considering the averaged system "as given by Grebenikov [2]", suitable for finding solution of the problem we get;

$$\frac{d\bar{x}_i^{(0)}}{dv} = P_{i1}^{(0)}\bar{x}_1^{(0)} + P_{i2}^{(0)}\bar{x}_2^{(0)} + P_{i3}^{(0)}\bar{x}_3^{(0)} + P_{i4}^{(0)}\bar{x}_4^{(0)} \tag{2.22}$$

$$P_{i,s}^{(0)} = \frac{1}{2\pi}\int_0^{2\pi} P_{is}(v)dv;\ i,s = 1,2,3,4 \tag{2.23}$$

$$\bar{x}_i^{(0)} = \sum_{s=1}^{4} C_s \bar{x}_{is}^{(0)}; \tag{2.24}$$

Where $C_s$ are arbitrary constants and $\{\bar{x}_{is}^{(0)}\}$ are the fundamental system of solution, of equation (2.22) and solution is given by equation (2.9) and equation (2.10).

After evaluation we get:

$$P_{11}^{(0)} = P_{12}^{(0)} = P_{14}^{(0)} = P_{21}^{(0)} = P_{22}^{(0)} = P_{23}^{(0)} = P_{33}^{(0)} = P_{44}^{(0)} = 0.\ P_{13}^{(0)} = 1\ \ P_{24}^{(0)} = 1\ \ P_{34}^{(0)} = 2\ \ P_{43}^{(0)} = -2. \tag{2.25}$$

$$P_{31}^{(0)} = \frac{1}{2\pi}\int_0^{2\pi} P_{31}dv = \frac{3}{4\sqrt{1-e^2}}\left[(Q + \mu(4-Q)(1-Q))\right]; \qquad P_{32}^{(0)} = \frac{1}{2\pi}\int_0^{2\pi} P_{32}dv = \pm\frac{3}{4\sqrt{1-e^2}}\left[\sqrt{Q(4-Q)}(1+\mu Q - 3\mu)\right] = P_{41}^{(0)}$$
$$P_{41}^{(0)} = \frac{1}{2\pi}\int_0^{2\pi} P_{32}dv = \pm\frac{3}{4\sqrt{1-e^2}}\left[\sqrt{Q(4-Q)}(1+\mu Q - 3\mu)\right];$$
$$P_{42}^{(0)} = \frac{1}{2\pi}\int_0^{2\pi} P_{42}dv = \frac{3}{4\sqrt{1-e^2}}\left[(4-Q-\mu(4-Q)(1-Q))\right]. \tag{2.26}$$

# 3. Stability of triangular equilibrium points

The characteristics equation for the system of equation of variations (3.22) is given by:

$$\lambda^4 - T\lambda^2 + R = 0; \tag{3.1}$$

Where, $T = P_{31}^{(0)} + P_{42}^{(0)} - 4;\ \ = \dfrac{3}{4\sqrt{1-e^2}} - 4.$ (3.2)

$$R = P_{31}^{(0)}\cdot P_{42}^{(0)} - P_{32}^{(0)}\cdot P_{41}^{(0)} = \frac{9}{4\sqrt{1-e^2}}(4-Q)\mu(1-\mu); \tag{3.3}$$

We obtained the characteristics roots given by:

$$\lambda = \pm\left(\frac{T}{2} \pm \frac{(T^2-4R)^{1/2}}{2}\right)^{1/2};$$

$$\lambda = \pm\sqrt{\left(\frac{3}{2\sqrt{1-e^2}}-2\right) \pm \sqrt{\Delta}}. \tag{3.4}$$

(3.5)

Where, $\Delta = \left(2-\frac{3}{2\sqrt{1-e^2}}\right)^2 - 4\frac{9\mu(1-\mu)(4-Q)}{4\sqrt{1-e^2}}.$ (3.6)

For the stability of Lagrangian points, the eccentricity satisfies the inequality given by

$$\left(\frac{3}{2\sqrt{1-e^2}}-2 \geq 0\right); \tag{3.7}$$

. Thus in case the eccentricity does not satisfy inequality (3.7), the characteristics roots will be either real or complex conjugate. In case of complex roots, there must be roots with positive real parts leading to instability of the equilibrium points in the first approximation.

