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On Generalized Quasi Metric Spaces

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Abstract

Quasi metrics have been used in several places in the literature on domain theory and the formal semantics of programming languages [1], [3]. In this paper we introduce the concept of generalized quasi metric(=gq) space and establish some fixed point theorems in gq metric spaces.

***Keywords**: Generalised Quasi metric, fixed point, Contractive Condition, CS Complete, CS Continous.*

1 Introduction

Pascel Hitzler presented Rutten Smyth Theorem[1] for quasi metric space and applied it to semantic analysis of logic programs. In this paper we prove the gq metric version of Rutten Smyth Theorem. Rhoades[4] collected a large number of variants of Banach's Contractive conditions on self maps on a metric space and proved various implications or otherwise among them. We pick up a good number of these conditions which ultimately imply Banach condition [4]. We prove that these implications hold good for self maps on a gq metric space and prove the gq metric version of Banach's result then by deriving the gq analogue's of fixed point theorems of Rakotch, Edelstein, Kannan ,Bianchini, Reich and Ciric.

We denote the set of non-negative real numbers by R^+ and set of natural numbers by N.

1.1:Let binary operation $\Diamond : R^+ \times R^+ \to R^+$ satisfies the following conditions:

(I) $\Diamond$ is Associative and Commutative,

(II) $\Diamond$ is continuous w.r.t to the usual metric R^+

A few typical examples are $a \Diamond b = \max\{a, b\}$, $a \Diamond b = a + b$, $a \Diamond b = a\, b$, $a \Diamond b = a\, b + a + b$ and

$a \lozenge b = \dfrac{ab}{\max\{a,b,1\}}$ for each a, $b \in R^+$.

In what follows we fix a binary operation $\lozenge$ that satisfies (I) and (II).

Definition 1.2[5]: A binary operation $\lozenge$ on R^+ is said to satisfy β-property if there exists a positive real number β such that $a \lozenge b \le \beta \max\{a, b\}$ for every a, $b \in R^+$.

Definition 1.3: Let X be a non empty set. A generalized quasi (simply gq) metric (or d^* metric) on X is a function d^***:** X $^2 \to R^+$ that satisfies the following conditions:

(1) $d^*(x,x) = 0$

(2) $d^*(x,y) = d^*(y,x) = 0$ Implies $x = y$

(3) $d^*(x,z) \le d^*(x,y) \lozenge d^*(y,z)$ for each $x, y, z \in$ X.

The pair (X, d^*) is called a generalized quasi (or simply d^*) metric space.

Definition 1.4: A sequence (x_n) in a gq metric space (X, d^*) is a (forward) Cauchy sequence if, for all $\in >0$, there corresponds $n_\varepsilon \in N$ such that for all $n \ge m \ge n_\varepsilon$ we have $d^*(x_n, x_m) < \in$. A Cauchy sequence (x_n) converges to $x \in X$ if, for all $y \in X$, $d^*(x, y) = \lim d^*(x_n, y)$. In this case we write $\lim x_n = x$. Finally X is called CS-complete if every Cauchy sequence in X converges.

Note : Let X be a gq metric space such that $\lozenge$ satisfies β- property with $\beta \le 1$,then limits of Cauchy sequence are unique.

Definition 1.5: Let X be a gq metric space. A function $f : X \to X$ is called

(1) CS-continuous if, for all Cauchy sequences (x_n) in X with $\lim x_n = x$, ($f(x_n)$) is a Cauchy sequence and $\lim f(x_n) = f(x)$.

(2) Non expanding if $d^*(f(x), f(y)) \le d^*(x, y)$ for all $x, y \in X$.

(3) Contractive if there exists some $0 \le c < 1$ such that $d^*(f(x), f(y)) \le c\, d^*(x, y)$ for all $x, y \in X$.

We present the gq metric version of Rutten Smyth theorem[1] for fixed points.

2 Main Results

Theorem 2.1 : Let (X , d^*) be a CS-complete gq metric space such that $\Diamond$ satisfies β- property with $\beta \le 1$. And let $f : X \to X$ be non expanding. If f is CS-continuous and contractive, then f has a unique fixed point.

