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Research paper

Optimal Recovery Method of the Solution of a One-Dimensional Wave Equation at Time T From Inaccurate Data at $t = 0$ And $t = T$

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Annotation

The article deals with the problem of optimal recovery of the solution of the wave equation at some time instant by known, but given with some error, functions determining the shape of the string at times *t* and *T*. The goal of the paper is to construct an optimal recovery method for the solution of the wave equation from inaccurate data. An important assumption used in the work is the possibility of representing the solution in the form of a Fourier series. The main solution method is the introduction of an auxiliary extremal problem for a conditional extremum, the solution of which determines the optimal recovery method. The result of the work is to find the optimal recovery method among all possible methods. The solution of the restoration problem and the value of the error of optimal recovery are obtained. Cases are indicated when it is possible to reduce the amount of initial information required for solving the problem.

Keywords: *optimal recovery, solution of the wave equation from inaccurate data.*

1. Introduction

When solving many problems of mathematical physics, and especially when they are being simulated numerically, problems arise related to the discretization of functions, the recovery of functions, functionals or operators from them according to some incomplete and inaccurate information about the function. Such tasks, which have been intensively studied recently (especially in connection with the development of information technologies), constitute the direction that has been called the optimal restoration. The problems studied in this area contain such important tasks as the construction of optimal methods for recovering functions specified exactly or approximately at a finite number of points, the construction of optimal quadrature formulas, the recovery of derivatives (numerical differentiation), the optimal choice of information that you need to know about the function in order to with the smallest error to reconstruct it, approximation of a function from its approximate Fourier coefficients or Fourier transform, etc. The results on these topics are contained in papers [4, 5]. If the classical approach, as a rule, sets the means of approximation (algebraic or trigonometric polynomials, rational functions, splines, wavelets, etc.), then in optimal recovery problems, the type of restoration method is not fixed in advance - it is searched for among all possible methods (algorithms) using values of the approximated function. The importance of such a statement is due to the fact that with fixed information the best way is chosen for approximating a function or functional (in the general case, an operator). It should be noted that the theory of approximation of functions originates from the works of P. L. Chebyshev. He introduced the concept of the best approximation of a function by polynomials, namely, he posed an extremal problem, the solution of which gives the polynomial of the best approximation. The next step is to consider the approximation of an objective function of a given class by functions from a previously specified set; an example of such a problem is the problem of approximating a function from a Sobolev space with polynomials of degree not higher than n. A. N. Kolmogorov also paid attention to the best approximation problems. In [2], the characteristics of a new type, called widths, which characterize the deviation of a set in a normalized space from a certain system of objects with a certain method of approximation, were determined. The idea of finding the best, in a given sense, width lies at the basis of the tasks formulated by S. A. Smolyak [9]. The approach proposed by Magaril-Il'yaev G. G., Osipenko K. Yu., Tikhomirov V. M. in [6, 7] is based on the Lagrange principle and is unified for solving optimal recovery problems. In the problem under consideration, the optimal method for reconstructing the solution of the wave equation at an intermediate time is searched for among all possible methods, without imposing any restrictions on them. The optimal recovery method is obtained, the optimal error value is obtained for different ratios of the initial data errors, an equation solution is found that determines the shape of the string at a selected time instant. In addition, it is indicated that for some ratios of the initial errors, to obtain the optimal method, it is not necessary to use data on the shape of the string at the initial (or final) moment. In practice, this allows to reduce the number of measurements required to solve the problem.