Thus, if the eccentricity satisfies the inequality $\frac{\sqrt{7}}{4} = e_0 \leq e < 1$, the triangular Lagrangian points are unstable. The result is in conformity with "that of Grebenikov [2]".

The characteristics roots will be purely imaginary if:

$$T < 0; \tag{3.8}$$

$$T^2 - 4R \geq 0; \tag{3.9}$$

From the inequality (3.8), it follows that:

$$\mu(1-\mu)(27+6\beta) \leq (3-4\sqrt{1-e^2})^2 \tag{3.10}$$

It is clear that when $e=0, \beta=0$ , equation (3.10) reduces to the well-known condition for stability of the triangular equilibrium points in the CR3BP, that is $27\mu(1-\mu) \leq 1.$

"Ammar [11]" found that the triangular equilibrium points are stable for $\mu \leq 0.5$ , satisfying the conditions $(27+6\beta)\mu(1-\mu) \leq (1+4e\cos v)^2$ , where $v$ is the true anomaly of the either primaries and $\beta$ the radiation pressure emanating from more massive primaries.

Since $\mu \leq 1/2$ , the inequality (3.10) is satisfied, when $0 < \mu < \mu*$

Where $\mu* = \frac{1}{2} - \frac{1}{2}\sqrt{1-\frac{4(4\sqrt{1-e^2}-3)^2}{27+6\beta}}.$

(3.11)

The value of $\mu^*$ "as per Ammar [11]" is $\mu* = \frac{1}{2} - \frac{1}{2}\sqrt{1-\frac{4+32e(1+2e)}{27+6\beta}}.$

Slight difference in the expression of $\mu^*$ is found"as compared to Ammar [11]". It is possible that the difference in result may be due to method of averaging adopted.

The two roots of equation (3.1) are represented as follows:

$$\alpha_1 = \sqrt{2-\frac{3}{2\sqrt{1-e^2}}-\sqrt{\Delta}}; \tag{3.12}$$

$$\alpha_2 = \sqrt{2-\frac{3}{2\sqrt{1-e^2}}+\sqrt{\Delta}}; \tag{3.13}$$

We have $\alpha_1^2 + \alpha_2^2 = \left(4-\frac{3}{\sqrt{1-e^2}}\right);$

(3.14)

Hence, taking the limit as $\lim\limits_{\substack{e\to 0\\ \beta\to 0}} (\alpha_1^2+\alpha_2^2) = 1.$ (3.15)

For values of the eccentricity close to $e_0$, both $\alpha_1$ and $\alpha_2$ are small quantities .For small eccentricities, when one of the quantities $\alpha_1$ is small, the others differ a little from unity.

The schematic behavior of the system can be investigated when the infinitesimal move around the triangular equilibriums points under the radiating bigger primaries by plotting the transition curves for different values of 'e' and parameters 'β' using MATLAB 7.1 version of software. We have plotted curves between different values of eccentricity of the orbit and $\mu^*$ (critical mass ratio) by varying radiation parameter. We observe that when the radiation parameter increases the region stability of the infinitesimal around triangular points decreases which is obvious from the figure2.

