International Journal of Engineering & Technology, 7 (4.8) (2018) 692-696

International Journal of Engineering & Technology

Website:www.sciencepubco.com/index.php/IJET

Research paper

Structural and Topological Properties of the Most Compact Toroidal-Lattice Communication Networks

Oleksii Tyrtyshnikov[1]*, Maryna Mavrina[2], Ilona Chernytska[3], Serhii Voloshko[4]

[1]Poltava National Technical Yuri Kondratyuk University, Ukraine
[2]Poltava National Technical Yuri Kondratyuk University, Ukraine
[3]Poltava National Technical Yuri Kondratyuk University, Ukraine
[4]Poltava National Technical Yuri Kondratyuk University, Ukraine
**Corresponding author E-mail: alexey_it@ukr.net*

Abstract

In this article a simple analytical description of the structural-topological properties of toroidal-lattice communication networks is proposed, which allows accurately estimate the main topological metrics of the network at the stage of its topological synthesis. It is shown, that with an increase in the size of a toroidal-lattice network, the number of possible variants for its construction (configurations) increases rapidly. Therefore, it is necessary to solve the problem of finding the most compact structure in the process of topological synthesis, taking into account restrictions on the topological cost of the network. A method for searching for such a structure is described.

Keywords: *boolean hypercube, communication network, topological metrics, topological synthesis, toroidal-lattice structure*

1. Introduction

In modern multiprocessor computer systems communication networks (CN), whose topologies form a class of toroidal-lattice structures (TLS) – n-dimensional tors based on rectangular n-lattice or hypertors, are widely used [1-6].
The attractiveness of TLS is due to the following properties:
- the configuration of n-lattice topological structures corresponds to the specifics of scientific and technical tasks that require processing of large arrays of various dimensions [2, 4, 6];
- non-univalent n-lattices are easily transformed into univalent n-tors by adding a relatively small number of additional connections to their structure [7, 8];
- TLS provide many opportunities for their optimization (choosing the ratio between the topological cost of the CN and the values of the metrics that characterize its reliability and the maximum delay in the transmission of messages) [7-10];
- the routing of messages in the CN TLS is carried out on the basis of a simple cubic function, which provides the possibility of a simple hardware implementation of an adaptive coordinate-by-order routing algorithm [2, 4, 11].
In modern multiprocessor computer systems, there is an obvious tendency to an increase in the dimension and complexity of the CN structure [1].
Accordingly, obtaining a generalized and formalized description of TLS with the aim of automating their topological synthesis, taking into account the features of various subclasses of these structures, looks quite relevant.
The aim of this work is to obtain the method of search for TLS, the best for the given size of the network and its topological cost. Achieving this aim involves the following tasks:
- obtaining a set of analytical expressions describing the main topological metrics of TLS;
- clarification of the formulation of the problem of synthesizing such structures, taking into account the possibility of optimizing them (achieving the desired ratio between the values of the main topological metrics characterizing the speed, reliability and cost of the network);
- development of a formal description of the method.

2. Statement of the problem of topological synthesis of a communication toroidal-lattice network

The task of synthesizing CN TLS can be formulated as the search for the optimal variant of distribution of a certain number of I connections between N network nodes with a fixed (or bounded above) order of nodes d while maintaining the cubic routing function [7, 8].
The best option is to minimize the maximum diameter D and maximize the width of the bisection B. The maximum diameter value is an estimate of the maximum message transfer delay in the network. The width of the bisection is equal to the minimum number of connections between any two halves of the structure; therefore, it can be used at the topological level to assess the reliability of the network [8]. The network size is most often chosen as $N = 2^n$, which ensures the addressing of network nodes using all possible combinations of an n-bit binary address code. The relationship between N, I, d for TLS, as for all univalent structures, is described by the simple relation $I = N*d/2$ [8]. The value of d determines the topological dimension of the univalent TLS [11, 12], and the value of I is a simple estimate of its topological cost [7, 8].
Note that the basic graphs for constructing TLS are non-univalent rectangular n-lattices, the dimension of which is determined by the minimum order value of their nodes [12, 13].

Obviously, rectangular n-lattices and n-tors can have the same or different number of nodes in each of their dimensions. The first will be called “cubic”, the second, respectively, “non-cubic”.

