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Research paper

Invariance of the Equations of the Theory of Penetration with Respect to Galileo's Algebra and its Extensions

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Abstract

In this work, the symmetry properties of the system of equations of the theory of penetration, which describes the adiabatic motion of an inviscid compressible fluid, are investigated. Maximal invariance algebras of a class of systems that describe the adiabatic motion of an inviscid compressible fluid in the absence of mass forces and in their presence are found. The paper shows that for a given system one can observe a similar effect of the absence and presence of axial symmetry the same as for the known equations of mathematical physics, for example, the Schrödinger equation. It was established that in the absence of axial symmetry, this system is invariant with respect to the generalized Galilean algebra AG2 (1, n) with a fixed power nonlinearity, and in the presence of axial symmetry the system is invariant with respect to the generalized Galilean algebra AG2 (1, n-1) with arbitrary power nonlinearity.

Keywords: *Lie algebra, group classification, quasilinear equations, symmetry operator, equations of evolution type, exact solutions*.

1. Introduction

In the study of various phenomena of nature often come to the mathematical models in the form of differential equations. With the emergence and subsequent development of the theory of differential equations, natural science received an effective means of modeling and researching various problems of science and technology.

Phenomena that are studied in hydrodynamics, the theory of elasticity, electrodynamics, the theory of heat conduction, quantum mechanics, atomic physics, etc., are described by equations of mathematical physics. The methods of integrating differential equations began to be intensively developed after the appearance of the "Mathematical Principles of Natural Philosophy" by I. Newton in the process of studying the problems of world wide and the theory of light. The heyday of the methods of classical mathematical physics is associated with the names of J. Lagrange, L. Euler, J.L. d'Alembert, P.S. Laplace, D. Bernoulli, J. Fourier, M.V. Ostrogradsky, A.M. Lyapunov, S. Lee and many other researchers. One of these methods is the Sophus Lie method, which is based on the principle of symmetry. The method is based on finding and applying the operators of the invariance algebra (Lie symmetry) of a differential equation to find its exact solutions. Many researchers used and developed the theory of S. Lee [1-8].

The principles of symmetry play a fundamental role in natural science. The laws of conservation of energy, momentum, angular momentum are the result of homogeneity, isotropy of four-dimensional space - time. In relation to differential equations, symmetry can be considered as a principle, with the help of which only those with wide symmetry are selected from the most different logically acceptable models (equations, relations). This is primarily due to the fact that the basic physical laws, equations of motion, various mathematical models possess explicit or implicit, geometric or non-geometric, local or non-local symmetries. All classical equations of mathematical physics (Newton, Laplace, d'Alembert, Shredin-Ger, Liouville, Hole, Maxwell, etc. equations) are invariant with respect to fairly wide transformation groups. It is this property that distinguishes them from a variety of other differential equations.

The construction of a constructive mathematical apparatus capable of identifying various types of symmetries is one of the most important tasks of the qualitative theory of differential equations. No less important is the task, in a certain sense, the inverse formulated above: for a given group of transformations, construct mathematical models (equations or systems) possessing the specified symmetry. This work is devoted to solving such urgent problems.

2. Main body

Consider the adiabatic motion of an inviscid compressible fluid with axial symmetry in the absence of mass forces. In this case, the equation of motion, non-intervension, and conservation of entropy of a particle have the form ([9,10]):

$$\frac{\partial w_x}{\partial t} + w_x\frac{\partial w_x}{\partial x} + w_y\frac{\partial w_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{\partial w_y}{\partial t} + w_x\frac{\partial w_y}{\partial x} + w_y\frac{\partial w_y}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

$$\frac{\partial \rho}{\partial t} + \rho\left(\frac{\partial w_y}{\partial x} + \frac{\partial w_y}{\partial y}\right) + w_x\frac{\partial \rho}{\partial x} + w_y\frac{\partial \rho}{\partial y} + \rho\frac{w_y}{y} = 0, \quad (1)$$

$$\frac{\partial S}{\partial t}+w_x\frac{\partial S}{\partial x}+w_y\frac{\partial S}{\partial y}=0,$$

where w_x, w_y - particle velocity components, ρ - density, p - pressure, S – is entropy, t – is time, x, y – are spatial variables.

