



A variety of exact analytical solutions of extended shallow water wave equations via improved (G'/G) -expansion method

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Abstract

In this present work, we have established exact solutions for (2+1) and (3+1) dimensional extended shallow-water wave equations involving parameters by applying the improved (G'/G) -expansion method. Abundant traveling wave solutions with arbitrary parameter are successfully obtained by this method, and these wave solutions are expressed in terms of hyperbolic, trigonometric, and rational functions. The improved (G'/G) -expansion method is simple and powerful mathematical technique for constructing traveling wave, solitary wave, and periodic wave solutions of the nonlinear evaluation equations which arise from application in engineering and any other applied sciences. We also present the 3D graphical description of the obtained solutions for different cases with the aid of MAPLE 17.

Keywords: Extended Shallow Water Wave Equations; Exact Solutions; Improved (G'/G) -Expansion Method; Nonlinear Partial Differential Equations.

1. Introduction

A great deal of work has been carried out in order to determine exact traveling wave solutions for nonlinear evaluation equations (NLEEs) in all areas of Science and Engineering. In the nonlinear sciences, it is well known that many NLEEs are widely used as to describe the complex physical phenomena, such as Plasma physics, Chemical physics, Solid-state physics, Chemical kinematics, Biology, Optical fibers, Fluid mechanics, etc., It is well known that searching for an exact solution of nonlinear evolution equation arising in mathematical physics plays an important role in the study of nonlinear physical phenomena. In the recent past, many researchers have implemented various analytical methods [1-24] to seek traveling wave solutions of different nonlinear partial differential equations. Many powerful methods have been successfully developed by diverse groups of mathematicians and physicists, such as, Hirota's bilinear method [1], the tanh-function method [2], [3], the extended than-function method [4], the exp -function method [5-7], sine-cosine method [8], the Inverse scattering transforms [9], the Jacobi elliptic function expansion [10], [11], the homogeneous balance method [12], the Homotopy perturbation methods [13], auxiliary equation method [14], the first integral method [15-17], the tanh-coth method [18], the Cole-Holf transformation method [19], the (G'/G) -expansion method [20-22], the improved (G'/G) -expansion method [23], the Enhance (G'/G) -expansion method [24] and so on. The improved (G'/G) -expansion method [23] is a direct and effective algebraic method for handling nonlinear evaluations equations, and the author [23] have been applied the improved (G'/G) -expansion method for solving modified Korteweg-de Vries equation.

The objective of this study is to apply the improved (G'/G) -expansion method to find the exact traveling wave solutions of the (2+1) and (3+1) dimensional extended shallow-water wave

equations involving parameters. The extended model of shallow-water waves equations important partial differential equations of nonlinear dispersive waves. The (2+1) dimensional shallow-water wave equations [7], [22-23].

$$u_{xt} + u_{xxy} - 2u_{xt}u_y - 4u_xu_{xy} = 0 \quad (1)$$

$$u_{xt} + u_{xxy} - 3u_{xt}u_y - 3u_xu_{xy} = 0 \quad (2)$$

Were studied. Both equations reduce to potential KdV equation for $y = x$. The difference between the two models (1) and (2) is that x replaces y in the term u_{xt} and in the coefficients of the other terms. The (3+1) dimensional shallow water wave equations [7], [22-23]

$$u_{xxt} + u_{xxyt} - 2(u_{xx}u_{yt} + u_yu_{xxt}) - (4u_xu_{xyt} + u_{xt}u_{xy}) = 0 \quad (3)$$

$$u_{xxt} + u_{xxyt} - 6u_xu_{xyt} - 6u_{xt}u_{xy} = 0 \quad (4)$$

Were studied. Both equations reduce to potential KdV equation for $y = x$. The difference between the two models (3) and (4) is that x replaces y in the term u_{xxt} .

In this present work, we will solve four extended (2+1) and (3+1) dimensional shallow water wave equations in that were solved by authors [7], [18-19], [22].

$$u_{xt} + u_{xxy} - 3u_{xt}u_y - 3u_xu_{xy} + cu_{xy} = 0 \quad (5)$$

$$u_{xt} + u_{xxy} - 2u_{xt}u_y - 4u_xu_{xy} + cu_{xy} = 0 \quad (6)$$

$$u_{xxt} + u_{xxyt} - 6u_xu_{xyt} - 6u_{xt}u_{xy} + cu_{xyt} = 0 \quad (7)$$

$$u_{xz} + u_{xyx} - 2(u_{xx}u_{yz} + u_{xy}u_{xz}) - 4(u_{xy}u_{yz} + u_{xz}u_{xy}) + cu_{xyz} = 0 \quad (8)$$

These shallow-water wave equations play a major role in the study of nonlinear dispersive waves. The general idea of dispersive waves oriented from the problem of water waves. The shallow-water wave equations describe the evaluation of incompressible flow, neglecting density change along the depth; shallow-water wave equations are applicable to cases where the horizontal scale of the flow is much bigger than the depth of fluid [21].

The article is arranged as follows: In section 2, the methodology is discussed. In section 3, we apply this method to solve the nonlinear evaluation equations point out above. We will discuss the solution of the result and its discussion in section 4. In section 5, we present the graphical representation, and finally we conclude this article in section 6.

2. Methodology

Suppose the general nonlinear partial differential equation in three independent variables x and t is given by

$$N(u, u_x, u_y, u_z, u_{xx}, u_{yy}, u_{zz}, u_{xy}, u_{yz}, u_{xz}, \dots) = 0 \quad (9)$$

Where $u = u(x, y, t)$ is an unknown function, N is a polynomial in $u = u(x, y, t)$ and the subscripts signify the partial derivatives and the nonlinear term are involved.