3. Estimation of metrics of rectangular n-lattices and n-tors on their basis

A set of known expressions [2, 6, 10] for determining metrics of cubic n-lattices of dimension $n = 1\div3$ (the linear structure is considered here as a one-dimensional lattice) is presented in table 1 (rows 1-3).
The size of the n-dimensional cubic lattice here is determined by the parameter $m \geq 3$ (the number of nodes in the edge of the lattice) to the power n.
It is easy to obtain expressions for the metrics of a generalized n-dimensional cubic lattice by using the method of mathematical induction:

$$N = m^n,$$
$$d = n \div 2n,$$
$$I = nm^{n-1}(m-1), \qquad (1)$$
$$D = n(m-1),$$
$$B = m^{n-1} = N/m.$$

Similarly, expressions for the metrics of d-dimensional tors based on cubic lattices of dimension $n = d/2$ were obtained. The initial expressions for TLS of dimension d = 2, 4, 6 are presented in table 2 (rows 1-3) [2, 6, 10].

$$N = m^{d/2},$$
$$I = \frac{d}{2} m^{d/2},$$
$$D = \frac{d}{2}\left[\frac{m}{2}\right], \qquad (2)$$
$$B = 2m^{\frac{d}{2}-1} = 2N/m.$$

It should also be noted, that in the framework of the previously formulated approach, the introduction of toroidal connections into the base n-lattice leads to a doubling of its dimension. However, d-dimensional TLS are often called d/2-dimensional tors [2, 4, 6, 12, 13], which is due to the peculiarities of the visual presentation of these structures and, accordingly, may be the source of some misunderstandings.
If a rectangular n-lattice is “non-cubic” its size can be defined as

$$N = \prod_{i=1}^{n} m_i;\ m_i \geq 3.$$

The set of expressions for the metrics of such n-lattice of dimension n = 1÷3 is presented in table 1 (rows 1, 4, 5).
Expressions for metrics of n-dimensional non-cubic lattice:

$$N = \prod_{i=1}^{n} m_i,$$
$$d = n \div 2n,$$
$$I = \left(n - \sum_{i=1}^{n}\frac{1}{m_i}\right)\prod_{i=1}^{n} m_i = \left(n - \sum_{i=1}^{n}\frac{1}{m_i}\right)N, \qquad (3)$$
$$D = \sum_{i=1}^{n} m_i - n,$$
$$B = \min\left(\frac{N}{m_1},...,\frac{N}{m_n}\right).$$

Accordingly, the expressions for the metrics of the d-dimensional tor based on the n-dimensional non-cubic lattice will be obtained on the basis of the set of expressions presented in rows 1, 4, 5 of table 2:

$$N = \prod_{i=1}^{d/2} m_i;\ m_i \geq 3,$$
$$I = \frac{d}{2}\prod_{i=1}^{d/2} m_i,$$
$$D = \sum_{i=1}^{d/2}[m_i/2], \qquad (4)$$
$$B = 2\min\left(\frac{N}{m_1},...,\frac{N}{m_n}\right).$$

In fig. 1. 2D-torus is shown ($N = 16$). Note that in reality this structure has the dimension $d = 4$.

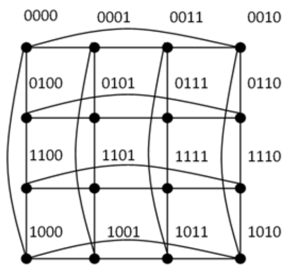

Fig. 1: 2D-tor ($N = 16$)

Table 1: N-lattice metrics (n=1÷3)

№	n	N	I	D	d	B	K
1	1	m	$m-1 = m\left(1-\frac{1}{m}\right)$	m-1	1, 2	1	$1-\frac{1}{m}$
2	2	m^2	$2m(m-1)$	$2(m-1)$	2÷4	M	
3	3	m^3	$3m^2(m-1)$	$3(m-1)$	3÷6	m^2	
4	2	m_1m_2	$m_1m_2\left(2-\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\right)$	m_1+m_2-2	2÷4	$\min(m_1, m_2)$	$1-\frac{\frac{1}{m_1}+\frac{1}{m_2}}{2}$
5	3	$m_1m_2m_3$	$m_1m_2m_3\left(3-\left(\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3}\right)\right)$	$m_1+m_2+m_3-3$	3÷6	$\min(m_1m_2, m_2m_3, m_1m_3)$	$1-\frac{\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3}}{3}$

