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Research paper

Dynamical characteristics and modulation instability (MI) analysis of sharp slope bell soliton and kink wave solutions to the perturbed space-time fractional Boussinesq equation

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Abstract

To attain soliton results for the nonlinear fractional progression equations (NLFEEs) like the space-time fractional Boussinesq equation, we have utilized the advanced expansion technique. In this study, solutions are explicitly determined as kink, soliton, and sharp slope bell soliton types. These obtained singular wave solutions might play a significant role in finding the mathematical model of the realistic corporal phenomena. The results illustrated that the advanced-exp $(-\phi(\xi))$ expansion technique is a simple, straightforward, and actual mathematical model for searching extensive wave solutions with suitable parameters of higher-dimensional NLFEEs. We have also applied the modulation instability analysis (MI) to deliberate the consistency scrutiny of the achieved solutions, and the driving character of the obtained waves is inspected, which ensures that all achieved solutions are explicit, reliable, exact, and stable.

Keywords: *Space-Time Fractional Perturbed Boussinesq Equation; Conformable Derivative (CD); The Advanced-* $exp\ (-\phi(\xi))$ *Expansion Technique; Modulation Instability MI Analysis; Soliton Solutions.*

1. Introduction`

Nature's physical concepts and processes are generally intricate and nonlinear. Nonlinear dynamics is a key science area which includes studies of nonlinear processes in general. Nowadays, nonlinear fractional evolution equations can be used to express almost any tangible complicated phenomenon (NLFEEs). The (NLFEEs) have been applied to many various submissions in water wave mechanics and shallow water waves in the contemporary period. The influence of precise and N-soliton solutions for nonlinear PDEs, which has been a primary priority for both applied mathematics and theoretical physicists [1], [2], has seen significant progress. The concern of the exact and explicit N-Soliton solutions3 for NLFEEs have plenty of significance in penetrating the nonlinear natural dynamics. [4], [5] The NLFEEs have noteworthy submissions in many disciplines, such as, optical fiber communication system, nonlinear optics, plasma science, solid-state physics, mathematical physic, biological chemistry, water wave mechanics, meteorology, electromagnetic theory, control theory, chemical kinematics, theoretical mechanics, biogenetics, system identification, etc. Because of the appearance in various applications in fiber optic communication systems, signal processing, mathematical physics, and control theory, the explicit solutions to NLFEEs have been the topic of attention in numerous studies. [6], [7].

There are various investigations that offer explicit, exact, and numerical solutions for the NLFEEs to be espoused. The analysis of explicit and exact solutions of NLFEEs takes advanced and exceptional attention and qualities with extra findings by mathematicians and physicists. Different types of computational tools, namely MATLAB, Maple, and Mathematica mark it far easier for mathematicians, physicists, as well as engineers to form an opportunity to progress several analytical and numerical approaches range of newly defined nonlinear PDE solutions. The approaches are the generalized Kudryashov scheme, [8] csch-function method and traveling wave hypothesis, [9] improved simple equation technique, [10] improved tanh 〖(ϕ(ξ)⁄2)〗 -expansion and tan 〖(ϕ(ξ)⁄2)〗 approache, [11] multiple Exponential functiona system, [12], [13] altered extended tanh-function (mETF) scheme, [14] novel exponential rational expression technique,15 extended tanh method,16 sub-equation manner, [17] the exp-function method,18 modified simple equation outline, [19] modified trial equation technique, [20] darboux transform scheme, [21] adomian decomposition scheme, [22] the unified technique, [23], [24] Hirota's bilinear scheme, [25] Bäcklund transformation and inverse scattering scheme, [26] exp(-ϕ(ξ))- expansion and advanced schemes, [27], [28] (G^'/G)-propagation system, [29] extended sinh-Gordon expansion technique, [30] sine-Gordon expansion technique, [31] (G^'/G, 1/G)-expansion scheme,[32-36] improved (G'/G) and $(1/G')$ −expansion approaches, [37] etc. To the best of our understanding, the advanced exp(-ϕ(ξ))-

expansion method we propose for studying the Boussinesq equation within the framework of the conformable derivative appears to be unexplored, particularly concerning its geometric and physical implications. This study primarily aims to derive new, exact, and explicit conditions for the space-time perturbed Boussinesq equation (BE) using the advanced exp(-ϕ(ξ)) expansion scheme. With the help of the progressive exp(-ϕ(ξ)) expansion system, we have considered all the obtained solutions in the case of conformable derivatives. Our obtained method is an equivalent method of (G'/G) -expansion. Here we have also found a diversity between these two methods in case of nonlinear equation solving. The Boussinesq equation is well-known for describing the circulation of gravity waves on the water's surface, particularly the head-on crash of oblique waves. Moreover, our studied perturbed Boussinesq equation (BE) rise in flexibility for longitudinal waves in extended, plasma waves, water waves, acoustic waves, bars, nonlinear optics, quantum mechanics, etc. In this work, our proposed advanced exp(-ϕ(ξ))- expansion scheme is efficient, reliable, and provides precise soliton solutions in a novel way to illustrate the movement of gravity waves across the water's surface