The main steps of the improved (G'/G) -expansion method as follows:

Step 1: Consider the traveling wave variable

$$u(x, y, t) = u(\xi), \xi = x + y - ct \quad (10)$$

Where c is the speed of the traveling wave.

Now using equation (10), the equation (9) is transformed into an ordinary differential equation for $u = u(\xi)$ in the form

$$P(u, u', u'', u''', \dots) = 0 \quad (11)$$

Where P is a polynomial in $u(\xi)$ and its total derivatives. The superscripts stand for the ordinary derivatives with respect to ξ .

Step 2: Suppose that the traveling waves solution of equation (12) can be expressed by a polynomial in $F(\xi)$ as the following form:

$$u(\xi) = \sum_{i=0}^n \alpha_i (m + F(\xi))^i + \sum_{i=1}^n \beta_i (m + F(\xi))^{-i} \quad (12)$$

Where $F(\xi) = \frac{G'(\xi)}{G(\xi)}$, α_n and β_n are not zero simultaneously.

Also, the function $G(\xi)$ is the solution of the linear ordinary differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0 \quad (13)$$

Where $G'' = \frac{d^2G}{d\xi^2}$, $G' = \frac{dG}{d\xi}$ and λ, μ are real constants to be determined later, n is a positive integer to be determined later and n can be determined by considering the homogeneous balance between the highest order derivatives and the highest order nonlinear terms appearing in ODE equation (11).

The solutions of equation (13) can be written as follows:

2.1. Hyperbolic function solution: (when $\Omega = \lambda^2 - 4\mu > 0$)

$$F_1 = \frac{\sqrt{\Omega}}{2} \coth\left(A + \frac{\sqrt{\Omega}}{2}\xi\right) - \frac{\lambda}{2}$$

$$F_2 = \frac{\sqrt{\Omega}}{2} \tanh\left(A + \frac{\sqrt{\Omega}}{2}\xi\right) - \frac{\lambda}{2}$$

2.2. Trigonometric function solution: (when $\Omega = \lambda^2 - 4\mu < 0$)

$$F_3 = \frac{\sqrt{-\Omega}}{2} \cot\left(A + \frac{\sqrt{-\Omega}}{2}\xi\right) - \frac{\lambda}{2}$$

$$F_4 = \frac{\sqrt{-\Omega}}{2} \tan\left(A - \frac{\sqrt{-\Omega}}{2}\xi\right) - \frac{\lambda}{2}$$

2.3. Rational function solution: (when $\Omega = \lambda^2 - 4\mu = 0$)

$$F_5 = \frac{B}{A + B\xi} - \frac{\lambda}{2}$$

Step 3: Substituting equation (12) into (11) and using (13), with the value of n obtained in step 2, then we obtain polynomials in $F(\xi)^i$ and $F(\xi)^{-i}$ ($i = 0, 1, 2, 3, \dots$), separate all terms with same order of $F(\xi)^i$ and $F(\xi)^{-i}$, then equating each coefficient of the polynomials to zero, yield a system of algebraic equations for α_n , β_n , c , λ and μ .

Step 4: Solve the system of algebraic equations which are found in step - 3 with the aid of algebraic software Maple to obtain values for α_n, β_n, m and c .

Step 5: Substitute obtained values in step - 4, in equation (12) along with equation (13) with the value of n , we obtain some exact traveling wave solutions of the nonlinear equation (9).

3. Application of the method

In this section, we implement the method described in section 2 to find the exact analytical solutions of (2+1) dimensional first and second extended shallow water wave equations via improved (G'/G) -expansion method.

3.1. The first extended shallow water wave equation

In this present work, we consider the extended shallow water wave equation in (2+1) dimensions of the following form [7], [22].

$$u_{xz} + u_{xyx} - 2u_{xx}u_y - 4u_xu_{xy} + \beta u_{xy} = 0 \quad (14)$$

Where β is a non-zero constant. According to the method by using the traveling wave variable

$$u(x, t) = u(\xi), \xi = kx + ry - ct, \quad (15)$$

Where c is the speed of the traveling wave, transform equation (14) into an ordinary differential equation, integrating ODE and neglecting constants of integration, we have

$$k^2nu'' + (r\beta - c)u' - 3kr(u')^2 = 0 \quad (16)$$

Where the prime denotes differential with respect to ξ . Balancing u'' with $(u')^2$, we get $n+3=2(n+1)$ Hence $n=1$

Hence for $n=1$ Eq. (12) reduces to

$$u(\xi) = \alpha_0 + \alpha_1(m + F) + \beta_1(m + F)^{-1} \tag{17}$$

Now substituting Eq. (17) along with Eq. (13) into Eq. (16), we get a polynomial in $F(\xi)$. Equating the coefficient of same power of $F(\xi)$, we attain the following system of algebraic equations:

$$6k^2r\alpha_1 + 3kr\alpha_1^2 = 0$$

$$24k^2mr\alpha_1 + 12k^2r\lambda\alpha_1 + 12kmr\alpha_1^2 + 6kr\lambda\alpha_1^2 = 0$$

$$36k^2m^2r\alpha_1 + 48k^2mr\lambda\alpha_1 + 7k^2r\lambda^2\alpha_1 + 18km^2r\alpha_1^2 + 24kmr\lambda\alpha_1^2 + 3kr\lambda^2\alpha_1^2 + 8k^2r\mu\alpha_1 + 6k\mu r\alpha_1^2 - 6kr\alpha_1\beta_1 + r\beta_1\alpha_1 - c\alpha_1 = 0$$