Table 2: Metrics of d-tors based on rectangular n-lattices (n=1÷3)

№	*d*	*N*	*I*	*D*	*B*
1	2	m	m	$[m/2]$	2
2	4	m^2	$2m^2$	$2[m/2]$	$2m$
3	6	m^3	$3m^3$	$3[m/2]$	$2m^2$
4	4	m_1m_2	$2m_1m_2$	$[m_1/2]+[m_2/2]$	$2\min(m_1, m_2)$
5	6	$m_1m_2m_3$	$3m_1m_2m_3$	$[m_1/2]+[m_2/2]+[m_3/2]$	$2\min(m_1m_2, m_2m_3, m_1m_3)$

From the sets of expressions (4) and (3) it is easy to get (2) and (1) respectively, which confirms their correctness. Thus, the topological metrics of TLS and their basic lattices, cubic and non-cubic, can be accurately estimated in the process of topological synthesis of CN using simple analytical expressions.

4. Boolean hypercube as a toroidal--lattice structure

Boolean hypercube in CN topology is called n-cube, which has only two nodes in each of the *n* edges. Its metrics are described by well-known expressions:

$$N = 2^d \,,$$
$$I = d2^{d-1} \,,$$
$$D = d \,, \qquad (5)$$
$$B = 2^{d-1} = N/2 \,.$$

From comparison of (5) and (2) it can be seen that for boolean hypercube of any even dimension *d*, there exists an equivalent cubic TLS with $m = 4$, since equality $2^d = m^{d/2}$ holds only for the specified value *m.*
Accordingly, for a boolean hypercube of any odd dimension d, there exists an equivalent non-cubic TLS, which has 2 nodes in only one of *d/2* dimensions, and 4 in each of the others. On this basis, boolean hypercubes of various dimensions can be considered as TLS for which the "hypercubicity condition" is satisfied – $d = \log_2 N$.
As an example, in fig. 2 shows a 4D hypercube of size N = 16. It is easy to verify that this structure and the 2D torus shown in Fig. 1. are equivalent.

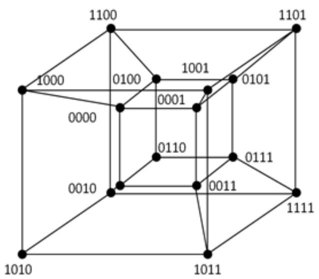

Fig. : 4D hypercube (*N = 16)*

Note that the "hypercubic" method of scaling (increase in size) TLS leads to an increase in the order of all nodes of the structure by one for each doubling of their number [2]. At the same time, the width of the binary address code also increases by one, the topological cost of the network and the hardware cost of routing increase significantly, but the cubic routing rule retains its classic look (there are no forbidden ways to send messages) [11]. In fact, this method requires an increase in the number of ports of the message nodal processors as the number of network nodes increases, which makes it extremely difficult to scale the hypercubic structure.
Scaling homogeneous TLS while maintaining a fixed order of nodes requires only breaking and then reinstalling some part of the network connections.
In this case, topological cost and hardware costs for routing increase much slower [11], however, prohibited message transfer paths appear, and the values of *D, B* metrics are always worse in general than the hypercubic structure of the same size *N*.