Moreover, the advanced exp(-ϕ(ξ))-expansion method is an efficient approach for devising the explicit, itinerant wave solutions of nonlinear fractional single, joined, and arrangement of equations arising in numerous areas of water wave procedure, theoretical physics, fluid dynamics, and wave propagation problems as well. The progressive exp(-ϕ(ξ))-expansion procedure has grown much significance due to its universal alleged and correctness. The progressive exp(-ϕ(ξ))-expansion scheme is a more effective and steadfast system as equated to the G'/G -expansion scheme. The results derived from the mentioned procedure can be represented as hyperbolic, trigonometric, and balanced functions. These solution arrangements are suitable for revising and interpreting real-world physical phenomena. We compared the accomplished solutions established by the G'/G -expansion [37-39] to our proposed progressive exp(-ϕ(ξ))-expansion technique. To the best of our understanding, the kink, soliton, and sharp slope bell soliton solutions for the perturbed BE introduced through our progressive exp(-ϕ(ξ))-expansion scheme are unique and not previously documented. [33], [36-38] Here, our findings are that our mentioned method gives much more accurate novel soliton solutions than its parallel G'/G -expansion system of perturbed BE. It is also significant to recognize that most of the scrutinized solutions in this paper have various assemblies over the conditions available in the literature. Because of wave circulation, the achieved method is completely new for this considered perturbed BE. We have added the MI analysis for showing the achieved solutions, and the movement character of the obtained waves is inspected, which ensures that all achieved solutions are more accurate and stable. Thus, the established exact and explicit solutions could assist the authors in further research to better understand the practical scenarios in mathematical physics and shallow water waves. Furthermore, we assert that the perturbed BE we presented is appropriate in the context of the conformable derivative, and the remaining findings are presented as follows.

2. Preliminaries and approaches

Understanding Conformable Derivatives (CD): Concepts and Applications

Khalil et al. [38] explained conformable derivatives (CD) through the use of limits.

Definition By considering a mapping $f:(0,\infty)\to\Re$, the CD of f order ω is inscribed as follows

$$T_t^{\omega} f(t)=\lim_{\varepsilon\to 0}\left(\frac{f\left(t+\varepsilon t^{1-\omega}\right)-f(t)}{\varepsilon}\right),\quad \text{for all } t>0,\ 0<\omega\le 1.$$

Fourier transformation, Laplace transform, the exponential function rule, the chain rule, Gronwall's inequality, integration by slices, and Taylor series expansions for conformable derivatives (CD) with fractional orders have all been established by the renowned researcher Abdeljawad [39]. The objective of the currently adopted Riemann Liouville derivative [39] definition is readily overshadowed by the explanation of the CD. Newly, Bashar et al. [40] and Mamun et al. [41], [42] reused this conformable derivative (CD) theory to resolve some time fractional PDEs.

Theorem 1: *Let us assume* $\omega\in(0,1]$, *and function* $f=f(t)$, and $g=g(t)$, *be* ω*-CD at the point* $t>0$, *then we may express*

i) $T_t^{\omega}(cf+dg)=cT_t^{\omega}f+dT_t^{\omega}g$, for all $c,d\in\Re$

ii) $T_t^{\omega}(t\gamma)=\gamma t^{\gamma-\omega}$, for all $\gamma\in\Re$

iii) $T_t^{\omega}(fg)=gT_t^{\omega}(f)+f\,T_t^{\omega}(g)$,

iv) $T_t^{\omega}\left(\frac{f}{g}\right)=\frac{gT_t^{\omega}(f)-f\,T_t^{\omega}(g)}{g^2}$.

Moreover, if this function f is differentiable, then $T_t^{\omega}(f(t))=t^{1-\omega}\frac{df}{dt}$.

Theorem 2: *Assume* $f:(0,\omega)\to R$, *be a real-valued function, where* f *is differentiable and* ω*-conformable derivable. Furthermore, consider g as a differentiable function defined on the range of* f *. Then* $T_t^{\omega}(fog)(t)=t^{1-\omega}g(t)^{\omega-1}\,g'(t)T_t^{\omega}(f(t))_{t=g(t)}$, *where the prime marks the general derivative concerning the point* t .

We exercised caution in our study regarding the selected equation and the idea of conformable derivatives. Many functions do not extend the order of Taylor expansions at specific points in basic calculus, yet they diminish in presence when using the conformable order derivative technique. While complex concepts emerge in the framework of basic fractional geometry, CD operates effectively with the product and chain rule. In cases where the Riemann derivative of fractional order is not considered, the CD of a constant function is zero. Mittag-Leffler functions play a significant role in fractional order calculus, serving as a generalization of the exponential function. The fractional order form of an exponential-type function, $f(t)=e\frac{t^{\alpha}}{\alpha}$, emerges in the context of the CD.

3. 3. The enlargement of advanced $\exp(-\phi(\xi))$ -expansion techniques

This part provides a concise step-by-step presentation of our advanced $\exp(-\phi(\xi))$ -expansion scheme. Let us examine a nonlinear time-fractional NPD equation in the following form

$$\Re\,(\Pi,\ \Pi_x\ ,\ T_t^{\theta}\Pi, \Pi_{xx}\ ,\ T_{tt}^{2\theta}\Pi\ , \Pi_{xxx}\ ,\ldots\ldots) = 0, \tag{3.1}$$

Where $\Pi = \Pi(x,t)$, is treated as an unknown function, and $\Re$ indicates the polynomial in terms of Π . This represents a distinct type of partial differential equation, where both the nonlinear components and the highest order of derivatives are involved.
Step-1. We introduce a traveling variable to transform the given equation into a non-dimensional form. A traveling variable is introduced to rewrite the given equation in non-dimensional terms.

$$\Pi(x,t) = u(\xi)\ ,\ \ \xi = k\,\frac{x^{\eta}}{\eta} \pm V\ \frac{t^{\theta}}{\theta}. \tag{3.2}$$