$$24k^2m^3r\alpha_1 + 72k^2m^2r\lambda\alpha_1 + 28k^2mr\lambda^2\alpha_1 + k^2r\lambda^3\alpha_1 + 12km^3r\alpha_1^2 + 36km^2r\lambda\alpha_1^2 + 12kmr\lambda^2\alpha_1^2 + 32k^2m\mu r\alpha_1 + 8k^2\mu r\lambda\alpha_1 + 24km\mu r\alpha_1^2 + 6k\mu r\lambda\alpha_1^2 - 12kmr\alpha_1\beta_1 - 12kr\lambda\alpha_1\beta_1 + 4mr\beta_1\alpha_1 + r\beta_1\lambda\alpha_1 - 4cm\alpha_1 - c\lambda\alpha_1 = 0$$

$$3km^4\alpha_1^2 + 3kr\mu^2\alpha_1^2 + 6k^2m^4\alpha_1 - 6k^2m^2\beta_1 + 2k^2r\mu^2\alpha_1 - k^2r\lambda^2\beta_1 - 2k^2r\mu\beta_1 + 6m^2r\beta_1\alpha_1 - 4cm\lambda\alpha_1 + \mu r\beta_1\alpha_1 - 6cm^2\alpha_1 - c\mu\alpha_1 - r\beta_1\beta_1 + 3kr\beta_1^2 + c\beta_1 + 32k^2m\mu\lambda\alpha_1 + 24km\mu\lambda\alpha_1^2 - 24km\lambda\alpha_1\beta_1 - 12kr\mu\alpha_1\beta_1 + 48k^2m^3\lambda\alpha_1 + 42k^2m^2\lambda^2\alpha_1 + 4k^2m\lambda^3\alpha_1 + 48k^2m^2\mu\alpha_1 + k^2r\mu\lambda^2\alpha_1 + 6k^2m\lambda\beta_1 + 4m\beta_1\lambda\alpha_1 + 24km^3\lambda\alpha_1^2 + 18km^2\lambda^2\alpha_1^2 + 36km^2\mu\alpha_1^2 - 6km^2\alpha_1\beta_1 - 6kr\lambda^2\alpha_1\beta_1 = 0$$

$$6kr\lambda\beta_1^2 - k^2r\lambda^3\beta_1 + 4m^2r\beta_1\alpha_1 - 6cm^2\lambda\alpha_1 - 4cm\mu\alpha_1 - 2mr\beta_1\beta_1 - r\beta_1\lambda\beta_1 - 4cm^3\alpha_1 + 2cm\beta_1 + c\lambda\beta_1 - 24km\mu\alpha_1\beta_1 - 12kr\mu\lambda\alpha_1\beta_1 + 48k^2m^2\mu\lambda\alpha_1 + 4k^2m\mu\lambda^2\alpha_1 + 36km^2\mu\lambda\alpha_1^2 - 12km^2\lambda\alpha_1\beta_1 - 12km\lambda^2\alpha_1\beta_1 + 12k^2m^2\lambda\alpha_1 + 28k^2m^3\lambda^2\alpha_1 + 6k^2m^2\lambda^3\alpha_1 + 32k^2m^3\mu\alpha_1 - 12k^2m^2\lambda\beta_1 + 8k^2m\mu^2\alpha_1 + 10k^2m\lambda^2\beta_1 + 8k^2m\mu\beta_1 - 8k^2r\mu\lambda\beta_1 + 6m^2r\beta_1\alpha_1 + 4m\mu r\beta_1\alpha_1 + 6km^4\lambda\alpha_1^2 + 12km^3\lambda^2\alpha_1^2 + 24km^3\mu\alpha_1^2 + 12km\mu^2\alpha_1^2 = 0$$

$$3kr\lambda^2\beta_1^2 + 6kr\mu\beta_1^2 - 8k^2r\mu^2\beta_1 + m^4r\beta_1\alpha_1 - 4cm^3\lambda\alpha_1 - 6cm^2\mu\alpha_1 - m^2r\beta_1\beta_1 + 2cm\lambda\beta_1 - \mu r\beta_1\beta_1 - 24km\mu\lambda\alpha_1\beta_1 - cm^4\alpha_1 + cm^2\beta_1 + c\mu\beta_1 - 12km^2\mu\alpha_1\beta_1 + 32k^2m^3\mu\lambda\alpha_1 + 6k^2m^2\mu\lambda^2\alpha_1 + 20k^2m\mu\lambda\beta_1 + 24km^3\mu\lambda\alpha_1^2 - 6km^2\lambda^2\alpha_1\beta_1 - 6kr\mu^2\alpha_1\beta_1 + 7k^2m^4\lambda^2\alpha_1 + 4k^2m^3\lambda^3\alpha_1 + 8k^2m^4\mu\alpha_1 + 12k^2m^2\mu^2\alpha_1 - 7k^2m^2\lambda^2\beta_1 + 4k^2m\lambda^3\beta_1 - 8k^2m^2\mu\beta_1 - 7k^2r\mu\lambda^2\beta_1 + 4m^3r\beta_1\lambda\alpha_1 + 6m^2\mu r\beta_1\alpha_1 - 2mr\beta_1\lambda\beta_1 + 3km^4\lambda^2\alpha_1^2 + 6km^4\mu + \alpha_1^2 + 18km^2\mu^2\alpha_1^2 = 0$$