We introduce the potential of the velocity field $w_x=\frac{\partial u}{\partial x}$, $w_y=\frac{\partial u}{\partial y}$. Then the system (1) will be written in this form:

$$\frac{\partial^2 u}{\partial t\partial x}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x},$$
$$\frac{\partial^2 u}{\partial t\partial y}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}=-\frac{1}{\rho}\frac{\partial p}{\partial y},$$
$$\frac{\partial \rho}{\partial t}+\rho\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial u}{\partial x}\frac{\partial \rho}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial \rho}{\partial y}+\rho\frac{\partial u}{y\partial y}=0, \qquad (2)$$
$$\frac{\partial S}{\partial t}+\frac{\partial u}{\partial x}\frac{\partial S}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial S}{\partial y}=0.$$

Assuming that $p=f(\rho)$ and integrate the first two equations of system (2) with respect to x and y, respectively

$l=x_0,\ x=x_1,\ y=r,\ S=v$,

as well as redefine then the system will take the form:

$$u_0+\frac{1}{2}\left(u_1^2+u_r^2\right)=F(\rho),$$
$$v_o+u_1v_1+u_rv_r=0,$$
$$\rho_0+u_1\rho_1+u_r\rho_r+\rho\left(u_{11}+u_{rr}+\frac{1}{r}u_r\right)=0, \qquad (3)$$

where $F=F(\rho)$— smooth function such that $F_\rho=-\rho^{-1}f_\rho$. The three-dimensional Laplace operator in cylindrical coordinates is $\Delta=\partial_{11}+\partial_{rr}+\frac{1}{r}\partial_r+\frac{1}{r^2}\partial_{\theta\theta}$, where $r=\sqrt{x_2^2+x_3^2}$.

In the case when the function has axial (cylindrical) symmetry, then $\Delta_u=u_{11}+u_{rr}+\frac{1}{r}u_r$.

Using these formulas, we write the system (3) for the case of three spatial variables, when u does not possess cylindrical symmetry:

$$u_0+\frac{1}{2}u_au_a=F(\rho),$$
$$v_0+u_av_a=0,$$
$$\rho_0+u_a\rho_a+\rho\Delta u=0,\ a=\overline{1,3}. \qquad (4)$$

Obviously from the system (4) when $u_\theta=0$ the system (3) is obtained.

2.1. Adiabatic motion of an inviscid compressible fluid in the absence of mass forces.

Let us generalize system (3) for the case of an arbitrary number of spatial variables with such a system:

$$u_0+\frac{1}{2}\left(\vec{\nabla}u\right)^2=F(p),$$
$$v_0+\vec{\nabla}u\vec{\nabla}v=0, \qquad (5)$$
$$\rho_0+\vec{\nabla}u\vec{\nabla}\rho+\rho\left(\Delta u+\frac{N}{x_n}u_n\right)=0,$$

where $\vec{\nabla}=(\partial_1,\partial_2,\ldots,\partial_n)$, N - arbitrary constant.

Consider and solve the problem: to investigate the symmetry properties of systems (4) and (5) depending on the type of function $F(\rho)$ in the case of an arbitrary number of spatial variables $\vec{x}\in R_n$.

First consider the case $N=0$.

Theorem 1. The maximum invariance algebra of the system of equations:

$$u_0+\frac{1}{2}\left(\vec{\nabla}u\right)^2=F(p),$$
$$v_0+\vec{\nabla}u\vec{\nabla}v=0, \qquad (6)$$
$$\rho_0+\vec{\nabla}u\vec{\nabla}\rho+\rho\cdot\Delta u=0$$

is an algebra:

1. $A=\langle\partial_\mu=\frac{\partial}{\partial x_\mu},\quad \partial_u=\frac{\partial}{\partial u},\ J_{ab}=x_a\partial_b-x_b\partial_a$
$G_a=x_0\partial_a+x_a\partial_u,\ D=x_\mu\partial_\mu+u\partial_u,\ Q_1=\psi\partial_\upsilon\rangle,$
where $\mu=\overline{0,n},\ a=\overline{1,n},\ b=\overline{1,n}$, if $F(\rho)$ arbitrary smooth function;
2. $A_1=\left\langle A,\quad Q_2=x_0\partial_u+\frac{1}{\lambda}\rho\partial_\rho\right\rangle$, if $F(\rho)=\lambda\ln\rho,\ \lambda\neq0;$
3. $A_2=\left\langle A,\quad D_1=2x_0\partial_0+x_a\partial_a-\frac{2}{m}\rho\partial_\rho\right\rangle$, if $F(\rho)=\lambda\rho^m,\ \lambda\neq0\ m\neq0,\ \frac{2}{n};$
4. $A_3=\langle A,\quad D_2=2x_0\partial_0+x_a\partial_a-n\rho\partial_\rho,$
$\Pi=x_0^2\partial_0+x_0x_a\partial_a-nx_0\rho\partial_\rho$ if $F(\rho)=\lambda\rho^{\frac{2}{n}},\ \lambda\neq0;$
5. $A_4=\langle A_3,\quad Q_3=\psi^2\rho\partial_\rho\rangle$, if $F(\rho)=0,$