Proof: For any $x \in X$ the sequence of iterates satisfies $d^*\left(f^n(x), f^{n+1}(x)\right) \le \alpha^n d^*(x, f(x))$ where α is any contractive constant. Consequently if $n < m$

$$\begin{aligned} d^*(f^n(x), f^m(x)) &\le d^*(f^n(x), f^{n+1}(x)) \Diamond d^*(f^{n+1}(x), f^{n+2}(x)) \Diamond \ldots \Diamond d^*(f^{m-1}(x), f^m(x)) \\ &\le \beta \max\{d^*(f^n(x), f^{n+1}(x)), d^*(f^{n+1}(x), f^{n+2}(x)), \ldots, d^*(f^{m-1}(x), f^m(x))\} \\ &\le d^*(f^n(x), f^{n+1}(x)) + d^*(f^{n+1}(x), f^{n+2}(x)) + \ldots + d^*(f^{m-1}(x), f^m(x)) \\ &\le \alpha^n d^*(f^n(x), f^{n+1}(x)) + \alpha^{n+1} d^*(f^{n+1}(x), f^{n+2}(x)) + \ldots + \alpha^{m-1} d^*(f^{m-1}(x), f^m(x)) \\ &\le \left(\alpha^n + \alpha^{n+1} + \ldots + \alpha^{m-1}\right) d^*(x, f(x)) \\ &= \alpha^n \frac{\left(1-\alpha^{m-n}\right)}{1-\alpha} d^*(x, f(x)) \end{aligned}$$

Hence { $f^n(x)$ } is Cauchy sequence in (X, d^*) ,hence convergent.

Let $\xi = \lim_n f^n(x)$ Then $f^{n+1}(x)$ is Cauchy and $f(\xi) = \lim_n f^{n+1}(x)$

$$d^*(\xi, f(\xi)) = \lim_n d^*(f^n(x)\, f(\xi)) = \lim d^*(f^{n+1}(x)\, f(\xi)) = 0$$

$$d^*(f(\xi), \xi) = \lim d^*(f^n(x), f(\xi)) = \lim d^*(f^{n+1}(x)\, f(\xi)) = 0$$

$\Rightarrow f(\xi) = \xi$

Uniqueness: Suppose $f(\xi) = \xi$ *and* $f(\eta) = \eta$

$$d^*(\xi, \eta) = \lim d^*(f^n(x), \eta) = \lim d^*(f^{n+1}(x), \eta) = 0$$

Similarly, $d^*(\eta, \xi) = 0$ *Hence* $\xi = \eta$.

Theorem 2.2: Let (X , d^*) be a CS-complete gq metric space such that $\Diamond$ satisfies β -property with $\beta \le 1$ and $f : X \to X$ be CS-continuous. Assume that there exist non-negative constants a_i satisfying $a_1 + a_2 + a_3 + 2a_4 < 1$ such that for each $x, y \in X$ *with* $x \ne y$

$$d^*\left(f(x), f(y)\right) \le a_1 d^*(x, y) \Diamond a_2 d^*(x, f(x)) \Diamond a_3 d^*(y, f(y)) \Diamond a_4 d^*(x, f(y)) \Diamond a_5 d^*(y, f(x))$$

Then f has a unique fixed point.

Proof: For any $x \in X$

$$d^*(f(x), f^2(x)) \le a_1 d^*(x, f(x)) \Diamond a_2 d^*(x, f(x)) \Diamond a_3 d^*(f(x), f^2(x)) \Diamond a_4 d^*(x, f^2(x)) \Diamond a_5 d^*(f(x), f(x))$$

$\leq \beta \max\{a_1 d^*(x, f(x)), a_2 d^*(x, f(x)), a_3 d^*(f(x), f^2(x)), a_4 d^*(x, f^2(x)), a_5 d^*(f(x), f(x))\}$

$\leq a_1 d^*(x, f(x)) + a_2 d^*(x, f(x)) + a_3 d^*(f(x), f^2(x)) + a_4 d^*(x, f^2(x)) + a_5 d^*(f(x), f(x))$

$$\leq (a_1 + a_2 + a_4)\, d^*(x, f(x)) + (a_3 + a_4)\, d^*(f(x), f^2(x))$$

$\Rightarrow d^*(f(x), f^2(x)) \leq \gamma\, d^*(x, f(x))$ Where $\gamma = \dfrac{a_1 + a_2 + a_4}{1 - a_3 - a_4}$. *clearly* $0 \leq \gamma < 1$