$$8k^2m^4\mu\lambda\alpha_1 + 12km^3\mu^2\alpha_1^2 + 6kr\mu\lambda\beta_1^2 - k^2m^2\lambda^3\beta_1 + k^2m^4\lambda^3\alpha_1 + 8k^2m^2\mu\lambda\beta_1 - cm^4\lambda\alpha_1 + 10k^2m\mu\lambda^2\beta_1 + 4m^3\mu r\beta_1\alpha_1 - 4cm^3\mu\alpha_1 - 12k^2r\mu^2\lambda\beta_1 + 6km^4\mu\lambda\alpha_1^2 - 12km\mu^2\alpha_1\beta_1 - m^2r\beta_1\lambda\beta_1 + 2cm\mu\beta_1 + 8k^2m^3\mu^2\alpha_1 - 8k^2m^2\mu\lambda\beta_1 - 12km^2\mu\lambda\alpha_1\beta_1 + cm^2\lambda\beta_1 - 2m\mu r\beta_1\beta_1 + m^4r\beta_1\lambda\alpha_1 + 4k^2m^3\mu\lambda^2\alpha_1 = 0$$

$$-6km^2\mu^2\alpha_1\beta_1 - 6k^2r\mu^2\beta_1 + cm^2\mu\beta_1 + k^2m^4\mu\lambda^2\alpha_1 + 3km^4\mu^2\alpha_1^2 + 3kr\mu^2\beta_1^2 + 2k^2m^4\mu^2\alpha_1 + 6k^2m\mu^2\lambda\beta_1 - k^2m^2\mu\lambda^2\beta_1 - 2k^2m^2\mu^2\beta_1 - cm^4\mu\alpha_1 + m^4\mu r\beta_1\alpha_1 - m^2\mu r\beta_1\beta_1 = 0$$

Solving the above system of equations for $\alpha_0, \alpha_1, \beta_1, m$ and c with MAPLE 17, we get the following values:

Set 1:

$$c = k^2r\lambda^2 - 4k^2\mu r + r\beta, m = m, \alpha_0 = \alpha_0, \alpha_1 = 0, \beta_1 = 2km^2 - 2km\lambda + 2k\mu \tag{18}$$

Set 2:

$$c = 4k^2r\lambda^2 - 16k^2\mu r + r\beta, m = \frac{1}{2}\lambda, \alpha_0 = \alpha_0, \alpha_1 = -2k, \beta_1 = -\frac{1}{2}k\lambda^2 + 2k\mu \tag{19}$$

Set 3:

$$c = 4k^2r\lambda^2 - 16k^2\mu r + r\beta, m = \frac{1}{2}\lambda, \alpha_0 = \alpha_0, \alpha_1 = -2k, \beta_1 = -\frac{1}{2}k\lambda^2 + 2k\mu \tag{20}$$

3.1.1. Hyperbolic function solutions

When, $\Omega = \lambda^2 - 4\mu > 0$ we obtain the hyperbolic function solutions of Eq. (14)

Family 1:

$$u_1(\xi) = \alpha_0 - 2k \left(m + \frac{1}{2}\sqrt{\Omega} \coth \left(A + \frac{1}{2}\sqrt{\Omega}\xi \right) - \frac{1}{2}\lambda \right) \tag{21}$$

$$u_2(\xi) = \alpha_0 - 2k \left(m + \frac{1}{2}\sqrt{\Omega} \tanh \left(A + \frac{1}{2}\sqrt{\Omega}\xi \right) - \frac{1}{2}\lambda \right) \tag{22}$$

Where $\xi = kx + ry - (k^2r\lambda^2 - 4k^2\mu r + r\beta)t$

Family 2:

$$u_3(\xi) = \alpha_0 + \frac{2km^2 - 2km\lambda + 2k\mu}{m + \frac{1}{2}\sqrt{\Omega} \coth \left(A + \frac{1}{2}\sqrt{\Omega}\xi \right) - \frac{1}{2}\lambda} \tag{23}$$

$$u_4(\xi) = \alpha_0 + \frac{2km^2 - 2km\lambda + 2k\mu}{m + \frac{1}{2}\sqrt{\Omega} \tanh \left(A + \frac{1}{2}\sqrt{\Omega}\xi \right) - \frac{1}{2}\lambda} \tag{24}$$

Where $\xi = kx + ry - (k^2r\lambda^2 - 4k^2\mu r + r\beta)t$

Family 3:

$$u_5(\xi) = \alpha_0 - k\sqrt{\Omega} \coth \left(A + \frac{1}{2}\sqrt{\Omega}\xi \right) + \frac{2 \left(-\frac{1}{2}k\lambda^2 + 2k\mu \right)}{\sqrt{\Omega} \coth \left(A + \frac{1}{2}\sqrt{\Omega}\xi \right)} \tag{25}$$

$$u_6(\xi) = \alpha_0 - k\sqrt{\Omega} \tanh \left(A + \frac{1}{2}\sqrt{\Omega}\xi \right) + \frac{2 \left(-\frac{1}{2}k\lambda^2 + 2k\mu \right)}{\sqrt{\Omega} \tanh \left(A + \frac{1}{2}\sqrt{\Omega}\xi \right)} \tag{26}$$

Where $\xi = kx + ry - (4k^2r\lambda^2 - 16k^2\mu r + r\beta)t$

3.1.2. Trigonometric function solutions

When $\Omega = \lambda^2 - 4\mu < 0$, we obtain the trigonometric function solutions of Eq. (14)

Family 4:

$$u_7(\xi) = \alpha_0 - 2k \left(m + \frac{1}{2}\sqrt{-\Omega} \cot \left(A + \frac{1}{2}\sqrt{-\Omega}\xi \right) - \frac{1}{2}\lambda \right) \tag{27}$$

$$u_8(\xi) = \alpha_0 - 2k \left(m + \frac{1}{2} \sqrt{-\Omega} \tan \left(A - \frac{1}{2} \sqrt{-\Omega} \xi \right) - \frac{1}{2} \lambda \right) \quad (28)$$