where m, n, λ - arbitrary constants, in cases 1 and 5 of the $\psi^1=\psi^1(v),\psi^2=\psi^2(v)$ - arbitrary smooth function.

Consider the case when $N\neq0$. The result of research on the symmetry properties of system (5) is the following theorem.

Theorem 2. The maximum invariance algebra of the system of equations (5) is algebra:

1. $A=\Big\langle\partial_a=\frac{\partial}{\partial x_a},\quad \partial_u=\frac{\partial}{\partial u},\quad J_{ij}=x_i\partial_j-x_j\partial_i,\quad G_i=x_0\partial_i+x_i\partial_u,$
$D=x_a\partial_a+x_n\partial_n+u\partial_u,\quad Q_1=\psi^1\partial_\upsilon\Big\rangle$, where $a=\overline{0,n-1}$, $i=\overline{1,n-1}$, $j=\overline{1,n-1}$, if $F(\rho)$ - arbitrary smooth function;
2. $A_1=\left\langle A,\quad Q_2=x_0\partial_u+\frac{1}{\lambda}\rho\partial_\rho\right\rangle$, if $F(\rho)=\lambda\ln\rho,\ \lambda\neq0;$
3. $A_2=\left\langle A,\quad D_1=2x_0\partial_0+x_i\partial_i+x_n\partial_n-\frac{2}{m}\rho\partial_\rho\right\rangle,$
if $F(\rho)=\lambda\rho^m,\ \lambda\neq0,\ N\neq\frac{2}{m}-n;$

4. $A_3 = \langle A, \quad D_2 = 2x_0\partial_0 + x_i\partial_i + x_n\partial_n - (n+N)\rho\partial_\rho,$

$\Pi = x_0^2\partial_0 + x_0x_i\partial_i + \frac{\vec{x}^2}{2}\partial_u - (n+N)x_0\rho\partial_\rho \rangle,$

if $F(\rho) = \lambda\rho^m, N = \frac{2}{m} - n, \lambda \neq 0;$

5. $A_4 = \langle A_3, \quad Q_3 = \psi^2\rho\partial_\rho \rangle$, if $F(\rho) = 0$.

The proofs of Theorems 1 and 2 will be carried out simultaneously. According to the Lie criterion, acting the second continuation of the infinitesimal operator on each of the equations of system (5), taking into account the transition to the variety of system (5), after splitting by derivatives, we obtain the system of defining equations:

$$\xi^0 = \xi^0(x_0), \quad \xi^a = \xi^a(x), \ a = \overline{1,n},$$
$$\xi^a_b + \xi^b_a = 0, \ b = \overline{1,n}, \ a \neq b, \ \xi^1_1 = \xi^2_2 = \ldots = \xi^n_n,$$
$$\eta^1_\rho = \eta^2_\rho = \eta^1_v = 0, \ n^2 = \eta^2(v), \tag{7}$$

$$\xi^0_0 + \eta^1_u = 2\xi^1_1, \ \eta^1_a = \xi^a_0,$$
$$\eta^0_u + \rho\eta^1_{uu} = 0, \ \rho\eta^0_\rho = \eta^0 = 0, \ \eta^0_u + 2\rho\eta^1_{uu} = 0, \tag{8}$$