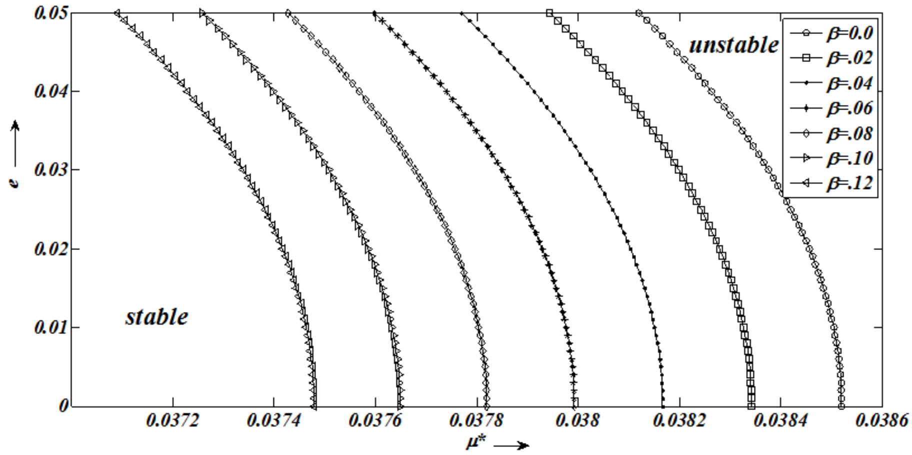


**Fig. 2:** Correlation between μ* (critical mass ratio) and e (eccentricity) for $\beta \in [0:0.14)$

Thus, the triangular equilibrium points are stable if the eccentricity 'e' satisfies the condition (3.7) and the mass ratio μ* obeys the equation (3.11).The dependence of μ* on the eccentricity are plotted in graph for different values of radiation pressure. It is observed that as radiation pressure is increasing the range of stability is decreasing.

# 4. Solutions of the un averaged equations

We analyze further, that the Fourier expansion of the co-efficient of the Eq. (2.20) can be written as:

$$P_{is} = P_{is}{}^{(0)} + \sum_{k=1}^{\infty} P_{is}{}^{(k)}, i, s = 1,2,3,4. \tag{4.1}$$

$$P_{11}{}^{(k)} = P_{12}{}^{(k)} = P_{13}{}^{(k)} = P_{14}{}^{(k)} = P_{21}{}^{(k)} = P_{22}{}^{(k)} = P_{23}{}^{(k)} = P_{24}{}^{(k)} = P_{33}{}^{(k)} = P_{34}{}^{(k)} = P_{43}{}^{(k)} = P_{44}{}^{(k)} = 0,$$
$$k = 1,2,3,...... \tag{4.2}$$

$$P_{31}{}^{(k)} = \frac{3}{4}\left[\left(Q + \mu(4-Q)(1-Q)\right)\right] a_k \cos kv;$$
$$P_{32}{}^{(k)} = \frac{3}{4}\left[\sqrt{Q(4-Q)}(1+\mu Q - 3\mu)\right] a_k \cos kv;$$
$$P_{41}{}^{(k)} = \frac{3}{4}\left[\sqrt{Q(4-Q)}(1+\mu Q - 3\mu)\right] a_k \cos kv; \tag{4.3}$$
$$P_{42}{}^{(k)} = \frac{3}{4}\left[\left(4-Q-\mu(4-Q)(1-Q)\right)\right] a_k \cos kv;$$
$$k = 1,2,3,4.$$

Where $a_k = \frac{2}{\pi}\int_0^{\pi} \frac{\cos kv dv}{1+e\cos v}.$ (4.4)

Using, the expansion of the elliptic function, it can be shown, that $a_k$ is a power series whose first term starts at $e^k$ and that:

$$|a_k| < 2e^k. \tag{4.5}$$

All the coefficients of Eq. (4.3) satisfy the inequality given as follows:

$$|P_{is}| < (9/2)e^k. \tag{4.6}$$

We look for an analytical solution of system of the form:

$$\bar{x}_i = \bar{x}_i{}^{(0)} + \sum_{k=1}^{\infty} \bar{x}_i{}^{(k)}; \tag{4.7}$$

Where $\{\bar{x}_i{}^{(0)}\}$ is the general solution of the homogenous system of Eq. (4.20). It is given by

$$\bar{x}_i{}^{(0)} = \sum_{s=1}^{4} C_s \bar{x}_{is}{}^{(0)}; \tag{4.8}$$

Where $C_s$ are arbitrary constants and $\{\bar{x}_{is}{}^{(0)}\}$ are the fundamental system of solution, of system of equation (2.20) given as follows:

$\bar{x}_{11}{}^{(0)} = l_1{}^{(1)}\cos\alpha_1 v - l_1{}^{(2)}\sin\alpha_1 v$, $\bar{x}_{12}{}^{(0)} = l_1{}^{(1)}\sin\alpha_1 v + l_1{}^{(2)}\cos\alpha_1 v$, $\bar{x}_{13}{}^{(0)} = l_1{}^{(3)}\cos\alpha_2 v - l_1{}^{(4)}\sin\alpha_2 v$, $\bar{x}_{14}{}^{(0)} = l_1{}^{(3)}\sin\alpha_2 v + l_1{}^{(4)}\cos\alpha_2 v$,
$\bar{x}_{21}{}^{(0)} = l_2{}^{(1)}\cos\alpha_1 v$, $\bar{x}_{22}{}^{(0)} = l_2{}^{(1)}\sin\alpha_1 v$, $\bar{x}_{23}{}^{(0)} = l_2{}^{(3)}\cos\alpha_2 v$,
$\bar{x}_{24}{}^{(0)} = l_2{}^{(3)}\sin\alpha_2 v$, $\bar{x}_{31}{}^{(0)} = l_3{}^{(1)}\cos\alpha_1 v - l_3{}^{(2)}\sin\alpha_1 v$, $\bar{x}_{32}{}^{(0)} = l_3{}^{(1)}\sin\alpha_1 v + l_3{}^{(2)}\cos\alpha_1 v$, $\bar{x}_{33}{}^{(0)} = l_3{}^{(3)}\cos\alpha_2 v - l_3{}^{(4)}\sin\alpha_2 v$,
$\bar{x}_{34}{}^{(0)} = l_3{}^{(3)}\sin\alpha_2 v + l_3{}^{(4)}\cos\alpha_2 v$, $\bar{x}_{41}{}^{(0)} = -l_4{}^{(2)}\sin\alpha_1 v$, $\bar{x}_{42}{}^{(0)} = l_4{}^{(2)}\cos\alpha_1 v$, $\bar{x}_{43}{}^{(0)} = -l_4{}^{(4)}\sin\alpha_2 v$, $\bar{x}_{44}{}^{(0)} = l_4{}^{(4)}\cos\alpha_2 v$. (4.9)

Where

$$l_1{}^{(1)} = \frac{3}{4}\left[\sqrt{Q(4-Q)}(1+\mu Q - 3\mu)\right],$$

$$l_1{}^{(3)} = -\frac{3}{4}\left[\sqrt{Q(4-Q)}(1+\mu Q - 3\mu)\right], \quad l_1{}^{(2)} = -2\alpha_1, \quad l_1{}^{(4)} = -2\alpha_2, \quad l_2{}^{(1)} = \frac{3}{4}+\alpha_1{}^2, \quad l_2{}^{(3)} = \frac{3}{4}+\alpha_2{}^2, \quad l_3{}^{(1)} = 2\alpha_1{}^2,$$

$$l_3{}^{(2)} = -\frac{3}{4}\left[\sqrt{Q(4-Q)}(1+\mu Q - 3\mu)\right]\alpha_1,$$

$$l_3{}^{(3)} = 2\alpha_2{}^2, \; l_3{}^{(4)} = -\frac{3}{4}\left[\sqrt{Q(4-Q)}(1+\mu Q - 3\mu)\right]\alpha_2, \; l_4{}^{(2)} = \left(\frac{3}{4}+\alpha_1{}^2\right)\alpha_1, \; l_4{}^{(4)} = \left(\frac{3}{4}+\alpha_2{}^2\right)\alpha_2, \tag{4.10}$$

The k[th] approximation is given by the system of equation:

$$\frac{d\bar{x}_i^{(k)}}{d\text{v}} = \sum_{s=1}^{4} P_{is}{}^{(0)}\bar{x}_i{}^{(k)} + \sum_{s=1}^{4}\sum_{m=0}^{k-1} P_{is}{}^{(k-m)}\bar{x}_s{}^{(m)}. \tag{4.11}$$