Where $\xi = kx + ry - (k^2 r \lambda^2 - 4k^2 \mu r + r \beta) t$

Family 5:

$$u_9(\xi) = \alpha_0 + \frac{2km^2 - 2km\lambda + 2k\mu}{m + \frac{1}{2} \sqrt{-\Omega} \cot \left(A + \frac{1}{2} \sqrt{-\Omega} \xi \right) - \frac{1}{2} \lambda} \quad (29)$$

$$u_{10}(\xi) = \alpha_0 + \frac{2km^2 - 2km\lambda + 2k\mu}{m + \frac{1}{2} \sqrt{-\Omega} \tan \left(A - \frac{1}{2} \sqrt{-\Omega} \xi \right) - \frac{1}{2} \lambda} \quad (30)$$

Where $\xi = kx + ry - (k^2 r \lambda^2 - 4k^2 \mu r + r \beta) t$

Family 6:

$$u_{11}(\xi) = \alpha_0 - k \sqrt{-\Omega} \cot \left(A + \frac{1}{2} \sqrt{-\Omega} \xi \right) + \frac{2 \left(-\frac{1}{2} k \lambda^2 + 2k\mu \right)}{\sqrt{-\Omega} \cot \left(A + \frac{1}{2} \sqrt{-\Omega} \xi \right)} \quad (31)$$

$$u_{12}(\xi) = \alpha_0 - k \sqrt{-\Omega} \tan \left(A - \frac{1}{2} \sqrt{-\Omega} \xi \right) + \frac{2 \left(-\frac{1}{2} k \lambda^2 + 2k\mu \right)}{\sqrt{-\Omega} \tan \left(A - \frac{1}{2} \sqrt{-\Omega} \xi \right)} \quad (32)$$

Where $\xi = kx + ry - (4k^2 r \lambda^2 - 16k^2 \mu r + r \beta) t$

3.1.3. Rational function solutions

When $\lambda^2 - 4\mu = 0$, we obtain the following solutions of Eq. (14).

Family 7:

$$u_{13}(\xi) = \alpha_0 - 2k \left(m + \frac{B}{B\xi + A} - \frac{1}{2} \lambda \right) \quad (33)$$

Where $\xi = kx + ry - (k^2 r \lambda^2 - 4k^2 \mu r + r \beta) t$

Family 8:

$$u_{14}(\xi) = \alpha_0 + \frac{2km^2 - 2km\lambda + 2k\mu}{m + \frac{B}{B\xi + A} - \frac{1}{2} \lambda} \quad (34)$$

Where $\xi = kx + ry - (k^2 r \lambda^2 - 4k^2 \mu r + r \beta) t$

Family 9:

$$u_{15}(\xi) = \alpha_0 - \frac{2kB}{B\xi + A} + \frac{\left(-\frac{1}{2} k \lambda^2 + 2k\mu \right) (B\xi + A)}{B} \quad (35)$$

Where $\xi = kx + ry - (4k^2 r \lambda^2 - 16k^2 \mu r + r \beta) t$

3.2. The second extended shallow water wave equation

In this section, we consider the extended shallow water wave equation in (2+1) dimensions of the following form [7], [22].

$$u_{yy} + u_{xxx} - 3u_x u_y - 3u_x u_{xy} + \beta u_{xy} = 0 \quad (36)$$

Where β is a non-zero constant. According to the method by using Eq. (15), transform equation (36) into an ordinary differential equation, integrating ODE and neglecting constants of integration we have:

$$k^3 u''' + (k\beta - c)u' - 3k^2 (u')^2 = 0 \quad (37)$$

Where the prime denotes differential with respect to ξ . Balancing u''' with $(u')^2$, we get $n+3=2(n+1)$ hence $n=1$. Hence for $n=1$ Eq. (12) reduces to

$$u(\xi) = \alpha_0 + \alpha_1(m+F) + \beta_1(m+F)^{-1} \quad (38)$$

Now substituting Eq. (38) along with Eq. (14) into Eq. (37), we get a polynomial in $F(\xi)$. Equating the coefficient of same power of $F(\xi)$, we attain a system of algebraic equations that consists of nine algebraic equations (for simplicity, we omitted to display them). Solving these systems of equations for $\alpha_0, \alpha_1, \beta_1, m$ and c with the aid of MAPLE 17, the following sets of solutions are obtained:

Set- 01:

$$c = k^3 \lambda^2 - 4k^3 \mu + k\beta, m = m, \alpha_0 = \alpha_0, \alpha_1 = 0, \beta_1 = 2km^2 - 2km\lambda + 2k\mu \quad (39)$$

Set- 02:

$$c = k^3 \lambda^2 - 4k^3 \mu + k\beta, m = m, \alpha_0 = \alpha_0, \alpha_1 = -2k, \beta_1 = 0 \quad (40)$$

Set-3:

$$c = 4k^3 \lambda^2 - 16k^3 \mu + k\beta, m = \frac{1}{2} \lambda, \alpha_0 = \alpha_0, \alpha_1 = -2k, \beta_1 = -\frac{1}{2} k \lambda^2 + 2k\mu \quad (41)$$

3.2.1. Hyperbolic function solutions

When $\Omega = \lambda^2 - 4\mu > 0$ we obtain the hyperbolic function solutions of Eq. (36).