2. Problem formulation

Consider a wave equation with zero boundary conditions and zero initial velocity

(1)
$$u_{tt} = u_{xx},\quad u(0,t) = u(\pi,t) = 0,\quad u_t(x,0) = 0.$$

We assume that the approximate values of the function $u(x,t)$ at $t = 0$ and $t = T$ is:

$$u(x,0) \approx y_0(x),\ \|f_0(x) - y_0(x)\|_{L_2([0,\pi])} \le \delta_0,$$
$$u(x,T) \approx y_1(x),\ \|f_1(x) - y_1(x)\|_{L_2([0,\pi])} \le \delta_1,$$

where $f_0(x)$and $f_1(x)$- the exact values of the function $u(x,t)$ respectively, at$t = 0$ and $t = T$, with $f_{0,1}, y_{0,1} \epsilon L_2 [0,\pi]$. It is required to find the optimal approximation of the value of the function $u(x,t)$ at $t = \tau$, $0 < \tau < T$. As is known, the exact solution to this problem is

(2)
$$u(x,\tau) = \sum_{j=1}^{\infty} a_j \cos j\tau \sin jx,$$

where

$$a_j = \frac{2}{\pi}\int_0^{\pi} f_0(x) \sin jx\, dx$$

Fourier coefficients of the function $f_0(x)$.
As recovery methods, we will consider arbitrary operators.ϕ: $L_2([0,\pi]) \to L_2([0,\pi])$. *The recovery error* for this method ϕ is the value

$$e(T,\tau,\delta_0,\delta_1,\varphi) = \sup_{\substack{f_{0,1}\in L_2([0,\pi]),\ y_{0,1}\in L_2([0,\pi]) \\ \|f_0(x)-y_0(x)\|_{L_2([0,\pi])}\le\delta_0 \\ \|f_1(x)-y_1(x)\|_{L_2([0,\pi])}\le\delta_1}} \left\|u(\cdot,\tau) - \varphi(y_{0,1})(\cdot)\right\|_{L_2([0,\pi]).}$$

The value

$$e(T,\tau,\delta_0,\delta_1) = \inf_{\varphi:L_2([0,\pi])\to L_2([0,\pi])} e(T,\tau,\delta_0,\delta_1,\varphi)$$

is called *the optimal recovery error*, and the method, on which the lower bound is achieved, is called *the optimal recovery method*.

3. Main results

Choose $\tau = \frac{T}{n}$ and consider two cases:
1. $\frac{T}{\pi} \in Q$. In this case, the set of values of $\cos^2 jT$ is finite. Then one can set the sequence s_k,
$k = 0,1,\dots,r$, such that $\cos^2 s_kT$ and $\cos^2 s_k\tau$ - descending sequences,

$$\frac{\cos^2 s_k\tau - \cos^2 s_{k+1}\tau}{\cos^2 s_kT - \cos^2 s_{k+1}T} < \frac{\cos^2 s_{k+1}\tau - \cos^2 s_{k+2}\tau}{\cos^2 s_{k+1}T - \cos^2 s_{k+2}T},$$
$$k = 0,1,\dots,r-2\,,$$

for all j and k

$$\cos^2 j\tau \le \cos^2 s_k\tau + \frac{\cos^2 s_k\tau - \cos^2 s_{k+1}\tau}{\cos^2 s_kT - \cos^2 s_{k+1}T}(\cos^2 jT - \cos^2 s_kT),$$

$\cos^2 jT \ge \cos^2 s_rT$, $j = 0,1,\dots$, and for any j there is such s_k, that $\cos^2 jT \le \cos^2 s_kT$.
Theorem 1. If $\frac{T}{\pi} \in Q$, *then:*

(i) *At* $\frac{\delta_1^2}{\delta_0^2} > 1$

$$E(T,\tau,\delta_0,\delta_1) = \delta_0,$$

and method

$$u(x,\tau) \approx= \sum_{j=0}^{\infty} b_j(y_0) \cos j\tau \sin jx$$

is optimal;