If m > n then,

$$d^*(f^n(x), f^m(x)) \leq d^*(f^n(x), f^{n+1}(x)) \Diamond d^*(f^{n+1}(x), f^{n+2}(x)) \Diamond \ldots\ldots\ldots \Diamond d^*(f^{m-1}(x), f^m(x))$$
$$\leq \beta \max\{d^*(f^n(x), f^{n+1}(x)), d^*(f^{n+1}(x), f^{n+2}(x)) \ldots\ldots\ldots d^*(f^{m-1}(x), f^m(x))\}$$
$$\leq d^*(f^n(x), f^{n+1}(x)) + d^*(f^{n+1}(x), f^{n+2}(x)) + \ldots\ldots\ldots + d^*(f^{m-1}(x), f^m(x))$$
$$\leq \gamma^n d^*(x, f(x)) + \gamma^{n+1} d^*(x, f(x)) + \ldots\ldots\ldots\ldots\ldots + \gamma^{m-1} d^*(x, f(x))$$
$$\leq \gamma^n (1 + \gamma + \gamma^2 + \ldots\ldots + \gamma^{m-n-1}) d^*(x, f(x)) < \frac{\gamma^n}{1-\gamma} d^*(x, f(x))$$

Hence $\{ f^n(x) \}$ is Cauchy's sequence in (X, d^*) ,hence convergent.

Let $\xi = \lim_n f^n(x)$ Then $f^{n+1}(x)$ is Cauchy and $f(\xi) = \lim_n f^{n+1}(x)$

$d^*(\xi, f(\xi)) = \lim_n d^*(f^n(x) f(\xi)) = \lim d^*(f^{n+1}(x) f(\xi)) = 0$

$d^*(f(\xi), \xi) = \lim d^*(f^n(x), f(\xi)) = \lim d^*(f^{n+1}(x) f(\xi)) = 0$

$\Rightarrow f(\xi) = \xi$

Uniqueness: Suppose $f(\xi) = \xi$ *and* $f(\eta) = \eta$

$d^*(\xi, \eta) = \lim d^*(f^n(x), \eta) = \lim d^*(f^{n+1}(x), \eta) = 0$

Similarly, $d^*(\eta, \xi) = 0$ *Hence* $\xi = \eta$

Theorem 2.3: Let (X, d^*) be a CS-complete gq metric space such that $\Diamond$ satisfies β-property with $\beta \leq 1$ and $f : X \to X$ be a CS-continuous mapping, there exist real numbers $\alpha, \delta, \gamma \ni, 0 \leq \alpha < \frac{1}{2}, 0 \leq \delta < \frac{1}{2}, \gamma < \min\{\frac{1}{4}, \frac{1}{2} - \alpha, \frac{1}{2} - \delta\}$ and

For each $x, y \in X$ at least one of the following holds

i. $d^*(f(x), f(y)) \leq \alpha d^*(x, y)$

ii. $d^*(f(x), f(y)) \leq \delta\ \{d(x, f(x)) \Diamond d^*(y, f(y))\}$

iii. $d^*(f(x), f(y)) \leq \gamma\ \{d^*(x, f(y)) \Diamond d^*(y, f(x))\}$

Then f has a unique fixed point.

Proof: put $y = x$ in the above.

$i.\ d^*(f(x), f^2(x)) \le \alpha\ d^*(x, f(x))$

$ii.\ d^*(f(x), f^2(x)) \le \delta\{d^*(x, f(x)) \Diamond d^*(f(x), f^2(x))\}$

$\le \delta\beta \max\{d^*(x, f(x)), d^*(f(x), f^2(x))\}$

$\le \delta\{d^*(x, f(x)) + d^*(f(x), f^2(x))\}$

$\Rightarrow d^*(f(x), f^2(x)) \le \dfrac{\delta}{1-\delta}\ d^*(x, f(x))$ *Similarly*

$iii.\ d^*(f(x), f^2(x)) \le \dfrac{3\gamma}{1-\gamma}\ d^*(x, f(x))$

$h = \max\{\alpha, \dfrac{\delta}{1-\delta}, \dfrac{\gamma}{1-\gamma}\}$ then $0 \le h < 1$

and $d^*(f(x), f^2(x)) \le h\ d^*(x, f(x))$

Hence $\{f^n(x)\}$ is a Cauchy sequence.