$$\eta^0_i + 2\rho\eta^1_{iu} = \rho\left(\Delta\xi^i + \frac{N}{x_n}\xi^i_n\right),$$
$$n^0_n + 2\rho\eta^1_{nu} = \rho\left(\Delta\xi^n - \frac{N}{x_n}\xi^1_1 + \frac{N}{x_n^2}\xi^n\right),$$
$$\eta^0_0 + \eta^0_u F + \rho\left(\Delta\eta^1 + \frac{n}{x_n}\eta^1_n\right) = 0, \tag{9}$$
$$\eta^1_0 + \left(\eta^1_u - \xi^0_0\right)F = \eta^0 F,$$

where $a = \overline{1,n}, b = \overline{1,n}, i = \overline{1,n-1}$. The solution of equations (7) is the functions:

$$\xi^0 = \xi^0(x_0),$$
$$\xi^a = \text{ü}x_a + C_{ab} + d_a, \ C_{ab} + C_{ba} = 0,$$
$$\eta^1 = \left[2\text{ü}(x_0) - \xi^0_0\right]u + \frac{\ddot{\text{ü}}}{2}\vec{x}^2 + d_a(x_0)x_a + \gamma(x_0), \tag{10}$$
$$\eta^2 = \eta^2(v), \ \eta^0 = \Phi(x_0, x_1, \ldots, x_n, v)\rho,$$

where ù, $d_a, \gamma, \Phi -$ arbitrary functions of its arguments, $C_{ab}(a \neq b)$ - arbitrary constant.

Substituting functions (10) into system (8) we obtain:

$$\Phi = \frac{N}{x_n}C_{in}x_i - \frac{N}{x_n}d_n + \varphi(x_0, v), \tag{11}$$

where $\varphi(x_0, v) -$ arbitrary function.

If we substitute (10) and (11) into equation (9), then we obtain the equation:

$$\text{ù}''' = 0, \ \ddot{d}_a = 0, \ \xi^0_0 = 2\text{ü}, \ NC_{in}F = 0,$$
$$\varphi'_{x_0} + (n+N)\text{ü} = 0, \tag{12}$$
$$N\dot{d} = 0, \ \ Nd_n\dot{F} = 0, \ \varphi\rho\dot{F} = 2\left(\text{ü} - \xi^0_0\right)F + \dot{\gamma}.$$

We investigate the last equation (12). Having studied its structure with respect to the function F and the variable ρ, we conclude that there are four nonequivalent cases:

1. $F -$ arbitrary smooth function;
2. $F = \lambda \ln\rho, \ \lambda = const, \ \lambda \neq 0;$
3. $F = \lambda\rho^m, \ \lambda, m = const, \ \lambda, m \neq 0;$
4. $F = 0$.

We consider only cases 1 and 3. The proof for cases 2 and 4 is carried out similarly.

Let F be an arbitrary function. Then equation (12) sets the conditions:

$$\text{ù} = c_1x_0^2 + c_2x_0 + c_3, \ \xi^0 = 2c_1x_0^2 + (2c_2 + c_4)x_0 + c_5,$$
$$d_a = l_ax_0 + k_a, \ 2(n+N)c_1 = 0, \tag{13}$$
$$NC_{in} = 0, \ Nl_n = 0, \ Nk_n = 0, \ \varphi = 0, \ \text{ü} = \xi^0_0, \ \dot{\gamma} = 0.$$

It is obvious that solution (13) depends on the value of the constant N.

Let N = 0. Then from (13) we get:

$$c_1 = 0, \ c_1 = -c_2, \ \gamma = c_6. \tag{14}$$

Substituting (11), (13), (14) into (10), we obtain the coordinates of the infinitesimal operator $\xi^0, \ \xi^1, \ \eta^1, \ \eta^2, \ \eta^6$, that define the algebra A in Theorem 1.

When N ≠ 0. Then from equation (13) we get:

$$C_{in} = 0, \ l_n = k_n = 0, \ c_1 = 0, \ c_4 = -c_2, \ \gamma = c_6. \tag{15}$$

Substituting the functions (11), (13), (15) into equations (10), we obtain the functions $\xi^0, \ \xi^1, \ \eta^1, \ \eta^2, \ \eta^0,$ that define the algebra A in Theorem 2.

Let $F = \lambda\rho^m$ Then equations (12) set the conditions:

$$\text{ù} = c_1x_0^2 + c_2x_0 + c_3, \ \xi^0 = 2c_1x_0^2 + (2c_2 + c_4)x_0 + c_5, \ d_a = l_ax_0 + k_a,$$
$$NC_{in} = 0, \ Nl_n = 0, \ Nk_n = 0, \tag{16}$$

$$c_1\left(n + N - \frac{2}{m}\right) = 0, \quad \varphi = \frac{2}{m}(-2c_1x_0 - c_2 - c_4) \quad \dot{\gamma} = 0.$$

Solution (16) depends on the value of the constant N.