5. Calculation of metrics of toroidal-lattice structures and analysis of the obtained results.

Table 3 presents the results of calculating the metrics of the construction options (configurations) of some TLS (*N = 16÷4096*) based on expressions (1-4) for all possible values $\log_2 N \geq d \geq 4.$
From the analysis of the results obtained, the following conclusions can be made:
1. With an increase in the size of the TLS, the number of its possible configurations is rapidly increasing. So, if *N = 16* it can be represented only in a single "hypercubic" form, and if *N = 4096*, then 32 variants of presentation are possible.
2. From the point of view of achieving the best values of *D, B* the "hypercubic" version of its construction is optimal, however, it has the maximum topological value (the values in the rows of table 3 describing such options are underlined).
3. For TLS of any size $2^n \geq 32$, except for the indicated optimal version of construction, there are its "sub-optimal" configurations (the values in the rows of table 3 describing such options are in italics). With a sufficiently large *N*, these configurations provide, in comparison with the "hypercubic", a significant decrease in the order of nodes and the topological cost of the network with a slight increase in the maximum diameter and a two-fold reduction in the bisection width. It should be noted that when decreasing the order of nodes by a certain value, the maximum diameter increases by the same value. For example, the "hypercubic" TLS with a size of *N = 4096* nodes has *D = d = 12*, *B = 2048*, *I = 24576.* The "suboptimal" version of constructing TLS of the same size with *d = 8* (decrease by 4) has *D = 16* (increase by 4), *B = 1024*, *I = 16384*. It should be noted that the topological cost of the second TLS configuration is 1.5 times less than the first.
It should also be noted, that with a sufficiently large size of the network N, there are several possible configurations of TLS that have the same dimensionality and, accordingly, the topological cost. For example, for *N = 4096* and *d = 6*, six variants are possible (see Table 3), of which only one is the best, since it has the minimum value *D* and the maximum value *B*.
Consequently, in the process of network topological synthesis, it is necessary to solve the problem of choosing the best possible configuration of TLS for given values of *N, d*.
It is easy to see that for any given values of *N* and *d* (or limiting the topological cost *I*), the best option for constructing TLS will be a cubic configuration, and in cases where it does not exist, the one that comes closest to the cubic one, that is, as compact as possible. For such a TLS, the sequence of factors mi describing its configuration (see Table 3) has the maximum number of identical values, and the difference between the minimum and maximum factors is the smallest. For example, when *N = 4096* and *d = 9*, out of two configurations (32 * 4 * 4 * 4 * 2 and 8 * 8 * 8 * 4 * 2), having three identical factors, the second one is preferable.

Table 3: The results of the calculation of the metrics of possible configurations of TLS for *N = 16 ÷ 4096*

d	I	Structure	B	D
N=16				
4	32	2^4	8	4
N=32				

5	80	2^5	16	5
4	64	8*4	8	6
N=64				
6	192	2^6	32	6
5	160	8*4*2	16	7
4	128	16*4 8*8	8 16	10 8
N=128				
7	448	2^7	64	7
6	384	8*4*4	32	8
5	320	16*4*2 8*8*2	16 32	11 9
4	256	32*4 16*8	8 16	18 12
N=256				
8	1024	2^8	128	8
7	896	8*4*4*2	64	9
6	768	16*4*4 8*8*4	32 64	12 10
5	640	32*4*2 16*8*2	16 32	19 13
4	512	64*4 32*8 16*16	8 16 32	34 20 16
N=512				
9	2304	2^9	256	9
8	2048	8*4*4*4	128	10
7	1792	16*4*4*2 8*8*4*2	64 128	13 11
6	1536	32*4*4 16*8*4 8*8*8	32 64 128	20 14 12
5	1280	64*4*2 32*8*2 16*16*2	16 32 64	35 21 17
4	1024	128*4 64*8 32*16	8 16 32	66 36 24
N=1024				
10	5120	2^{10}	512	10
9	4608	8*4*4*4*2	256	11
8	4096	16*4*4*4 8*8*4*4	128 256	14 12
7	3584	32*4*4*2 16*8*4*2 8*8*8*2	64 128 256	21 15 13
6	3072	64*4*4 32*8*4 16*16*4 16*8*8	32 64 128 128	36 22 18 16
5	2560	128*4*2 64*8*2 32*16*2	16 32 64	67 37 25
4	2048	256*4 128*8 64*16 32*32	8 16 32 64	130 68 40 32
N=2048				
11	11264	2^{11}	1024	11
10	10240	8*4*4*4*2	512	12
9	9216	16*4*4*4*2 8*8*4*4*2	256 512	15 13
8	8192	32*4*4*4 16*8*4*4 8*8*8*4	128 256 512	22 16 14
7	7168	64*4*4*2 32*8*4*2 16*16*4*2 16*8*8*2	64 128 256 256	37 23 19 17
6	6144	128*4*4 64*8*4 32*16*4 16*16*8 32*8*8	32 64 128 256 128	68 38 26 20 24
5	5120	256*4*2 128*8*2 64*16*2 32*32*2	16 32 64 128	131 69 41 33
4	4096	512*4 256*8 128*16 64*32	8 16 32 64	258 132 72 48
N=4096				
12	24576	2^{12}	2048	12
11	22528	8*4*4*4*4*2	1024	13
10	20480	16*4*4*4*4 8*8*4*4*4	512 1024	16 14
9	18432	32*4*4*4*2 16*8*4*4*2 8*8*8*4*2	256 512 1024	23 17 15
8	16384	64*4*4*4 32*8*4*4 16*16*4*4 16*8*8*4 8*8*8*8	128 256 512 512 1024	38 24 20 18 16
7	14336	128*4*4*2 64*8*4*2 32*16*4*2 16*16*8*2 32*8*8*2	64 128 256 512 256	69 39 27 21 25
6	12288	256*4*4 128*8*4 64*16*4 64*8*8 32*16*8 16*16*16	32 64 128 128 256 512	132 70 42 40 28 24
5	10240	512*4*2 256*8*2 128*16*2 64*32*2	16 32 64 128	259 133 73 49
4	8132	1024*4 512*8 256*16 128*32 64*64	8 16 32 64 128	514 260 136 80 64