By using the mentioned variable in Eq. (3.2), we can eliminate Eq. (3.1) in the ODE for $\Pi(x,t) = u(\xi),$ in the type

$$P(....,u''',u'',u',u,) = 0. \tag{3.3}$$

Step-2. Assume that a polynomial can represent the solution of the ODE in Eq. (3.3) in terms of $\exp(-\phi(\xi))$ as $u = \sum_{i=0}^{N} A_i \exp(-\phi(\xi))^i$,

$$A_N \neq 0. \tag{3.4}$$

In this case, a positive integer N is chosen by matching the highest derivative order with the highest order of nonlinear terms in Eq. (3.3). Furthermore, the Riccati differential equation for $\phi(\xi)$ satisfies the ODE in the following form

$$\phi'(\xi) - \mu\exp\left(-\phi(\xi)\right) - \lambda\exp\left(\phi(\xi)\right) = 0 \tag{3.5}$$

Hence, the solutions to the ODE in Eq. (3.3) take the form
Case I:
The solutions are hyperbolic (when $\lambda\mu < 0$):

$$\phi(\xi) = \ln\left(\sqrt{\frac{\lambda}{-\mu}}\,\tanh\left(\sqrt{-\lambda\mu}\,(\xi + C)\right)\right),$$

And

$$\phi(\xi) = \ln\left(\sqrt{\frac{\lambda}{-\mu}}\,\coth\left(\sqrt{-\lambda\mu}\,(\xi + C)\right)\right).$$

Case II:
The solutions are trigonometric (when $\lambda\mu > 0$):

$$\phi(\xi) = \ln\left(\sqrt{\frac{\lambda}{\mu}}\,\tan\left(\sqrt{\lambda\mu}\,(\xi + C)\right)\right),$$

And

$$\phi(\xi) = \ln\left(-\sqrt{\frac{\lambda}{\mu}}\,\cot\left(\sqrt{\lambda\mu}\,(\xi + C)\right)\right).$$

Case III:
When $\lambda = 0$, and $\mu > 0$,

$$\varphi(\xi) = \ln\left(\frac{1}{-\mu(\xi + C)}\right)$$

Case IV:
When $\lambda \in \Re$, and $\mu = 0$,

$\varphi(\xi)=\ln\left(\lambda(\xi+C)\right)$.

Where C is defined as constant, $\lambda\mu<0$, or $\lambda\mu>0$ is conditional on the symbol of μ .

Step-3. Putting Eq. (3.4) in Eq. (3.3) and finally put the Eq. (3.1). Collecting all the similar term of order of $\exp(-k\,\phi(\xi))$, $k=0,\pm1,\pm2,\pm3,.....$, prearranged, a polynomial form of $\exp(-k\,\phi(\xi))$ is served. By setting each coefficient of the resulting polynomial to zero, we obtain a System of Algebraic Equations (SAE).

Step-4. The constants are determined by solving the mathematical conditions specified in phase 3, resulting in one or more solutions. By substituting the approximated constants into the form of Eq. (3.5), we derive a new and comprehensive exact traveling wave equation for the nonlinear evolution in Eq. (3.1).

4. Application of perturbed Boussinesq equation

We investigate the space-time fractional perturbed Boussinesq Equation (BE) within the framework of the conformable derivative, as given by the following expression (see for example [43]):

$$T_t^{2\theta}u-l^2T_x^{\eta}T_x^{\eta}u+mT_x^{\eta}T_x^{\eta}(u^{2n})+rT_x^{\eta}T_x^{\eta}T_x^{\eta}T_x^{\eta}u=\beta T_x^{\eta}T_x^{\eta}u+\rho T_x^{\eta}T_x^{\eta}T_x^{\eta}T_x^{\eta}u, \tag{4.1}$$

Where $u(x,t)=u(\xi)$, $\xi=\dfrac{x^{\eta}}{\eta}-\omega\dfrac{t^{\theta}}{\theta}$, (4.2)

With β being the dissipation coefficient and ρ corresponds to the higher-order stabilization term.

By plugging the wave transformation defined in Eq. (4.2) into Eq. (4.1) and finally integrating with respect to ξ , we get the formation of ODE like

$$\left(\omega^2-l^2-\beta\right)u+mu^2+\left(r-\rho\right)u''=0, \tag{4.3}$$

Here symbolize prime signifies the differential with regards to ξ. By matching nonlinear portion and uppermost order of derivative part in obtained ODE, we can get the value of N is equivalent to 2, yields the Eq.(3.4) receipts the resulting form.

$$u(\xi)=A_0+A_1\exp\left(-\phi(\xi)\right)+A_2\exp\left(-\phi(\xi)\right)^2. \tag{4.4}$$

Performing the differentiation of Eq. (4.4) with respect to ξ for finding the values of u, u', u'' and then putting these values in Eq. (4.3). After successfully putting the values we get a polynomial. Hence we collect all the coefficients of $\exp(-k\,\phi(\xi))$ from the obtained polynomial. Finally associating the obtained coefficients of $\exp(-k\,\phi(\xi))$ corresponding to zero, where $k=0,\pm1,\pm2,\pm3,.....$

We get the system of algebraic equation of the following form.

$$2rA_2\lambda^2-2\rho A_2\lambda^2-A_0l^2+mA_0^2+A_0\omega^2-A_0\beta=0,$$

$$2\lambda\mu rA_1-2\lambda\mu\rho A_1-l^2A_1+2mA_0A_1+\omega^2A_1-\beta A_1=0,$$

$$8\lambda\mu rA_2-8\lambda\mu\rho A_2-l^2A_2+2mA_0A_2+mA_1^2+\omega^2A_2-\beta A_2=0,$$

$$2\mu^2rA_1-2\mu^2\rho A_1+2mA_1A_2=0,$$

$$6\mu^2rA_2-6\mu^2\rho A_2+mA_2^2=0.$$

Solving those mentioned SAE, we accomplish two set of solutions as below. As the calculation is very complex to solve in hand so that we got help from the computational software Maple 17.