Family 1:

$$u_1(\xi) = \alpha_0 + \frac{2km^2 - 2km\lambda + 2k\mu}{m + \frac{1}{2} \sqrt{\Omega} \coth \left(A + \frac{1}{2} \sqrt{\Omega} \xi \right) - \frac{1}{2} \lambda} \quad (42)$$

$$u_2(\xi) = \alpha_0 + \frac{2km^2 - 2km\lambda + 2k\mu}{m + \frac{1}{2} \sqrt{\Omega} \tanh \left(A + \frac{1}{2} \sqrt{\Omega} \xi \right) - \frac{1}{2} \lambda} \quad (43)$$

Where $\xi = kx + ry - (k^3 \lambda^2 - 4k^3 \mu + k\beta) t$

Family 2:

$$u_3(\xi) = \alpha_0 - 2k \left(m + \frac{1}{2} \sqrt{\Omega} \coth \left(A + \frac{1}{2} \sqrt{\Omega} \xi \right) - \frac{1}{2} \lambda \right) \quad (44)$$

$$u_4(\xi) = \alpha_0 - 2k \left(m + \frac{1}{2} \sqrt{\Omega} \tanh \left(A + \frac{1}{2} \sqrt{\Omega} \xi \right) - \frac{1}{2} \lambda \right) \quad (45)$$

Where $\xi = kx + ry - (k^3 \lambda^2 - 4k^3 \mu + k\beta) t$

Family 3:

$$u_5(\xi) = \alpha_0 - k\sqrt{\Omega} \coth\left(A + \frac{1}{2}\sqrt{\Omega}\xi\right) + \frac{2\left(-\frac{1}{2}k\lambda^2 + 2k\mu\right)}{\sqrt{\Omega} \coth\left(A + \frac{1}{2}\sqrt{\Omega}\xi\right)} \quad (46)$$

$$u_6(\xi) = \alpha_0 - k\sqrt{\Omega} \coth\left(A + \frac{1}{2}\sqrt{\Omega}\xi\right) + \frac{2\left(-\frac{1}{2}k\lambda^2 + 2k\mu\right)}{\sqrt{\Omega} \coth\left(A + \frac{1}{2}\sqrt{\Omega}\xi\right)} \quad (47)$$

Where $\xi = kx + ry - (4k^3\lambda^2 - 16k^3\mu + k\beta)t$

3.2.2. Trigonometric function solutions

When $\Omega = \lambda^2 - 4\mu < 0$, we obtain the trigonometric function solutions of Eq. (36).

Family 4:

$$u_7(\xi) = \alpha_0 + \frac{2km^2 - 2km\lambda + 2k\mu}{m + \frac{1}{2}\sqrt{-\Omega} \tan\left(A - \frac{1}{2}\sqrt{-\Omega}\xi\right) - \frac{1}{2}\lambda} \quad (48)$$

$$u_8(\xi) = \alpha_0 + \frac{2km^2 - 2km\lambda + 2k\mu}{m + \frac{1}{2}\sqrt{-\Omega} \tan\left(A - \frac{1}{2}\sqrt{-\Omega}\xi\right) - \frac{1}{2}\lambda} \quad (49)$$

Where $\xi = kx + ry - (k^3\lambda^2 - 4k^3\mu + k\beta)t$

Family 5:

$$u_9(\xi) = \alpha_0 - 2k\left(m + \frac{1}{2}\sqrt{-\Omega} \cot\left(A + \frac{1}{2}\sqrt{-\Omega}\xi\right) - \frac{1}{2}\lambda\right) \quad (50)$$

$$u_{10}(\xi) = \alpha_0 - 2k\left(m + \frac{1}{2}\sqrt{-\Omega} \tan\left(A - \frac{1}{2}\sqrt{-\Omega}\xi\right) - \frac{1}{2}\lambda\right) \quad (51)$$

Where $\xi = kx + ry - (k^3\lambda^2 - 4k^3\mu + k\beta)t$

Family 6:

$$u_{11}(\xi) = \alpha_0 - k\sqrt{-\Omega} \cot\left(A + \frac{1}{2}\sqrt{-\Omega}\xi\right) + \frac{2\left(-\frac{1}{2}k\lambda^2 + 2k\mu\right)}{\sqrt{-\Omega} \cot\left(A + \frac{1}{2}\sqrt{-\Omega}\xi\right)} \quad (52)$$

$$u_{12}(\xi) = \alpha_0 - k\sqrt{-\Omega} \tan\left(A - \frac{1}{2}\sqrt{-\Omega}\xi\right) + \frac{2\left(-\frac{1}{2}k\lambda^2 + 2k\mu\right)}{\sqrt{-\Omega} \tan\left(A - \frac{1}{2}\sqrt{-\Omega}\xi\right)} \quad (53)$$

Where $\xi = kx + ry - (4k^3\lambda^2 - 16k^3\mu + k\beta)t$

3.2.4. Rational function solutions

When $\Omega = \lambda^2 - 4\mu = 0$, we obtain the following solutions of Eq. (36).