(ii) *If* $\cos^2 s_\tau T = 0$, *then with* $\cos^2 s_{k+1}T \le \frac{\delta_1^2}{\delta_0^2} < \cos^2 s_kT = 0$

$$E(T,\tau,\delta_0,\delta_1) = \sqrt{\frac{(\cos^2 s_{k+1}\tau\cos^2 s_kT - \cos^2 s_k\tau\cos^2 s_{k+1}T)\delta_0^2 + (\cos^2 s_k\tau - \cos^2 s_{k+1}\tau)\delta_1^2}{\cos^2 s_kT - \cos^2 s_{k+1}T}},$$

and the optimal method

$$u(x,\tau) \approx= \sum_{j=0}^{\infty} \frac{\hat\lambda_1 b_j(y_0) + \hat\lambda_2 c_j(y_1)}{\hat\lambda_1 + \hat\lambda_2\cos^2 jT}\cos j\tau \sin jx,$$
$$\hat\lambda_1 = \frac{\cos^2 s_{k+1}\tau\cos^2 s_kT - \cos^2 s_k\tau\cos^2 s_{k+1}T}{\cos^2 s_kT - \cos^2 s_{k+1}T},$$
$$\hat\lambda_2 = \frac{\cos^2 s_k\tau - \cos^2 s_{k+1}\tau}{\cos^2 s_kT - \cos^2 s_{k+1}T}.$$

$b_j(y_0), c_j(y_1) -$ *Fourier coefficients of the functiony*$_0$ *and* y_1.

(iii) *if* $\cos^2 s_\tau T > 0$, *then at* $0 < \frac{\delta_1^2}{\delta_0^2} \le \cos^2 s_r T$

$$E(T,\tau,\delta_0,\delta_1) = \frac{\cos s_r\tau}{\cos s_r T}\delta_1,$$

method

$$u(x,\tau) \approx= \sum_{j=0}^{\infty} \frac{c_j(y_1)}{\cos jT}\cos j\tau \sin jx -$$

is optimal.

Thus, the optimal method depends on the ratio of the errors with which the values of the function $u(x,0)$ and $u(x,T)$ are given. If the error at $t = 0$ is less than at $t = T$, the optimal method uses only the coefficients $b_j(y_0)$, which act as the Fourier coefficients at $t = \tau$ (case (*i*)). If the error at $t = 0$ is significantly larger than at $t = T$, then the construction of the optimal method requires only the coefficients $c_j(y_1)$, but with some smoothing factors (case (*iii*)). Finally, in the most general case (*ii*), the smoothing factor is constructed using both approximations.

Proof of Theorem 1. Denote $u_j = a_j^2$ and consider the extremal problem

(3)
$$\|u_j\|^2 \to min,\quad \sum_{j=1}^{\infty} u_j \le \delta_0^2, \sum_{j=1}^{\infty} u_j\cos^2 jT \le \delta_1^2, u_j \ge 0.$$

We introduce the Lagrange function for this problem

$$L(u,\lambda_1,\lambda_2) = \sum_{j=1}^{\infty}(\lambda_1 + \lambda_2\cos^2 jT - \cos^2 j\tau)u_j,$$

where $u = \{u_j\}_{j\in N}$, λ_1,λ_2 — Lagrange multipliers.
It follows from [3] (see also [8]), that if there are $\hat\lambda_1,\hat\lambda_2 \ge 0$ such that the following conditions are satisfied for the sequence $\hat u = \{\hat u_j\}_{j\in N}$ admissible in task (3)

(a)
$$\min_{u_j\ge 0} \mathcal{L}(u,\hat\lambda_1,\hat\lambda_2) = \mathcal{L}(\hat u,\hat\lambda_1,\hat\lambda_2),$$

(b)
$$\hat\lambda_1\left(\sum_{j=1}^{\infty}\hat u_j - \delta_0^2\right) + \hat\lambda_2\left(\sum_{j=1}^{\infty}\hat u_j\cos^2 jT - \delta_1^2\right) = 0,$$

then $\hat u$ is a solution to task (3), and its value is

$$\hat\lambda_1\delta_0^2 + \hat\lambda_2\delta_1^2.$$

If, moreover, for all $y \in L_2([0,\pi])$ there exists a solutionx_y of the extremal problem

(4) $$\hat\lambda_1\|x-y_0\|^2_{L_2([0,\pi])}+\hat\lambda_2\|x-y_1\|_{L_2([0,\pi])}\to min$$

then

(5) $$E(T,\tau,\delta_0,\delta_1)=\sqrt{\hat\lambda_1\delta_0^2+\hat\lambda_2\delta_1^2},$$

and method

$$\hat\varphi(y)=\sum_{j=1}^{\infty}(x_y)_j\cos j\,\tau$$

is optimal.