Set 1:

$$\omega=\pm\sqrt{-4\lambda\mu r+4\lambda\mu\rho+l^2+\beta},\quad A_0=-\frac{2\lambda\mu(r-\rho)}{m},\quad A_1=0,\quad A_2=-\frac{6\mu^2(r-\rho)}{m}.$$

Case 1: The following hyperbolic-type solutions are obtained when $\lambda\mu<0$.

Family 1

$$u_{1,2}(x,t)=\frac{12(r-\rho)}{m}-\frac{16(r-\rho)}{m\tanh\left(\sqrt{6}\left(\dfrac{x^{\eta}}{\eta}\pm\dfrac{\sqrt{l^2+\beta+24r-24\rho}\,t^{\theta}}{\theta}+C\right)\right)^2},$$

$$u_{3,4}(x,t)=\frac{12(r-\rho)}{m}-\frac{16(r-\rho)}{m\coth\left(\sqrt{6}\left(\frac{x^{\eta}}{\eta}\pm\frac{\sqrt{l^2+\beta+24r-24\rho}\,t^{\theta}}{\theta}+C\right)\right)^2}.$$

Where ξ takes the form $\frac{x^{\eta}}{\eta}-\omega\frac{t^{\theta}}{\theta}$ and C is arbitrary constant.

Case 2: The following trigonometric-type solutions are obtained when $\lambda\mu > 0$.
Family 2

$$u_{5,6}(x,t)=-\frac{12\,(r-\rho)}{m}-\frac{16\,(r-\rho)}{m\tan\left(\sqrt{6}\left(\frac{x^{\eta}}{\eta}\pm\frac{\sqrt{l^2+\beta-24r+24\rho}\,t^{\theta}}{\theta}+C\right)\right)^2},$$

$$u_{7,8}(x,t)=-\frac{12(r-\rho)}{m}-\frac{16(r-\rho)}{m\cot\left(\sqrt{6}\left(\frac{x^{\eta}}{\eta}\pm\frac{\sqrt{l^2+\beta-24r+24\rho}\,t^{\theta}}{\theta}+C\right)\right)^2}.$$

Where ξ takes the form $\frac{x^{\eta}}{\eta}-\omega\frac{t^{\theta}}{\theta}$, C is arbitrary constant.

Case 3: When $\mu=0,\quad \lambda\in\Re$,

$$u_{9,10}(x,t)=-\frac{24(r-\rho)\left(-\frac{2x^{\eta}}{\eta}\pm\frac{2\sqrt{l^2+\beta}\,t^{\theta}}{\theta}-2C\right)^2}{m}.$$

Case 4:
When $\lambda=0$, and $\mu>0$, the solution equation cannot be obtained, so we neglect this condition.
Set 2:

$$\omega=\pm\sqrt{4\lambda\mu r-4\lambda\mu\rho+l^2+\beta},\quad A_0=-\frac{6\lambda\mu(r-\rho)}{m},\ A_1=0,\ A_2=-\frac{6\mu^2(r-\rho)}{m}.$$

Case 1: The following hyperbolic-type solutions are obtained when $\lambda\mu < 0$.
Family 1

$$u_{11,12}(x,t)=\frac{36(r-\rho)}{m}-\frac{16(r-\rho)}{m\tanh\left(\sqrt{6}\left(\frac{x^{\eta}}{\eta}\pm\frac{\sqrt{l^2+\beta-24r+24\rho}\,t^{\theta}}{\theta}+C\right)\right)^2},$$

$$u_{13,14}(x,t)=\frac{36(r-\rho)}{m}-\frac{16(r-\rho)}{m\coth\left(\sqrt{6}\left(\frac{x^{\eta}}{\eta}\pm\frac{\sqrt{l^2+\beta-24r+24\rho}\,t^{\theta}}{\theta}+C\right)\right)^2}.$$

Where ξ takes the form $\frac{x^{\eta}}{\eta}-\omega\frac{t^{\theta}}{\theta}$, C is arbitrary constant.

Case 2: The following trigonometric-type solutions are obtained when $\lambda\mu > 0$.
Family 2

$$u_{15,16}(x,t)=-\frac{36(r-\rho)}{m}-\frac{16(r-\rho)}{m\tan\left(\sqrt{6}\left(\frac{x^{\eta}}{\eta}\pm\frac{\sqrt{l^2+\beta+24r-24\rho}\,t^{\theta}}{\theta}+C\right)\right)^2},$$

$$u_{17,18}(x,t)=-\frac{36(r-\rho)}{m}-\frac{16(r-\rho)}{m\cot\left(\sqrt{6}\left(\frac{x^{\eta}}{\eta}\pm\frac{\sqrt{l^2+\beta+24r-24\rho}\,t^{\theta}}{\theta}+C\right)\right)^2}.$$

Where ξ takes the form $\frac{x^{\eta}}{\eta}-\omega\frac{t^{\theta}}{\theta}$, C is arbitrary constant.

Case 3
When $\mu=0,\quad \lambda\in\Re$,

$$u_{19,20}(x,t)=-\frac{24(r-\rho)\left(-\frac{2x^{\eta}}{\eta}\pm\frac{2\sqrt{l^{2}+\beta}t^{\theta}}{\theta}-2C\right)^{2}}{m}.$$

Case 4:
When $\lambda=0,$ and $\mu>0,$ the solution equation cannot be obtained, so we negligence this condition.