Family 7:

$$u_{13}(\xi) = \alpha_0 + \frac{2km^2 - 2km\lambda + 2k\mu}{m + \frac{B}{B\xi + A} - \frac{1}{2}\lambda} \quad (54)$$

Where $\xi = kx + ry - (k^3\lambda^2 - 4k^3\mu + k\beta)t$

Family 8:

$$u_{14}(\xi) = \alpha_0 - 2k\left(m + \frac{B}{B\xi + A} - \frac{1}{2}\lambda\right) \quad (55)$$

Where $\xi = kx + ry - (k^3\lambda^2 - 4k^3\mu + k\beta)t$

Family 9:

$$u_{15}(\xi) = \alpha_0 - \frac{2kB}{B\xi + A} + \frac{\left(-\frac{1}{2}k\lambda^2 + 2k\mu\right)(B\xi + A)}{B} \quad (56)$$

Where $\xi = kx + ry - (4k^3\lambda^2 - 16k^3\mu + k\beta)t$

4. Results and discussion

In this present work, we have solved extended shallow water wave equations and found fifteen solutions, but author [22] have found only six solutions by using basic (G'/G) -expansion method. From our observation, we found that if we set $m = 0$ and leave out the portion $\sum_{i=1}^n \beta_i (m + F(\xi))^{-i}$ in our solution, then our obtained solution coincides with Biker [22] solution. Hence, we conclude that the basic (G'/G) -expansion method established by Wang et al. [20] is the particular case of the improved (G'/G) -expansion method. Comparing our obtained results from equations (14) and (36), it can be seen that the results are same with Biker [22] solutions when we set $m = 0$ and leave out the portion $\sum_{i=1}^n \beta_i (m + F(\xi))^{-i}$ in our solution. Similarly, for extended (3+1) dimensional shallow water wave equations, using the wave variable $\xi = kx + ry + sz - ct$ equations (7) and (8) must be carried into the ordinary differential equations (16) and (37) respectively. Thus the improved (G'/G) -expansion method is applied to these equations, it can be seen that the obtained solutions are the same with solutions of equations (16) and (37). From our obtained solutions we observe that solutions from family 1 to family 3 are hyperbolic function solutions for $\lambda^2 - 4\mu > 0$, from Family 4 to Family 6 are trigonometric function solutions for $\lambda^2 - 4\mu < 0$ and from Family 7 to Family 9 are rational function solutions for $\lambda^2 - 4\mu = 0$. When the parameters are taken as special values, the solitary wave solutions and periodic wave solutions are derived from the hyperbolic function solution and trigonometric function solution, respectively.

5. Graphical representations

In this section, we will discuss about the nature of some obtained solutions of the equation (14) and (36) by selecting particular values of the parameters existing in the exact solutions use the mathematical software Maple 17, which are represented in Fig. 1-9

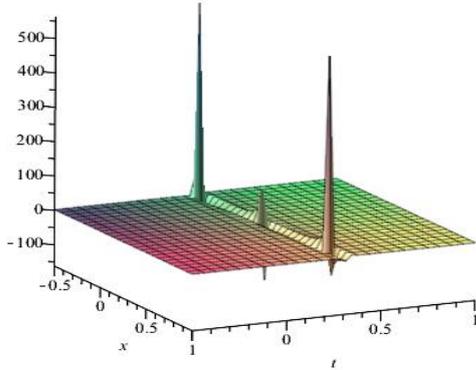


Fig. 1: 3D Graphics (Soliton Profile) of Solutions Eq. (21) when $\alpha_0 = 3, k = 0.8, r = 2, \beta = 5, \lambda = 3, \mu = 1, m = 1, y = 0, A = 5$ and $-0.5 \leq x, t \leq 1$

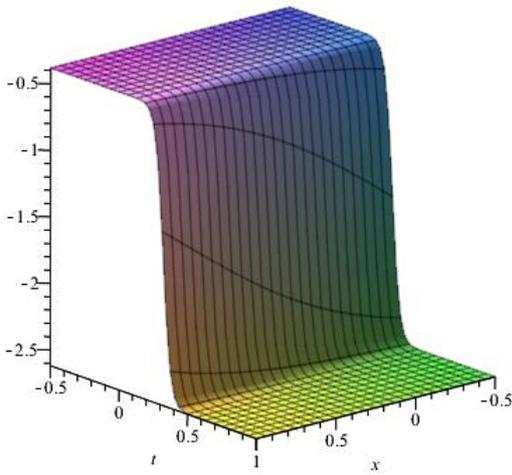


Fig. 2: 3D Graphics (Kink Profile) of Solutions Eq. (24) when $\alpha_0 = 3, k = 0.8, r = 2, \beta = 5, \lambda = 3, \mu = 1, m = 1, y = 0, A = 5$ and $-0.5 \leq x, t \leq 1$.

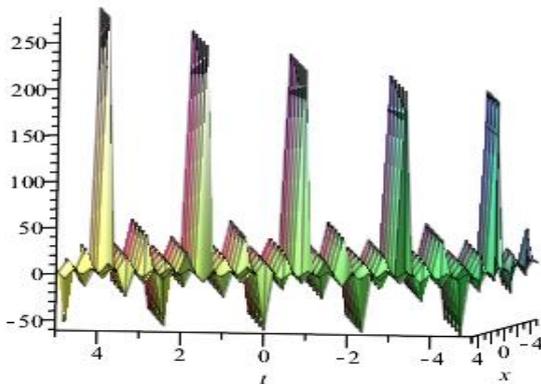


Fig. 3: 3D Graphics (Periodic Profile) of Solutions Eq. (27) when $\alpha_0 = 3, k = 1, r = 1, \beta = 2, \lambda = 1, \mu = 2, m = 0.5, y = 0, A = 5$ and $-5 \leq x, t \leq 5$.

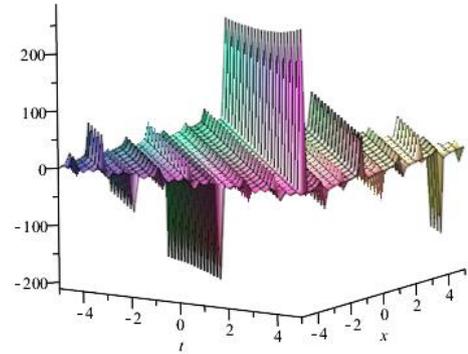


Fig. 4: 3D Graphics (Periodic Profile) of Solutions Eq. (49) when $\alpha_0 = 3, k = 1, r = 1, \beta = 3, \lambda = 1, \mu = 2, m = 0.5, y = 0, A = 5$ and $-5 \leq x, t \leq 5$.