6. Method of searching of the most compact toroidal-lattice structure

The number of "suboptimal" network configurations (differing in dimension *d*) also increases with an increase in its size *N*, although much slower than the number of all its possible configurations. So, for example, with *N = 32 (d = 5)* a single suboptimal configuration is possible *(d = 4)*, with *N = 4096* there are already four such configurations differing in dimension *(d = 8 ÷ 11)* and, accordingly, the topological cost. In fig. 3 shows the number of all possible configurations of TLS (K) for a given $N = 2^n$ with $n \leq 12$ and, accordingly, the number of suboptimal configurations (M).

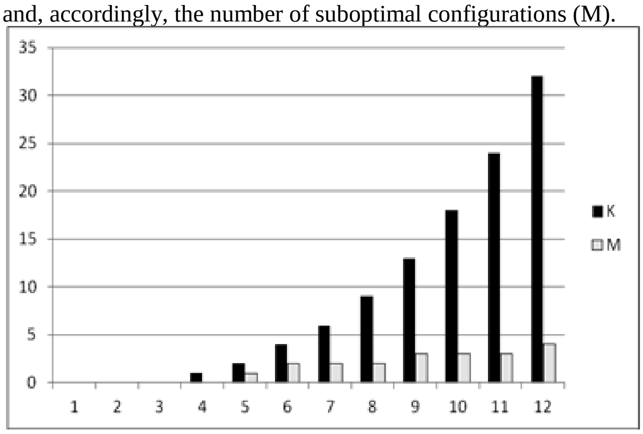

Fig. 3: The number of all possible (K) and suboptimal (M) TLS configurations for a given $N = 2^n$ with $n \leq 12$

To solve the problem of finding the most compact TLS for given values of N and even value of d, the following method is proposed.

1. Find the value $m = \sqrt[d/2]{N}$. If it is integer, the sought-for maximally compact structure is cubic, with the number of nodes in the "edge" m.
2. If the resulting value of m_1 is not an integer, this indicates that the desired structure is not cubic. In this case, m_1 is replaced by the closest value from the 2^n series and is considered the number of nodes in the first edge of the structure.
3. *N* into m_1 is divided to find the number of nodes of the structure smaller (by 2) in dimension, the copying of which is m1 times (with the connection of the corresponding nodes) leads to the initial TLS.
4. In the future the described recurrent procedure for the N / m_i structure, reducing the root degree by 1, d / 2-1 times, is repeated.
5. The last value of m_i by dividing *N* by the product of all previous m_i.