5. Results and analysis

5.1. Physical interpretation

The physical interpretation of the space-time fractional Perturbed BE, a notable variant of the exact influencing wave equation, is analyzed in this section. This subsection presents and examines the three-dimensional (3D) charts, density plots, and two-dimensional (2D) representations of the traveling wave solutions for the latest fractional perturbed BE equations. A three-dimensional plot illustrates the extent of divergence over time or highlights various wave phenomena. Wave phenomena are arranged systematically with evenly spaced disturbances, connected by a line to illustrate their interrelationships. The 3D style enhances the visual impact of the illustration. The density plot and 2D line chart are computed to clearly illustrate the variations in frequency and amplitude, both low and high. The charts are assembled with exclusive standards of $\eta,\theta\in(0,1]$ at several phases of time expending MATLAB for $u(x,t)$. The figures illustrate a variety of phenomena, such as singular kink type, kink type, sharp slope bell soliton, soliton solution, singular soliton solution, and other solution systems, generated by the correct physical interpretation based on different free parameter selections. In the context of mathematical physics, a solitary wave, or soliton, is defined as a self-sustaining wave packet that maintains its shape while propagating at a constant amplitude and velocity. Soliton is the comprehensive solution of a extensive sequence of feebly nonlinear diffusive PDEs relating to physical structures. Figures (1-10) illustrate the essential physical characteristics of these kink-type solutions, including their trajectories, phase shifts due to friction, and decay into separate single kink solitons. As the fractional parameter varies, both the frequency and amplitude of the wave are modified, causing the kink solution's shape to deform into a singular kink. Figure 1 depicts the multiple soliton solutions of the function $u_{1,2}(x,t)$ for the specified parameters $\lambda=3,\ \mu=-2,\ C=0.5,\ l=0.5,\ m=-1,\ \rho=-0.5,\ r=1.5,\ \beta=0.5,\ \eta=0.5,\ -10\le x,t\le 10$. In this expression l,β,r,ρ represents the free parameters while θ,η representing the fractional order derivative. As the free parameters vary, the solution graph of the function $u_{1,2}(x,t)$ alters only in height and frequency of the amplitude but when we change the fractional order derivative $\theta=0.5$ to $\theta=1$ the solution shape reformed into the two soliton solutions. Fig. 2 (set-1) shows the multiple soliton graph of $u_{3,4}(x,t)$ with respect to the parameters $\lambda=3,\ \mu=-2,\ C=0.5,\ l=0.5,\ m=-1,\ \rho=-0.5,\ r=1.5,\ \beta=0.5,\ \eta=0.75,\ -5\le x,t\le 5$. With the changes of free parameters, the solution graph of function $u_{3,4}(x,t)$ remain unchanged but when we change the fractional order parameter from $\theta=0.5$ to $\theta=1$ the solution shape reformed into the singular kink solution shape. Fig. 3-set-1 shows the one soliton solution shape of $u_{5,6}(x,t)$ with respect to the parameters $\lambda=3,\ \mu=2,\ C=0.5,\ l=0.5,\ m=-1,\ \rho=-0.5,\ r=1.5,\ \beta=0.5,\ \eta=0.5,\ -10\le x,t\le 10$. There have no dynamical diversity to the changes of free variables as well as fractional variable. Fig. 4 set-1 shows the singular soliton solution plot of $u_{7,8}(x,t)$ with respect to the parameters $\lambda=3,\ \mu=2,\ C=0.5,\ l=0.5,\ m=-1,\ \rho=0.5,\ r=1.5,\ \beta=0.5,\ \eta=0.5,\ -10\le x,t\le 10$. With the increasing of the value of fractional derivative from $\theta=0.5$ to $\theta=1$ the frequency of the soliton shape running small. Fig. 5 set-1 shows the sharp slope bell chart soliton of $u_{9,10}(x,t)$ with respect to the parameters $\lambda=0,\ \mu=2,\ C=0.5,\ l=0.5,\ m=-1,\ \rho=-0.5,\ r=1.5,\ \beta=0.5,\ \eta=0.5,$ $-10\le x,t\le 10$. There has no dynamical diversity to the changes of free variables as well as fractional variables. Fig. 6 and Fig.7 -Set-2 show the singular kink solutions of $u_{11,12}(x,t)$ and $u_{13,14}(x,t)$ respectively with respect to the parameters $\lambda=3,\ \mu=-2,\ C=0.5,\ l=0.5,$ $m=-1,\ \rho=-0.5,\ r=1.5,\ \beta=0.5,\ \eta=0.5,\ -10\le x,t\le 10$. With the variety of free parameters the solution shape reform, it height and amplitude only and have no changes with the variety of fractional parameters. Fig. 8, Fig. 9 Set-2 shows the singular soliton shape of $u_{15,16}(x,t)$, $u_{17,18}(x,t)$, respectively with respect to the parameter $\lambda=3,\ \mu=2,\ C=0.5,\ l=0.5,\ m=-1,\ \rho=0.5,\ r=1.5,\ \beta=0.75,\ \eta=0.85,\ -10\le x,t\le 10$. With the changes of free parameters, the solution graph of function $u_{17,18}(x,t)$ remain unchanged. Fig. 10 Set-2 represents the sharp slope bell soliton solution chart of $u_{19,20}(x,t)$ with respect to the parameters $\lambda=0,\ \mu=2,\ C=-0.5,\ l=0.5,\ m=-1,\ \rho=0.5,\ r=1.5,\ \beta=0.5,$ $\eta=0.75,\ -10\le x,t\le 10$. There have no dynamical diversity to the changes of free variables as well as fractional variables.

5.2. Graphical representations

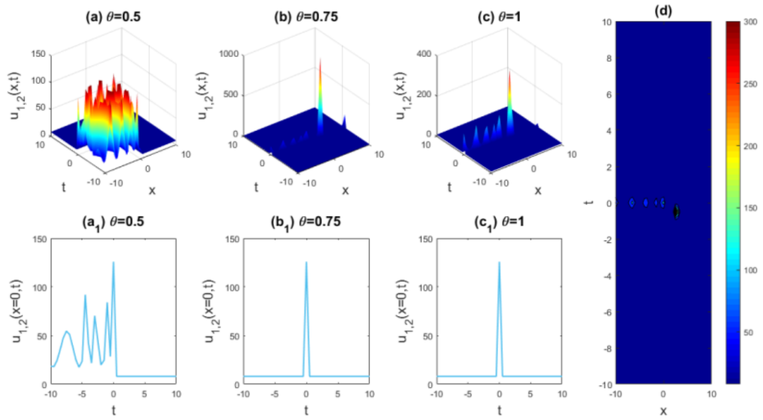

Fig. 1: Set-1: Dynamical Variation of the Function $u_{1,2}(x,t)$ Under the Influence of Fractional and Free Parameter Fluctuations.