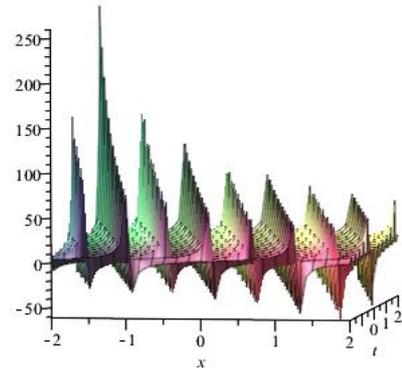


Fig. 5: 3D Graphics (Periodic Profile) of Solutions Eq. (28) when $\alpha_0 = 5, k = -0.8, r = -1, \beta = 2, \lambda = 0.2, \mu = 3, m = 1, y = 0, A = 0$ and $-2 \leq x, t \leq 2$

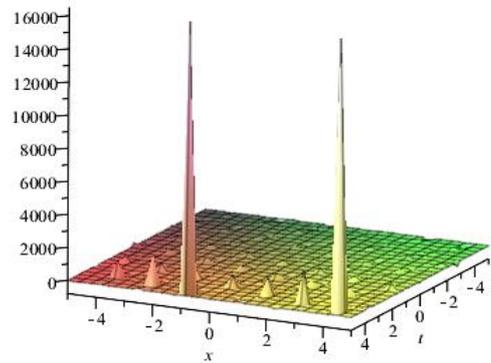


Fig. 6: 3D Graphics (Soliton Profile) of Solutions Eq. (53) when $\alpha_0 = 3, k = 1, r = 1, \beta = 3, \lambda = 1, \mu = 2, m = 0.5, y = 0, A = 5$ and $-5 \leq x, t \leq 5$.

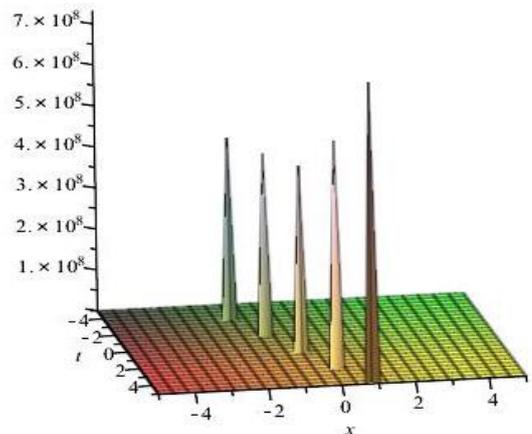


Fig. 7: 3D Graphics (Soliton Profile) of Solutions Eq. (55) when $\alpha_0 = 3, k = 1.5, r = -0.5, \beta = 3, \lambda = 2, \mu = 1, m = 1, y = 0, A = 5, B = 1$ and $-5 \leq x, t \leq 5$.

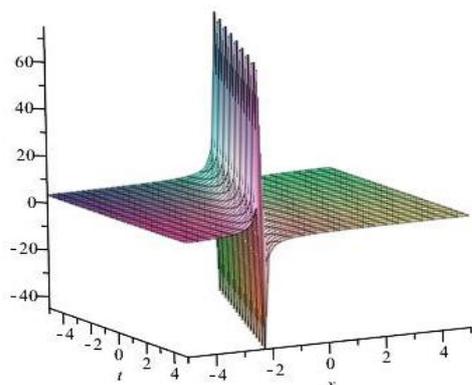


Fig. 8: 3D Graphics (Singular Kink Profile) of Solutions Eq. (34) when $\alpha_0 = 3, k = 1.5, r = -0.5, \beta = 1, \lambda = 2, \mu = 1, m = 1, y = 0, A = 1, B = 1$ And $-5 \leq x, t \leq 5$.

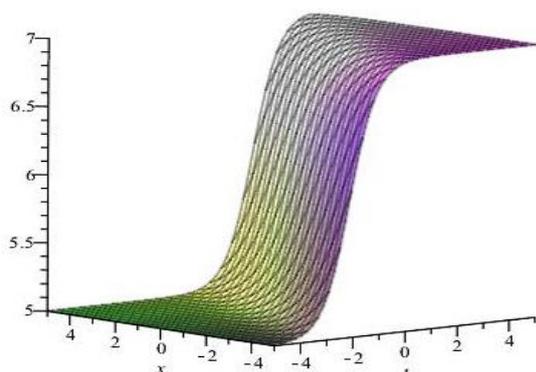


Fig. 9: 3D Graphics (Singular Kink Profile) of Solutions Eq. (56) when $\alpha_0 = 3, k = 1.5, r = -0.5, \beta = 3, \lambda = 2, \mu = 1, m = 1, y = 0, A = 5, B = 1$ and $-5 \leq x, t \leq 5$.

6. Conclusion

In this paper, we have examined the exact solutions for (2+1) and (3+1) dimensional extended shallow water wave equations involving parameters by applying the improved (G'/G) -expansion method. Some of these results are in agreement with the results reported by others in the literature, and new results are formally developed in this work. The solution procedure is very simple and the travelling wave solutions are expressed in terms of hyperbolic functions, trigonometric functions and rational functions. It is shown that the improved (G'/G) -expansion method provides a very effective and powerful mathematical tool for solving nonlinear equations in mathematical physics.

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