We look for a solution of a system of equation (4.11) which is represented in the form:

$$\bar{x}_i{}^{(k)} = (\Lambda_{i,0}^{(k)}v + \Lambda_{i,0}^{(k)})\cos\alpha_1 v + (\lambda_{i,0}^{(k)}v + \mathrm{B}_{i,0}^{(k)})\sin\alpha_1 v$$
$$+(\Gamma_{i,0}^{(k)}v + C_{i,0}^{(k)})\cos\alpha_2 v + (\gamma_{i,0}^{(k)}v + D_{i,0}^{(k)})\sin\alpha_2 v$$
$$+\sum_{s=1}^{k}[A_{i,s}^{(k)}\cos(\alpha_1+s)v + A_{i,-s}^{(k)}\cos(\alpha_1-s)v + B_{i,s}^{(k)}\sin(\alpha_1+s)v + B_{i,-s}^{(k)}\sin(\alpha_1-s)v +$$
$$C_{i,s}^{(k)}\cos(\alpha_2+s)v + C_{i,-s}^{(k)}\cos(\alpha_2-s)v + D_{i,s}^{(k)}\sin(\alpha_2+s)v + D_{i,-s}^{(k)}\sin(\alpha_2-s)v\,]. \tag{4.12}$$

The unknown coefficients of Eq. (12) $\Lambda_{i,0}^{(k)},\ldots\ldots\ldots,D_{i,-s}^{(k)}$ are determined by the algebraic equations given by:

$$\begin{aligned}
&(P_{31}{}^{(0)}+\alpha_1{}^2)\lambda_{1,0}{}^{(k)} + P_{32}{}^{(0)}\lambda_{2,0}{}^{(k)} + P_{34}{}^{(0)}\lambda_{4,0}{}^{(k)} = 0,\\
&P_{41}{}^{(0)}\lambda_{1,0}{}^{(k)} + (P_{42}{}^{(0)}+\alpha_1{}^2)\lambda_{2,0}{}^{(k)} + P_{43}{}^{(0)}\lambda_{3,0}{}^{(k)} = 0,\\
&-P_{34}{}^{(0)}\alpha_1{}^2\lambda_{2,0}{}^{(k)} + (P_{31}{}^{(0)}+\alpha_1{}^2)\lambda_{3,0}{}^{(k)} + P_{32}{}^{(0)}\lambda_{4,0}{}^{(k)} = 0,\; -P_{41}{}^{(0)}\alpha_1{}^2\lambda_{1,0}{}^{(k)} + P_{41}{}^{(0)}\alpha_1{}^2\lambda_{3,0}{}^{(k)} + (P_{42}{}^{(0)}+\alpha_1{}^2)\lambda_{4,0}{}^{(k)} = 0.
\end{aligned} \tag{4.13}$$

And

$$\begin{aligned}
&(P_{31}{}^{(0)}+\alpha_2{}^2)\gamma_{1,0}{}^{(k)} + P_{32}{}^{(0)}\gamma_{2,0}{}^{(k)} + P_{34}{}^{(0)}\gamma_{4,0}{}^{(k)} = 0,\\
&P_{41}{}^{(0)}\gamma_{1,0}{}^{(k)} + (P_{42}{}^{(0)}+\alpha_2{}^2)\gamma_{2,0}{}^{(k)} + P_{43}{}^{(0)}\gamma_{3,0}{}^{(k)} = 0,\\
&-P_{34}{}^{(0)}\alpha_2{}^2\gamma_{2,0}{}^{(k)} + (P_{31}{}^{(0)}+\alpha_2{}^2)\gamma_{3,0}{}^{(k)} + P_{32}{}^{(0)}\gamma_{4,0}{}^{(k)} = 0,\; -P_{41}{}^{(0)}\alpha_2{}^2\gamma_{1,0}{}^{(k)} + P_{41}{}^{(0)}\alpha_2{}^2\gamma_{3,0}{}^{(k)} + (P_{42}{}^{(0)}+\alpha_2{}^2)\gamma_{4,0}{}^{(k)} = 0.
\end{aligned} \tag{4.14}$$