Task (4) can be written as

$$\hat\lambda_1\sum_{j=1}^{\infty}\left(x_j-b_j(y_0)\right)^2+\hat\lambda_2\sum_{j=1}^{\infty}\left(x_j-c_j(y_1)\right)^2\to min,$$

where b_j and c_j are the Fourier coefficients of the functions y_0 and y_1. It is easy to verify that for fixed $\hat\lambda_1$ and $\hat\lambda_2$ its solution is

$$x_y=\sum_{j=1}^{\infty}\frac{\hat\lambda_1 b_j(y_0)+\hat\lambda_2 c_j(y_1)}{\hat\lambda_1+\hat\lambda_2\cos^2 jT}$$

Therefore, it suffices to find $\hat\lambda_1,\hat\lambda_2\geq 0$ nd the sequence admissible in (3) $\hat u=\{\hat u_j\}_{j\in N}$, for which conditions (*a*) and (*b*) are satisfied. The method

(6)

$$\hat\varphi(y)=\sum_{j=1}^{\infty}\frac{\hat\lambda_1 b_j(y_0)+\hat\lambda_2 c_j(y_1)}{\hat\lambda_1+\hat\lambda_2\cos^2 jT}\cos jT$$

is optimal.

Let $\frac{\delta_1^2}{\delta_0^2}>1$. Set

$$\hat\lambda_1=1,\hat\lambda_2=0,\hat u_0=\delta_0^2,\hat u_j=0,j\neq 0.$$

It is easy to verify that the sequence $\{\hat u_j\}$ — is admissible and conditions (*b*) are fulfilled. We have b

$$L(u,\hat\lambda_1,\hat\lambda_2)=\sum_{j=0}^{\infty}(1-\cos^2 j\tau)u_j\geq 0,$$

$$L(\hat u,\hat\lambda_1,\hat\lambda_2)=(1-\cos^2 0)\delta_0^2=0=\min_{u_j\geq 0}L(u,\hat\lambda_1,\hat\lambda_2)$$

condition (*a*) is fulfilled. In this case, from (5) we obtain

$$E(T,\tau,\delta_0,\delta_1)=\delta_0,\quad \hat\varphi(y)=\sum_{j=0}^{\infty}b_j(y_0)\cos j\tau$$

For the case $\cos^2 s_{k+1}T\leq\frac{\delta_1^2}{\delta_0^2}<\cos^2 s_kT$ we set

$$\hat\lambda_1=\frac{\cos^2 s_{k+1}\tau\cos^2 s_kT-\cos^2 s_k\tau\cos^2 s_{k+1}T}{\cos^2 s_kT-\cos^2 s_{k+1}T},$$

$$\hat\lambda_2=\frac{\cos^2 s_k\tau-\cos^2 s_{k+1}\tau}{\cos^2 s_kT-\cos^2 s_{k+1}T},\hat u_j=0,j\neq s_k,s_{k+1},$$

and $\hat u_{s_k}$ and $\hat u_{s_{k+1}}$ are chosen so that conditions (*b*) are fulfilled:

$$\hat u_{s_k}+\hat u_{s_{k+1}}=\delta_0^2,$$

$$\hat u_{s_k}\cos^2 s_kT+\hat u_{s_{k+1}}\cos^2 s_{k+1}T=\delta_1^2,$$

from where

$$\hat u_{s_k}=\frac{\delta_1^2-\delta_0^2\cos^2 s_{k+1}T}{\cos^2 s_kT-\cos^2 s_{k+1}T},\hat u_{s_k+1}=\frac{\delta_0^2\cos^2 s_kT-\delta_1^2}{\cos^2 s_kT-\cos^2 s_{k+1}T}.$$