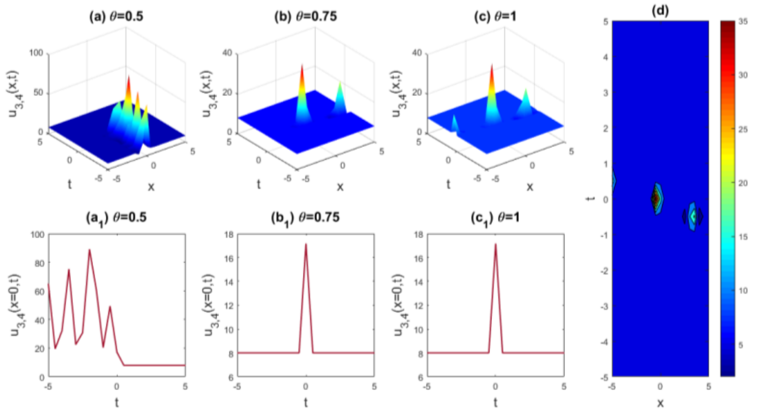

Fig. 2: Set-1-Dynamical Variation of the Function $u_{3,4}(x,t)$ Under the Influence of Fractional and Free Parameter Fluctuations.

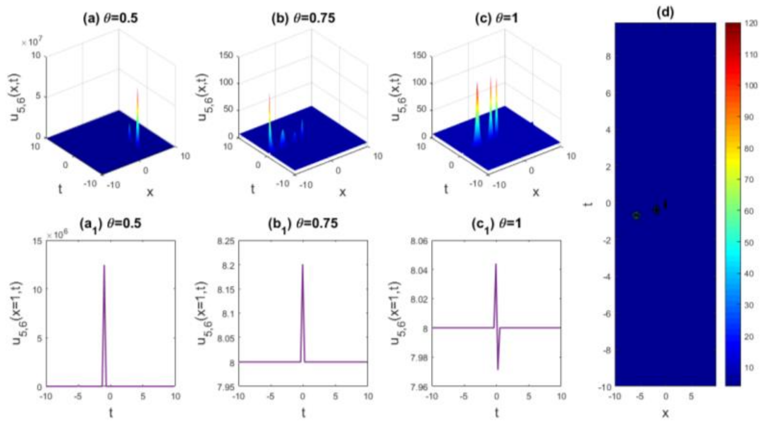

Fig. 3: Set-1-Dynamical Variation of the Function $u_{5,6}(x,t)$ Under the Influence of Fractional and Free Parameter Fluctuations.

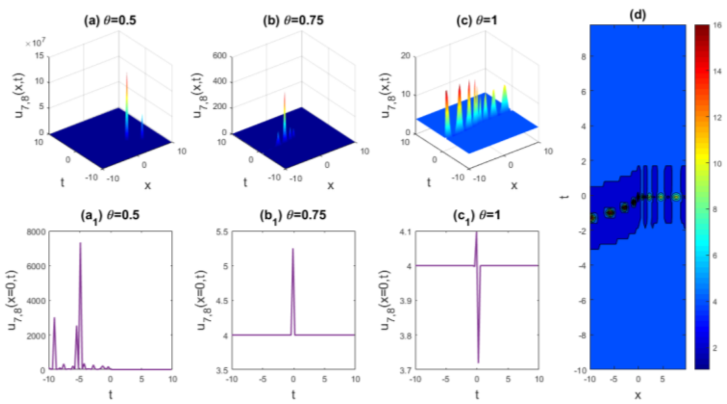

Fig. 4: Set-1-Dynamical Variation of the Function $u_{7,8}(x,t)$ Under the Influence of Fractional and Free Parameter Fluctuations.

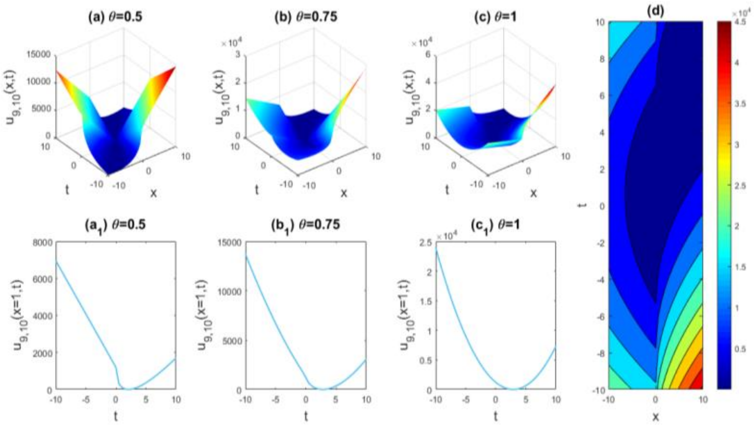

Fig. 5: Set-1-Dynamical Variation of the Function $u_{9,10}(x,t)$ Under the Influence of Fractional and Free Parameter Fluctuations.