When *N = 0* from (16) we get:

$$c_1\left(n - \frac{2}{m}\right) = 0, \quad \gamma = c_6. \tag{17}$$

Equation (17) defines two cases:

1. When $c_1 = 0$, then substituting functions (11), (16), (17) into (10) we obtain the coordinates of the infinitesimal operator, which define the algebra A_2 in Theorem 1;
2. At $n = \frac{2}{m}$ functions (11), (16), (17) в (16) define the algebra A3 in Theorem 1.

When $N \neq 0$ from (16) we get:

$$c_{in} = 0, \ l_n = k_n = 0, \ \gamma = c_6, \ c_1\left(n + N - \frac{2}{m}\right) = 0. \tag{18}$$

Equation (18) defines two cases:
1. When $c_1 = 0$, then substituting functions (11), (16), (17) into (10) we obtain the ξ^0, ξ^1, η^1, η^2, η^0, which define the algebra A_2 in Theorem 2;
2. At $n+N=\frac{2}{m}$ functions (11), (16), (18) в (10) set algebra A_3 in Theorem 2.
Theorems 1, 2 are proved.
Remarks. From the symmetry properties of the system (6) it appears that in the case when this system has the form:

$$u_0+\frac{1}{2}(\vec{\nabla}u)^2=\lambda\rho^{\frac{2}{n}},\ v_0+\vec{\nabla}u\vec{\nabla}v=0,\ \rho_0+\vec{\nabla}u\vec{\nabla}\rho+\rho\Delta u=0 \qquad (19)$$

it is invariant under the generalized Galilean algebra $AG_2(1,n)$.
From the symmetrical properties of system (5), it follows that in the case when this system has the form:

$$u_0+\frac{1}{2}(\vec{\nabla}u)^2=\lambda\rho^m,\ \lambda,m=const,$$
$$v_0+\vec{\nabla}u\vec{\nabla}v=0, \qquad (20)$$
$$p_0+\vec{\nabla}u\vec{\nabla}\rho+\rho\left(\Delta u\frac{n}{x_n}u_n\right)=0,\ N=\frac{2}{m}-n$$

it is invariant under a generalized Galilean algebra with an arbitrary degree m, but its invariant algebra is a generalized Galilean algebra $AG_2(1,n-1)$.

2.2. Adiabatic motion of an inviscid compressible fluid in the presence of mass forces.

The system of equations describing the adiabatic motion of an inviscid compressible fluid with axial symmetry in the presence of mass forces is similar, as in the previous paragraph, to the following system:

$$u_0+\frac{1}{2}(\vec{\nabla}u)^2=F(\rho),$$
$$v_0+\vec{\nabla}u\vec{\nabla}v=0, \qquad (21)$$
$$\rho_0+\vec{\nabla}u\vec{\nabla}\rho+\rho\left(\Delta u+\frac{N}{x_n}u_n\right)=G(\rho),$$