In this case, the following relation is valid

$$\hat\lambda_1+\hat\lambda_2\cos^2 jT-\cos^2 j\tau=\frac{\cos^2 s_kT(\cos^2 s_{k+1}\tau-\cos^2 j\tau)-\cos^2 s_{k+1}T(\cos^2 s_k\tau-\cos^2 j\tau)}{\cos^2 s_kT-\cos^2 s_{k+1}T}$$

$$+\frac{\cos^2 jT(\cos^2 s_k\tau-\cos^2 s_{k+1}\tau)}{\cos^2 s_kT-\cos^2 s_{k+1}T}\geq$$

$$\cos^2 s_{k+1}\tau-\cos^2 j\tau+\frac{\cos^2 jT(\cos^2 s_k\tau-\cos^2 s_{k+1}\tau)}{\cos^2 s_kT-\cos^2 s_{k+1}T}\geq$$

$$(\cos^2 s_k\tau-\cos^2 s_{k+1}\tau)\left(-1+\frac{\cos^2 s_kT}{\cos^2 s_kT-\cos^2 s_{k+1}T}\right)\geq 0.$$

Consequently, $L(u,\hat\lambda_1,\hat\lambda_2)\geq 0$. Respectively

$$L(u,\hat\lambda_1,\hat\lambda_2)=\left(\frac{\cos^2 s_{k+1}\tau\cos^2 s_kT-\cos^2 s_k\tau\cos^2 s_{k+1}T}{\cos^2 s_kT-\cos^2 s_{k+1}T}+\frac{\cos^2 s_k\tau\cos^2 s_kT-\cos^2 s_{k+1}\tau\cos^2 s_kT}{\cos^2 s_kT-\cos^2 s_{k+1}T}-\frac{\cos^2 s_k\tau\cos^2 s_kT-\cos^2 s_k\tau\cos^2 s_{k+1}T}{\cos^2 s_kT-\cos^2 s_{k+1}T}\right)\hat u_{s_k}$$
$$+\left(\frac{\cos^2 s_{k+1}\tau\cos^2 s_kT-\cos^2 s_k\tau\cos^2 s_{k+1}T}{\cos^2 s_kT-\cos^2 s_{k+1}T}+\frac{\cos^2 s_k\tau\cos^2 s_{k+1}T-\cos^2 s_{k+1}\tau\cos^2 s_{k+1}T}{\cos^2 s_kT-\cos^2 s_{k+1}T}-\frac{\cos^2 s_{k+1}\tau\cos^2 s_kT-\cos^2 s_{k+1}\tau\cos^2 s_{k+1}T}{\cos^2 s_kT-\cos^2 s_{k+1}T}\right)\hat u_{s_{k+1}}=0$$
$$=\min_{u_j\geq 0}L(u,\hat\lambda_1,\hat\lambda_2).$$

Thus, conditions (*a*) and (*b*) are fulfilled, and from ($E(T,\tau,\delta_0,\delta_1)=\sqrt{\frac{(\cos^2 s_{k+1}\tau\cos^2 s_kT-\cos^2 s_k\tau\cos^2 s_{k+1}T)\delta_0^2+(\cos^2 s_k\tau-\cos^2 s_{k+1}\tau)\delta_1^2}{\cos^2 s_kT-\cos^2 s_{k+1}T}}$. 5) we obtain

In the case when $\cos^2 s_rT>0$, we set

$$\hat\lambda_1=0,\hat\lambda_2=\frac{\cos^2 s_r\tau}{\cos^2 s_rT},\ \hat u_{s_r}=\frac{\delta_1^2}{\cos^2 s_rT},\hat u_j=0,j\neq s_r.$$