The determinants of system (13) and (14) are different from zero,

$$D_1(0) = 2.74 - 13.0938\beta - 7.3397e^2; \tag{4.15}$$

$$D_2(0) = 6.2202 + 34.88088\beta - 57.42e^2; \tag{4.16}$$

.Thus, it can be shown that the solution of the differential equations (4.11) giving the k[th] approximation contains only periodic terms if it is assumed that its non-homogeneous part is of the expanded form of (4.11). We will now show that the equations giving the (k+1)[th] approximation are of the same form. The (k+1)[th] approximation is obtained from the system. $\frac{d\bar{x}_i^{(k+1)}}{d\text{v}} = \sum_{s=1}^{4} P_{is}{}^{(0)}\bar{x}_s{}^{(k+1)} + \sum_{s=1}^{4}\sum_{m=0}^{k} P_{is}{}^{(k+1-m)}\bar{x}_s{}^{(m)}.$ (4.17)

The non-homogenous part of equation (17) is of the form:

$$\sum_{s=1}^{4}\sum_{m=0}^{k} P_{is}{}^{(k+1-m)}\bar{x}_s{}^{(m)}. = \sum_{s=1}^{4}[P_{is}{}^{(1)}\bar{x}_s{}^{(k)} + P_{is}{}^{(2)}\bar{x}_s{}^{(k-1)} + \ldots\ldots\ldots\ldots + P_{is}{}^{(k+1)}\bar{x}_s{}^{(0)}];$$

$$= \sum_{s=1}^{4}[\tilde{\wp}_{is}^{(1)}\cos v\bar{x}_s{}^{(k)} + \tilde{\wp}_{is}^{(2)}\cos 2v\bar{x}_s{}^{(k-1)} + \ldots\ldots\ldots\ldots + \tilde{\wp}_{is}^{(k+1)}\cos(k+1)v\bar{x}_s{}^{(0)}].$$

The coefficients $\tilde{\wp}_{is}^{(1)},\ldots\ldots\ldots\ldots,\tilde{\wp}_{is}^{(k+1)}$ are constants. The quantities $\cos v\bar{x}_s{}^{(k)}, \cos 2v\bar{x}_s{}^{(k-1)},\ldots\ldots\ldots\cos(k+1)v\bar{x}_s{}^{(0)}$ contain only variable periodic terms and so the solution of system (4.17) can be obtained in the form of (4.12). It should be noted that the coefficients of mixed terms $\Lambda_{i,0}{}^{(k+1)}, \lambda_{i,0}{}^{(k+1)}, \Gamma_{i,0}{}^{(k+1)}, \gamma_{i,0}{}^{(k+1)}$ are determined by the system of algebraic equations"given by Grebenikov [2]" consequently, they are zero. The other coefficients of (k+1)[th] approximation are determined by a system of equations in same manner.

Thus, we conclude that general solution of the system of differential equations (2.20) consists only of periodic term.

We further investigate the problem of the convergence of the above series. It follows from formula (3.12) and (3.13) that

$$\alpha_1 < \alpha_2. \quad (4.18)$$

It is easy to find that

$$0 < 1-\alpha_2 = \frac{\alpha_1^{\,2}}{1+\sqrt{\frac{1}{2}(1+\sqrt{\Delta})}} < \alpha_1. \quad (4.19)$$

So that of all the quantities $\alpha_k$ and $1-\alpha_k$, the smallest in absolute magnitude is $1-\alpha_2$. (4.20)

Suppose $1-\alpha_2 = \alpha$.

$$\bar{A} = \sup\left[\frac{D_{ij}(\pm s,e,\mu)}{D(\pm s,e,\mu)}\right], \quad (4.21)$$

Where $D_{ij}(\pm s,e,\mu)$ are the cofactors of the elements of the determinant $D(\pm s,e,\mu)$ ,encountered in the solution of systems of form (4.17). $\sup\left|D_{ij}/D\right|$ Is taken in the following regions of variation of parameters: $0 \le e \le \frac{\sqrt{7}}{4}$; $0 \le \mu \le 1/27$. ; $s = 0,1,2,3..........$

The quantity , $\bar{A}$ is bounded from above in view of the fact that $D(\pm s,e,\mu)$ is everywhere non zero, for sufficiently large values of s increases as $s^8$, while $D_{ij}(\pm s,e,\mu)$ increases as $s^6$ . $\bar{A}$ Is non-zero, because $D(0,0,0) = 2.74$ and $D_{ij}(0,0,0) = 0.1563$ .