where $F=F(\rho), G=G(\rho)\neq 0$ arbitrary smooth functions, N - arbitrary constant.
Let us classify the symmetry properties of system (21) depending on the type of the functions F and G. The following statements are true.
Theorem 3. The maximum algebra of invariance of the systems of equations (21) subject to the condition $N=0$ is algebra:
1. $A=\langle\partial_\mu=\frac{\partial}{\partial x_\varphi},\ \partial_u=\frac{\partial}{\partial u},\ J_{ab}=x_a\partial_b-x_b\partial_a,\ G_a=x_0\partial_a+x_a\partial_0,$ $Q_1=\psi\partial_n\rangle$, where $\mu=\overline{0,n},\ a=\overline{1,n},\ b=\overline{1,n}$, if $F(\rho)$, $G(\rho)$ arbitrary functions;
2. $A_1=\langle A,\ \ D_1=x_a\partial_a+2u\partial_u\rangle$, if $F(\rho)=0$, $G(\rho)-$ arbitrary functions;
3. $A_2=\langle A,\ \ Q_2=e^{\lambda_3x_0}(\lambda_1\partial_u+\lambda_3\rho\partial_\rho)\rangle$,
if $F(\rho)=\lambda_1\ln\rho+\lambda_2,\ \ G=\lambda_3\rho\ln\rho+\lambda_4\rho,\ \lambda_1,\ \lambda_3\neq 0$;
4. $A_3\langle A_1,\ \ Q_3=e^{\lambda_3x_0}\varphi\rho\partial_\rho\rangle$,
if $F(p)=0,\ G=\lambda_1\rho\ln\rho+\lambda_2\rho,\ \lambda_2\neq 0$
5. $A_4=\left\langle A,\ \ D_2=x_0\partial_0+x_a\partial_a+\left(u+\frac{\lambda_1x_0}{1-m}\right)\partial_u=\frac{1}{m-1}\rho\partial_\rho\right\rangle$,
if $F(\rho)=\lambda_1\ln\rho,\ G=\lambda_2p^m,\ \lambda_1,\ \ \lambda_2\neq 0,\ m\neq 0;1;$
6. $A_5=\left\langle A,\ \ D_3=2x_0\partial_0+x_a\partial_a-\frac{2}{m}\rho\partial_\rho\right\rangle$,
if $F(\rho)=\lambda_1p^m,\ G=\lambda_2p^{m+1},\ \lambda_1,\ \lambda_2\neq 0,\ m\neq 0,\ \frac{2}{n};$
7. $A_6=\left\langle A, D_4=(1-m_1)x_0\partial_0+\left(1-m_1+\frac{m}{2}\right)x_a\partial_a-(m_1-m-1)u\partial_u+\rho\partial_\rho\right\rangle$,
if $F(\rho)=\lambda_1p^m,\ G=\lambda_2p^{m_1},\ \lambda_1,\ \lambda_2\neq 0,\ m,\ m_1\neq 0,1;$
8. $A_7=\left\langle A_1,\ \ D_5=2x_0\partial_0+x_a\partial_a+\frac{2}{1-m}\rho\partial_\rho\right\rangle$,
if $F(\rho)=0,\ \ G=\lambda p^m,\ \lambda\neq 0,\ \ m\neq 0,1,\frac{2}{n}+1;$
9. $A_8=\left\langle A,\ \ D_6=2x_0\partial_0+x_a\partial_a-n\rho\partial_\rho,\ \Pi=x_0^2\partial_0+x_0\partial_a+\frac{\vec{x}^2}{2}\partial_u-nx_0p\partial_\rho\right\rangle$,
if $F(\rho)=\lambda_1p^{\frac{2}{n}},\ G=\lambda_2p^{\frac{2}{n}+1},\ \lambda_1,\ \ \lambda_2\neq 0;$
10. $A_9=\langle A_1,\ \ D_6,\ \ \Pi\rangle$, if $F(\rho)=0,\ G=\lambda_2p^{\frac{2}{n}+1},\ \lambda\neq 0,$
where $m,\ m_1,\ \lambda,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4$ – arbitrary constants, in cases 1 and 4 $\psi=\psi(v)-$ arbitrary smooth function.
Theorem 4. The maximum invariance algebra of systems of equations (21), provided that $N\neq 0$ it is an algebra:

1. $A=\langle\partial_a=\frac{\partial}{\partial x_a},\ \ \partial_u=\frac{\partial}{\partial u},\ \ J_{ij}=x_i\partial_j-x_j\partial_i,\ \ G_i=x_0\partial_i+x_i\partial_0,$ $Q_1=\psi\partial_v\rangle$, where $a=\overline{0,n-1},\ i=\overline{1,n-1},\ \ j=\overline{1,n-1},$
if $F(\rho)$, $G(\rho)-$ arbitrary smooth functions;
2. $A_1=\langle A,\ \ D_1=x_i\partial_i+x_n\partial_n+2u\partial_u\rangle$, if $F(\rho)=0$, $G(\rho)$ $-$ arbitrary smooth functions;
3. $A_2=\langle A,\ \ Q_2=e^{\lambda_3x_0}(\lambda_1\partial_u+\lambda_3\rho\partial_\rho)\rangle$,
if $F(\rho)=\lambda_1\ln\rho+\lambda_2,\ G=\lambda_3\rho\ln\rho+\lambda_4\rho,\ \ \lambda_1,\lambda_3\neq 0;$
4. $A_3\langle A_1,\ \ Q_3=e^{\lambda_2x_0}\varphi\rho\partial_\rho\rangle$, if
$F(p)=0,\ G=\lambda_1\rho\ln\rho+\lambda_2\rho,\ \lambda_2\neq 0$
5. $A_4=\left\langle A,\ \ D_2=x_0\partial_0+x_i\partial_i+x_n\partial_n+\left(u+\frac{\lambda_1x_0}{1-m}\right)\partial_u=\frac{1}{m-1}\rho\partial_\rho\right\rangle$,
if $F(\rho)=\lambda_1\ln\rho,\ G=\lambda_2p^m,\ \ \lambda_1,\lambda_2\neq 0,\ m\neq 0;1;$
6. $A_5=\left\langle A,\ \ D_3=2x_0\partial_0+x_i\partial_i+x_n\partial_n-\frac{2}{m}\rho\partial_\rho\right\rangle$,
if $F(\rho)=\lambda_1p^m,\ \ G=\lambda_2p^{m+1},\ \lambda_1,\lambda_2\neq 0,\ m\neq 0,\frac{2}{n};$
7. $A_6=\langle A,\ \ D_4=(1-m_1)x_0\partial_0+\left(1-m_1+\frac{m}{2}\right)(x_i\partial_i+x_n\partial_n)-$ $-(m_1-m-1)u\partial_u+\rho\partial_\rho\rangle$,
if $F(\rho)=\lambda_1p^m,\ \ G=\lambda_2p^{m_1},\ \lambda_2\neq 0,\ m,m_1\neq 0,1;$
8. $A_7=\left\langle A_1,\ \ D_5=2x_0\partial_0+x_i\partial_i+x_n\partial_n+\frac{2}{1-m}\rho\partial_\rho\right\rangle$,
if $F(\rho)=0,\ G(\rho)=\lambda p^m,\ \lambda\neq 0,\ \ m\neq 0,1,\frac{2}{n+N}+1;$

9. $A_8 = \langle A, \quad D_6 = 2x_0\partial_0 + x_i\partial_i + x_n\partial_n - (n+N)\rho\partial_\rho,$

$\Pi = x_0^2\partial_0 + x_0(x_a\partial_a + x_n\partial_n) + \frac{\vec{x}^2}{2}\partial_u - (n+N)x_0\, p\partial_\rho \rangle,$

if $F(\rho) = \lambda_1 p^m,\ G = \lambda_2 p^{m+1},\ N = \frac{2}{m} - n,\ \lambda_1, \lambda_2 \neq 0;$

10. $A_9 = \langle A_1, \quad D_6, \quad \Pi \rangle$, if

$F(\rho) = 0, \quad G = \lambda p^{m+1},\ N = \frac{2}{m} - n,\ \lambda \neq 0,$

where $m,\ m_1,\ \lambda,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4$ – arbitrary constants, in cases 1 and 4 $\psi = \psi(v)$ — arbitrary smooth function.

The proofs of Theorems 3 and 4 will be carried out simultaneously. The system of defining equations for system (21) will consist of equations (7), (8) and equations:

$$\eta_0^1 + (\eta_u^1 - \xi_0^0)F = \eta^0 F,$$
$$\eta_0^0 + \eta_u^0 G + \rho\left(\Delta\eta^1 + \frac{N}{x_n}\eta_n^1\right) + \left(\eta_u^1 + \frac{\eta^0}{\rho} - 2\xi_1^1\right)G = \eta^0 G. \tag{22}$$

As shown in the proof of Theorems 1 and 2 by solving equations (22) there are functions:

$$\begin{aligned}
&\xi^0 = 2c_1x_0^2 + (2c_2 + c_4)x_0 + c_5,\\
&\xi^a = (2c_1x_0 + c_2)x_a + C_{ab}x_b + l_ax_0 + k_a,\ C_{ab} + C_{ba} = 0,\\
&\eta^1 = -c_4u + c_1\vec{x}^2 + l_ax_a + \gamma(x_0),\\
&\eta^2 = \eta^2(v),\\
&\eta^0 = \left[\frac{N}{x_n}C_{in}x_i - \frac{N}{x_n}d_n + \varphi(x_0, v)\right]\rho,
\end{aligned} \tag{23}$$

where $c_1, c_2, c_3, c_4, c_5, C_{ab}, (a \neq b), l_a, k_a$ – arbitrary constants, γ, η^2, φ – arbitrary functions of their arguments.