Since X is CS-Complete, there exists z such that

$\lim_n f^n(x) = z$ in (X, d^*)

Hence $\lim_n f^{n+1}(x) = f(z)$ in (X, d^*)

$\because\ 0 \le d^*(z, f(z)) \le d^*(z, f^{n+1}(x)) \Diamond d^*(f^{n+1}(x), f(z))$

$\le \beta \max\{d^*(z, f^{n+1}(x)), d^*(f^{n+1}(x), f(z))\}$

$\le d^*(z, f^{n+1}(x)) + d^*(f^{n+1}(x), f(z))$

$\therefore d^*(z, f(z)) = 0$ similarly $d^*(f(z), z) = 0$

Hence $f(z) = z$ Which proves the theorem.

B.E Rhodes [4] presented a list of definitions of contractive type conditions for a self map on a metric space (X, d^*) and established implications and non implications among them ,there by facilitating to check the implication of any new contractive condition through any one of the condition mentioned in [4] so as to derive a fixed point theorem. We now present the gq metric versions of some of them.

Let (X, d^*) be a gq metric space and $f : X \to X$ be a mapping and x, y be any elements of X. Consider the following conditions.

1. (Banach) : there exists a number a, $0 \le a \le 1$ such that $d^*(f(x), f(y)) \le a\ d^*(x, y)$.

2. (Rakotch) : there exists a monotone decreasing function $\alpha : [0, \infty) \to [0,1)$ such that

$d^*(f(x), f(y)) \le \alpha\ d^*(x, y)$ whenever $d^*(x, y) \ne 0$.

3. (Edelstein) : $d^*(f(x), f(y)) < d^*(x, y)$ whenever $d^*(x, y) \ne 0$.

4. (Kannan) :there exists a number a, $0 < a < \frac{1}{2}$ such that

$d^*(f(x), f(y)) <$ a{ $d^*(x, f(x)) \Diamond d^*(y, f(y))$}

5. (Bianchini): there exists a number h ,0 $\le h < 1$ such that

$d^*(f(x), f(y)) \le$ h max{ $d^*(x, f(x))$, $d^*(y, f(y))$ }

6. (Reich) : there exist nonnegative numbers a, b, c satisfying a + b + c < 1 such that

$d^*(f(x), f(y)) \le$ a $d^*(x, f(x)) \Diamond$ b $d^*(y, f(y)) \Diamond$ c $d^*(x, y)$

7. (Reich) : there exist monotonically decreasing functions a, b, c from $(0, \infty)$ to [0 ,1) satisfying
a(t) + b(t) + c(t) < 1 such that ,

$d^*(f(x), f(y)) <$ a(t) $d^*(x, f(x)) \Diamond$ b(t) $d^*(y, f(y)) \Diamond$ c(t) t

where t = $d^*(x, y)$.

8. $d^*(f(x), f(y)) \le a(x,y)\ d^*(x, f(y)) \Diamond b(x,y)\ d^*(y, f(x)) \Diamond c(x,y)\ d^*(x, y)$

$$\sup_{x,y\in X}\{a(x,y)+b(x,y)+c(x,y)\} \le \lambda < 1$$

9. (Ciric): For each $x, y \in X$

$$d^*(f(x), f(y)) \le q(x,y)\ d^*(x, y) \Diamond r(x,y)\ d^*(x, f(x)) \Diamond s(x,y)\ d^*(y, f(y) \Diamond$$
$$t(x,y)[d^*(x, f(y)) \Diamond d^*(y, f(x))]$$
$$\sup_{x,y\in X}\{q(x,y)+r(x,y)+s(x,y)+2t(x,y)\} \le \lambda < 1$$

Theorem 2.4: Let (X , d^*) be a CS-complete gq metric space and let f : $X \rightarrow X$ be non expanding.
If f satisfies one of the above mentioned conditions, then f has a unique fixed point.

Proof: It now follows from Theorem 2.1 that f has a fixed point.

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References

[1] P. Hitzler: Generalized metrics and topology in logic programming semantics, Ph. D Thesis, School of Mathematics, Applied Mathematics and Statistics, National University Ireland, University college Cork, 2001.

[2] J. L. Kelley: General topology, D. Van Nostrand Company, Inc., 1960.

[3] S.G. Mathews: Metric domains for completeness, Technical report 76, Department of computer science, University of Warwick, U.K, Ph.D Thesis 1985.

[4] B.E.Rhoades : A comparison of various definitions of contractive mappings, Trans of the Amer.Math.Society ,vol 226(1977) P.P.257-290.

[5] S.Sedghi: fixed point theorems for four mappings in d*-metric spaces, Thai journal of mathematics,vol 7(2009) November 1 P.P.9-19.