The described procedure, due to the replacement of fractional mi values with "rounding" to the nearest integer 2^n both up and down, is guaranteed to result in a set of factors that differ from each other by no more than two times. For example, for *N = 1024* and *d = 8*, a sequence 4 * 8 * 8 * 4 is obtained, that is, indeed, the description of the most compact TLS (the sequence of obtaining the factors does not matter).
The described procedure is easily modified for TLS of odd dimensionality by simple preparation of the initial data. First, the value of *N* is divided by 2 (the first factor, in the description of the structure, respectively, is equal to 2), the dimension d is reduced by one. Further, the procedure described above is applied.

7. Conclusions

In this article, a simple analytical description of the structural-topological properties of generalized TLS, cubic and non-cubic, was obtained, which allows to accurately estimate the main topological metrics of the CN TLS at the stage of their topological synthesis.
It was shown that the boolean hypercubes of various dimensions can be considered as the most compact TLS of the same dimension, and the "hypercubic" (not necessarily boolean) representation of TLS exists only if the condition is also satisfied $d \geq \log_2 N$.
Also, based on the analysis of the results of calculating the values of the main topological metrics for the versions of constructing some TLS it was demonstrated that in the process of network topological synthesis, it is necessary to solve the problem of finding the best possible TLS configuration, namely, the most compact.
Accordingly, the direction of further research is to develop a method for finding such a configuration, generalized TLS. A method of searching for such a structure, which is formalized in the form of a simple recurrent procedure, is proposed.

References

[1] Cameron Patterson J (2012), *Managing a real-time massively-parallel neural architecture*, School of Computer Science.
[2] Melnyk AO (2011), *Computer Architecture: Scientific Edition: Text-book*, Volyn Regional Printing House.
[3] Orlov SA & Tsylker BY (2011), *Organization ECM and Systems: a textbook for high schools*, Piter, St. Petersburg.
[4] Dally WJ (2004), *Principles and practices of interconnection net-works*, Elsevier.
[5] Andrew S Tanenbaum & Todd Austin (2013), *Structured Computer Organization: translation from English*, Piter, St. Petersburg.
[6] Korneev VV, (1999), *Parallel computing systems*, Knowledge, Moscow.
[7] Kotsis G, (1992), *Interconnection topologies and routing for parallel processing systems*, Wien: ACPC Technical Report Series, ACPC / TR 92-19.
[8] Artamonov HT & Tiurin VD, (1991), *Topology of computer net-works and microprocessor systems*, Radio and communication, Moscow.
[9] Tyrtyshnikov OI, Botvin OO & Senko VV (2015), "The evaluation of asymmetry degree and topological value of static communication networks", *Information Processing Systems*, Vol.1, No.126, pp. 162-165, available online: http://www.hups.mil.gov.ua/periodic-app/article/4300, last visit:15.09.2018.
[10] Tyrtyshnikov OI & Dodukh IV (2014), "The properties of static communication network of torus-cub topologies", *Control, Navigation and Communication Systems. Academic Journal. Poltava: PNTU*, Vol.2, No.30, pp. 60-63, available online: http://www.irbis-nbuv.gov.ua/cgi-bin/irbis_nbuv/cgiirbis_64.exe?C21COM=2&I21DBN=UJRN&P21DBN=UJRN&IMAGE_FILE_DOWNLOAD=1&Image_file_name=PDF/suntz_2014_2_17.pdf, last visit:15.09.2018.
[11] Tyrtyshnikov OI, Kurchanov VM, Mavrina MO & Korzh YM (2017), "Features of hardware routing in the toroidal-latticed communication networks", *Control, Navigation and Communication Systems. Academic Journal. Poltava: PNTU*, Vol.2, No.42, pp.150-153, available online: http://journals.pntu.edu.ua/sunz/article/view/690, last vis-it:15.09.2018.
[12] Tyrtyshnikov OI, Korzh YM & Botvin OO (2016), "Classification of communication networks multiprocessor computer systems which based on the specified concept of dimension", Information Processing Systems, Vol.2, No.139, pp.126–131, available online: http://www.hups.mil.gov.ua/periodic-app/article/16234, last visit:15.09.2018.
[13] Tyrtyshnikov OI, Mavrina MO & Korzh YM (2017), "Practical classification of communication networks for multiprocessor computer systems", *Computer Science and Cybersecurity*, Vol.2, pp.18–23, available online: https://periodicals.karazin.ua/cscs/article/view/8719, last vis-it:15.09.2018.