Then

$$\hat\lambda_1+\hat\lambda_2\cos^2 jT-\cos^2 j\tau=\frac{\cos^2 s_r\tau}{\cos^2 s_rT}\cos^2 jT-\cos^2 j\tau=\cos^2 jT\left(\frac{\cos^2 s_r\tau}{\cos^2 s_rT}-\frac{\cos^2 j\tau}{\cos^2 jT}\right)\geq 0$$

that is

$$L(u,\hat\lambda_1,\hat\lambda_2)\geq 0;\ L(\hat u,\hat\lambda_1,\hat\lambda_2)=0=\min_{u_j\geq 0}L(u,\hat\lambda_1,\hat\lambda_2)-$$

condition (*a*) is fulfilled. The fulfillment of condition (*b*) is obvious, therefore,

$$E(T,\tau,\delta_0,\delta_1)=\frac{\cos s_r\tau}{\cos s_rT}\delta_1.$$

The theorem is proved completely.

2. If the ratio $\frac{T}{\pi}$ is an irrational number, then the set of values $\cos^2 jT$ is everywhere dense on the interval [0,1]. Therefore, for any $x\in[0,1]$, one can find a sequence x_n, $x_n=\cos^2 nT$, such that $\lim_{n\to\infty}x_n=x$.

Theorem 2. *At* $\frac{\delta_1^2}{\delta_0^2}>1$

$$E(T,\tau,\delta_0,\delta_1)=\delta_0,$$

method

$$u(x,\tau)\approx=\sum_{j=0}^{\infty}b_j(y_0)\cos j\,\tau\sin j\,x$$

is optimal;

at $\frac{\delta_1^2}{\delta_0^2}\leq 1$

$$E(T,\tau,\delta_0,\delta_1)=\sqrt{\frac{\delta_1\delta_0+\delta_0^2}{2}},$$

method

$$u(x,\tau)\approx=\sum_{j=0}^{\infty}\frac{\hat\lambda_1 b_j(y_0)+\hat\lambda_2 c_j(y_1)}{\hat\lambda_1+\hat\lambda_2\cos^2 jT}\cos j\tau\sin jx,$$

where

$$\hat{\lambda}_1 = \cos^2\left(\frac{1}{n}\arccos\frac{\delta_1}{\delta_0}\right) - \frac{\frac{\delta_1}{\delta_0}\sin\left(\frac{2}{n}\arccos\frac{\delta_1}{\delta_0}\right)}{2\sqrt{1-\frac{\delta_1^2}{\delta_0^2}}},$$

$$\hat{\lambda}_2 = \frac{\sin\left(\frac{2}{n}\arccos\frac{\delta_1}{\delta_0}\right)}{2\frac{\delta_1}{\delta_0}\sqrt{1-\frac{\delta_1^2}{\delta_0^2}}},$$

is optimal.

As in Theorem 1, to construct the optimal method, it is necessary to use the approximate values of the function $u(x,t)$ at the points $t=0$ and $t=T$, if the error at $t=T$ is less than at $t=0$. Otherwise, it suffices to use only $b_j(y_0)$ values.

Proof of the theorem 2. The proof for the case $\frac{\delta_1^2}{\delta_0^2} > 1$ coincides with the proof of the corresponding statement of Theorem 1.

For the case of $\frac{\delta_1^2}{\delta_0^2} \le 1$, consider the curve

(7) $$y = \cos^2\left(\frac{1}{n}\arccos\sqrt{x}\right).$$

The set of points with coordinates

$$x = \cos^2 jT, \ \ y = \cos^2 j\tau$$

is everywhere dense on this curve. Choose a point on (7) with the abscissa $x_0 = \frac{\delta_1^2}{\delta_0^2}$. If there is a value $j = j_0$, such that $\cos^2 j_0 T = x_0$, then we set $\hat{\lambda}_1$ and $\hat{\lambda}_2$ such that the line $y = \hat{\lambda}_1 + \hat{\lambda}_2 x$ is tangent to (7) at $x = x_0$. Then, by virtue of the convexity of the curve (7), all its points will lie below the tangent, that is, for any j, condition $\cos^2 j\tau \le \hat{\lambda}_1 + \hat{\lambda}_2\cos^2 jT$, will be fulfilled, whence it follows that $\mathcal{L}(u,\hat{\lambda}_1,\hat{\lambda}_2) \ge 0$.