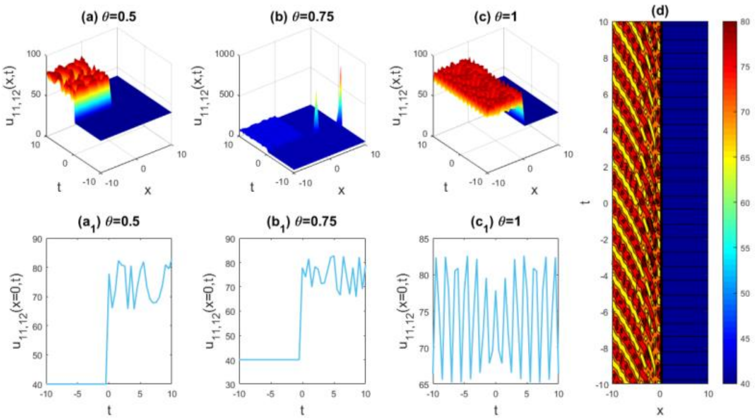

Fig. 6: Set-2-Dynamical Variation of the Function $u_{11,12}(x,t)$ Under the Influence of Fractional and Free Parameter Fluctuations.

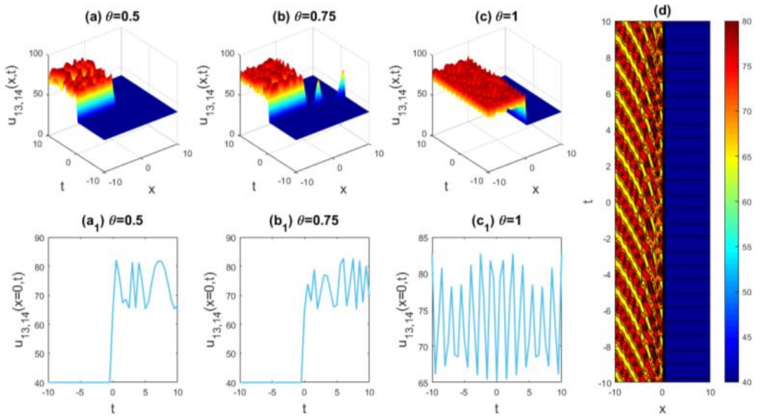

Fig. 7: Set-2-Dynamical Variation of the Function $u_{13,14}(x,t)$ Under the Influence of Fractional and Free Parameter Fluctuations.

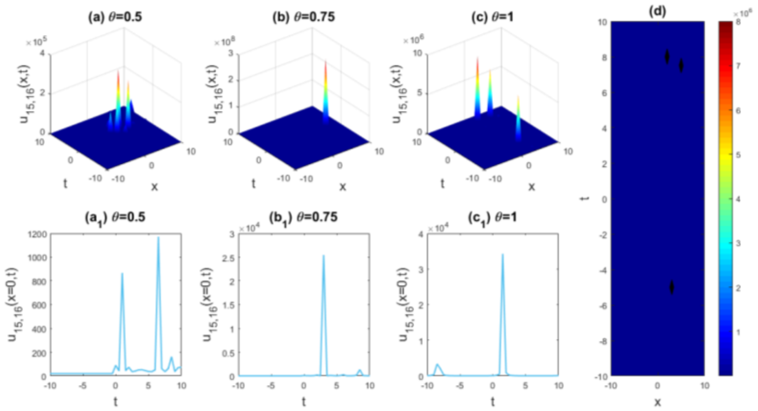

Fig. 8: Set-2-Dynamical Variation of the Function $u_{15,16}(x,t)$ Under the Influence of Fractional and Free Parameter Fluctuations.

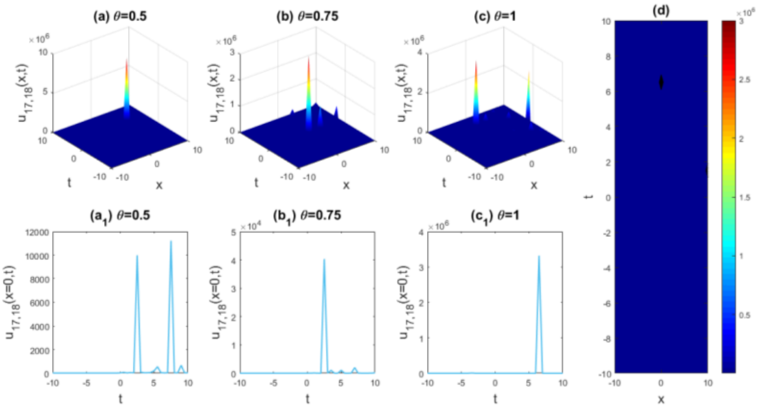

Fig. 9: Set-2-Dynamical Variation of the Function $u_{17,18}(x,t)$ Under the Influence of Fractional and Free Parameter Fluctuations.

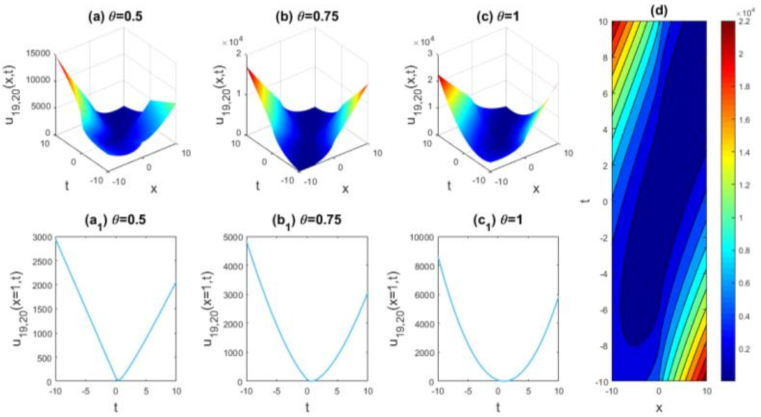

Fig. 10: Set-2-Dynamical Variation of the Function $u_{19,20}(x,t)$ Under the Influence of Fractional and Free Parameter Fluctuations.