The k$^{th}$ approximation contains $(8k+4)$ periodic terms each of which has a factor .It should be noted that in the k$^{th}$ approximation, the exponent of $\alpha^m$ will not be greater than k and, therefore, taking into account estimates (4.6), (4.18), (4.19) and (4.21), we find

$$\left|\bar{x}_i^{(k)}\right| < \left(\frac{9}{2}\right)^k \left(\frac{e}{\alpha}\right)^k \bar{A}^k C\,(8k+4). \quad (4.22)$$

Where, C is a positive constant.
Introducing the abbreviation,

$$\left(\frac{9}{2}\right)^k \left(\frac{e}{\alpha}\right)^k \bar{A}^k C\,(8k+4) = C\sigma^k (8k+4). \quad (4.23)$$

Where $\sigma = \left(\frac{9e}{2\alpha}\bar{A}\right)$.

The series $\sum_0^\infty \sigma^k (8k+4)$, converges for $|\sigma| < 1$ , so that for convergence of the series representing the solution of the problem is given by:

$$\left(\frac{9e}{2\alpha^*}\bar{A}\right) < 1, \quad (4.24)$$

Where $\alpha^* = \inf_{\{e\}} \alpha = \frac{\alpha}{e} = 0.$ (4.25)

$$\alpha^* = 1 - \sqrt{2 - \frac{3}{2\sqrt{1-e^2}} + \sqrt{\left(2 - \frac{3}{2\sqrt{1-e^2}}\right)^2 - \frac{(27+6\beta)\mu(1-\mu)}{4\sqrt{1-e^2}}}}; \quad (4.26)$$

$$\alpha^* = \frac{(27+6\beta)\mu(1-\mu)}{8}, \text{ if } e = 0 \quad (4.27)$$

When $\beta = 0, e = 0$ , the result agrees with “that of Grebenikov [2]”.

The series will converge for:

$$0 \le e < \bar{e} = \frac{2\alpha^*}{9\bar{A}}. \quad (4.28)$$

The approximated value of $\bar{e}$ is given by:

$$\bar{e} \cong \frac{(9+2\beta)\mu(1-\mu)}{12\bar{A}}. \quad (4.29)$$

When β=0, the result is in conformity with “those of Grebenikov [2]”.

The condition for the convergence of the series representing the solution of the problem is given by equation (4.31) .When β=0 equation (4.30) and equation (4.32) agree with“those of Grebenikov [2]”. It is observed that the eccentricity of the orbit as well as condition of convergence of the series representing the solution of problem is highly affected by the radiation pressure.

## 5. Discussion and conclusion

The stability of infinitesimal around the triangular equilibrium points of elliptical restricted three body problem in which bigger primaries is a source of radiation is studied, and the analysis of stability is investigated using the method of averaging "due to Grebenikov[2]". It is shown that for $\mu \le 0.5$ , satisfying the conditions $(27+6\beta)\mu(1-\mu) \le (3-4\sqrt{1-e^2})^2$, where he is eccentricity of the either primaries, the triangular points are stable. The simulation technique is exploited to study the linear stability of triangular equilibrium points, using Matlab 7.1 Software. It is observed that the range of stability decreases as the radiation pressure parameters increases. Also the condition for convergence of the series representing the solution of the problem is studied by using Fourier series analysis.

The critical value of mass ratio given by equation (3.11), agree with "those of Grebenikov [2]", provided $\beta = 0$. When $e = \beta = 0$, the result reduces to the well-known condition for stability of the triangular Lagrangian points in the CR3BP, that is $27\mu(1-\mu) \le 1$. The approximated value of eccentricity given by equation (4.29) is in conformity with those "given by Grebenikov [2]" provided $\beta = 0$.

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