Considering that at $j_0 T = \arccos\frac{\delta_1}{\delta_0}$ $j_0\tau = \frac{1}{n}\arccos\frac{\delta_1}{\delta_0}$, we obtain a system of equations for $\hat{\lambda}_1$ and $\hat{\lambda}_2$

(8)

$$\begin{cases} \hat{\lambda}_1 + \hat{\lambda}_2\frac{\delta_1}{\delta_0} = \cos^2\left(\frac{1}{n}\arccos\frac{\delta_1}{\delta_0}\right) \\ \hat{\lambda}_2 = \frac{\sin\left(\frac{2}{n}\arccos\frac{\delta_1}{\delta_0}\right)}{2\frac{\delta_1}{\delta_0}\sqrt{1-\frac{\delta_1^2}{\delta_0^2}}}, \end{cases}$$

from where

(9)

$$\hat{\lambda}_1 = \cos^2\left(\frac{1}{n}\arccos\frac{\delta_1}{\delta_0}\right) - \frac{\sin\left(\frac{2}{n}\arccos\frac{\delta_1}{\delta_0}\right)}{2\sqrt{1-\frac{\delta_1^2}{\delta_0^2}}}.$$

Set $\hat{u}_{j_0} = \delta_0^2$, $\hat{u}_j = 0$, $j \ne j_0$. Then

$$L(u,\hat{\lambda}_1,\hat{\lambda}_2) = 0 = \min_{u_j \ge 0} L(u,\hat{\lambda}_1,\hat{\lambda}_2) -$$

condition (*a*) is fulfilled. Check the condition (*b*).

$$\sum_{j=0}^{\infty}\hat{u}_j - \delta_0^2 = \hat{u}_{j_0} - \delta_0^2 = 0,$$

$$\sum_{j=0}^{\infty}\hat{u}_j\cos^2 jT - \delta_1^2 = \hat{u}_{j_0}\cos^2 j_0 T - \delta_1^2 = \delta_0^2\frac{\delta_1^2}{\delta_0^2} - \delta_1^2 = 0,$$

that is, condition (*b*) is fulfilled.

In the case when $x_0 \ne \cos^2 jT$, choose a sequence $\cos^2 jT$ such that $\lim_{k\to\infty}\cos^2 kT = x_0$, and consider the corresponding sequence $\{u_k\}$, $u_k = \frac{\delta_1^2}{\cos^2 kT}$. Then for $\hat{\lambda}_1, \hat{\lambda}_2$, given as (9), (8),

$$\lim_{k\to\infty} L(u_k,\hat{\lambda}_1,\hat{\lambda}_2) = 0,$$

$$\lim_{k\to\infty}\left(\hat{\lambda}_1\left(\sum_{j=1}^{\infty}\hat{u}_k - \delta_0^2\right) + \hat{\lambda}_2\left(\sum_{j=1}^{\infty}\hat{u}_k\cos^2 jT - \delta_1^2\right)\right) = 0.$$

Then it follows from Theorem 4 ([8]) that the error of optimal recovery and the optimal method are given, as in the previous case, by the formulas

$$E(T,\tau,\delta_0,\delta_1) = \sqrt{\frac{\delta_1\delta_0 + \delta_0^2}{2}}$$

and

$$u(x,\tau) \approx= \sum_{j=0}^{\infty}\frac{\hat{\lambda}_1 b_j(y_0) + \hat{\lambda}_2 c_j(y_1)}{\hat{\lambda}_1 + \hat{\lambda}_2\cos^2 jT}\cos j\tau\sin jx.$$

Conclusion

In this paper, we consider a method for reconstructing the solution of a one-dimensional oscillation equation with Dirichlet conditions and for an inaccurately given initial form and zero initial velocity. We constructively constructed recovery methods in the form of a Fourier series in eigenfunctions of the corresponding Dirichlet problem. The conditions under which the constructed methods are optimal are formulated and proved. The errors of optimal recovery are calculated.

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