6. Modulation instability (MI) analysis

Frequent nonlinear PDEs of sophisticated order show an uncertainty that companions to measure the variation of the stable state as a consequence of an interface between the linear and nonlinear effects. To generate the inflection uncertainty of the perturbed Boussinesq equation (4.1) by expanding upon the standard linear stability analysis.[44] The study state equation of modifying unstable NLSE has the form

$$u(x,t) = (\aleph(x,t) + \sqrt{R})e^{\vartheta(t)}, \ \vartheta(t) = R\varepsilon\tau t. \tag{6.1}$$

Where R is regularized visual control. The perturbation $\aleph(x,t)$ is examine by applying linear stability investigation. Placing Eq. (6.1) into Eq. (4.1) and linearizing, we get the form as below.

$$\frac{\partial^2\aleph}{\partial t^2} - l^2\frac{\partial^2\aleph}{\partial x^2} + m\frac{\partial^2\aleph}{\partial x^2} + r\frac{\partial^4\aleph}{\partial x^4} - \beta\frac{\partial^2\aleph}{\partial x^2} - \rho\frac{\partial^4\aleph}{\partial x^4} = 0. \tag{6.2}$$

The solution of Eq. (6.2) is analyzed within the system

$$\aleph(x,t) = \alpha_1 \exp\left(i\left(k\frac{x^\eta}{\eta} - \omega\frac{t^\theta}{\theta}\right)\right) + \alpha_2 \exp\left(-i\left(k\frac{x^\eta}{\eta} - \omega\frac{t^\theta}{\theta}\right)\right). \tag{6.3}$$

In this context, k represents the regularized wave number, and ω denotes the frequency of the perturbation. The dispersion relation $k = k(\omega)$ of a linear PDE with continuous coefficients governs the connection between time fluctuations and dimensional fluctuations $e^{ik\frac{x^\eta}{\eta}}$ of the wave number. Upon substituting Eq. (6.3) into Eq. (6.2), the resulting scattering relation is obtained as follows.

$$R\delta\varepsilon k^2 rt(k^2 + \frac{1}{k^2} - \rho - \frac{\rho}{k^4} + \frac{2}{k^2}(l^2 + \beta - m) = \omega^2(2R\tau\varepsilon t),$$

$$\omega^2 = \frac{R\tau\varepsilon k^2 rt(k^2 + \frac{1}{k^2} - \rho - \frac{\rho}{k^4} + \frac{2}{k^2}(l^2 + \beta - m))}{2R\tau\varepsilon t},$$

$$\omega = \frac{\sqrt{R\tau\varepsilon k^2 rt(k^2 + \frac{1}{k^2} - \rho - \frac{\rho}{k^4}) + R\delta\varepsilon k^2 rt\frac{2}{k^2}(l^2 + \beta - m)}}{\sqrt{2R\tau\varepsilon t}} \tag{6.4}$$

Equation (6.4) of the spreading relation reveals that the steady-state stability depends on the self-phase inflection, stimulated Raman Scattering, group velocity, and wave number dispersion. If

$R\tau\varepsilon k^2 rt(k^2+\frac{1}{k^2}-\rho-\frac{\rho}{k^4})+R\delta\varepsilon k^2 rt\frac{2}{k^2}(l^2+\beta-m)>0$, and $2R\delta\varepsilon t\neq 0$ and must be positive, i.e. the ω is real for altogether wave number k then the steady state is steady against the small distresses. On the further side, the steady state solution develops unstable if $R\delta\varepsilon k^2 rt\left(k^2+\frac{1}{k^2}-\rho-\frac{\rho}{k^4}\right)+R\tau\varepsilon k^2 rt\frac{2}{k^2}(l^2+\beta-m)<0$,
and $2R\tau\varepsilon t\neq 0$, and must be positive. Where the ω is an imaginary term since the perturbation results exponentially. It is easy to observe that, for the incidence of MI stability when $R\tau\varepsilon k^2 rt\left(k^2+\frac{1}{k^2}-\rho-\frac{\rho}{k^4}\right)<-R\tau\varepsilon k^2 rt\frac{2}{k^2}(l^2+\beta-m)$ and $2R\tau\varepsilon t>0$. Given this situation, the development rate of the MI stability improvement spectrum f(spec.) could be articulated as

$$f(\text{spec.})=2\text{lm}(\varphi)=\frac{\sqrt{R\tau\varepsilon k^2 rt(k^2+\frac{1}{k^2}-\rho-\frac{\rho}{k^4})+R\tau\varepsilon k^2 rt\frac{2}{k^2}(l^2+\beta-m)}}{\sqrt{2R\tau\varepsilon t}}$$

7. Conclusion

This research focused on the closed-form kink, soliton, and sharp slope bell soliton solutions of the space-time fractional perturbed Boussinesq equation, which are influenced by various free and fractional parameters. The obtained soliton results of the studied equations are commonly used to express different types of wave transmission in natural situations, especially in shallow water kinematics. In addition, the stability analysis of the attained results was reviewed, and the driving role of the sprays was examined, ensuring that all developed responses are explicit, exact, dependable, and stable. The varying magnitudes of the partial and variable parameters from a distinct function of significant solutions can be fully determined based on our chosen fractional and free parameters. Moreover, we contend that the traveling solutions we identified are novel in the context of the conformable derivative and may serve as valuable tools in the investigation of nonlinear physical processes. The approach we present is efficient, reliable, and easy to use, providing accurate solutions for NLFEEs in areas like engineering, applied mathematics, nonlinear dynamics, and mathematical physics.

Availability of data

Supporting data for the findings of this study are provided in the article.

Conflicting interest

The authors declare no conflicts of interest.

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