If we substitute (23) into equation (22). then we get the conditions:

$$\begin{aligned}
&Nl_n\dot{F} = 0,\ Nk_n\dot{F} = 0,\ \varphi\rho\dot{F} = 2(2c_1x_0 + c_2 + c_4)F + \dot{\gamma},\\
&NC_{in}(G - \rho\dot{G}) = 0,\ Nl_n(G - \rho\dot{G}) = 0,\ Nk_n(G - \rho\dot{G}) = 0,\\
&\varphi'_{x_0}\rho + 2(n+N)c_1\rho + (-Ac_1x_0 - 2c_2 - c_4 + \varphi)G = \varphi\rho\dot{G}.
\end{aligned} \tag{24}$$

The second equation of system (24) was investigated in the proof of Theorems 1, 2, where it was shown that the symmetry of the original system differs significantly only if:

1. F – arbitrary smooth function.
2. $F = \lambda\ln\rho, \quad \lambda = const, \quad \lambda \neq 0.$
3. $F = \lambda\rho^m, \quad \lambda,\ m = const, \quad \lambda,\ m \neq 0.$
4. $F = 0$.

We consider only cases 1 and 3. The proof of cases 2 and 4 is carried out similarly.

Let F be an arbitrary function. Then equation (24) set the conditions:

$$\begin{aligned}
&Nl_n = Nk_n = \varphi = c_1 = c_2 + c4 = \dot{\gamma} = 0,\\
&NC_{in}(G - \rho\dot{G}) = 0, \quad c_2G = 0.
\end{aligned} \tag{25}$$

It is obvious that solution (25) depends on the value of the constant N.

Let be $N = 0$. Since the case $G = 0$ considered in Theorems 1 and 2, in what follows, we assume that. $G \neq 0$ .Then from (25) we get:

$$c_1 = c_2 = c_4 = \varphi = 0, \quad \gamma = c_6, \tag{26}$$

substituting N = 0 and (26) in (23), get the coordinates of the infinitesimal operator $\xi^0, \xi^1, \eta^1, \eta^2, \eta^0$, that defines the algebra A3 in Theorem 3.

At $N \neq 0$ then from (25) it follows that:

$$l_n = k_n = \varphi = c_1 = c_2 = 0, \gamma = c_6,\ C_{in}(G - \rho\dot{G}) = 0. \tag{27}$$

When solving the second equation (27) there are two cases:

1. $C_{in} = 0$, and function G is arbitrary. Then using (27), will get $\xi^0, \xi^1, \eta^1, \eta^2, \eta^0$, defining algebra A in Theorem 4

2. $G - \rho\dot{G} = 0$, When $G = \lambda\rho$. In this case, the replacement $\rho = e^{\lambda x_0}\rho$, system (21) reduces to system (5), whose symmetry properties are investigated in Theorem 2.

Let $F = \lambda\rho^m,\ \lambda, m \neq 0$. Then equation (24) sets the following conditions:

$$\begin{aligned}
&Nl_n = Nk_n = \dot{\gamma} = 0,\ NC_{in}(G - \rho\dot{G}) = 0,\\
&Nl_n(G - \rho\dot{G}) = 0,\ Nk_n(G - \rho\dot{G}) = 0,\\
&\varphi = \frac{2}{m}(-2c_1x_0 - c_2 - c_4),\\
&c_1\left[\frac{1}{m}\rho\dot{G} - \left(\frac{1}{m} + 1\right)G\right] = 0,\\
&\frac{2}{m}(c_2 + c_4)\rho\dot{G} = \left[\left(\frac{2}{m} + 2\right)c_2 + \left(\frac{2}{m} + 1\right)c_4\right]G - 2c_1\left(n + N - \frac{2}{m}\right)\rho.
\end{aligned} \tag{28}$$

We investigate the structure of the last equation (28) with respect to the function G and the variable. Depending on the relationship between the coefficients k_1, k_2, k_3 of the structural equation $k_1\rho\dot{G} = k_2G + k_3\rho$, we get three nonequivalent cases:

1. G — довільна функція
2. $G = \lambda_1\rho\ln\rho + \lambda_2\rho, \quad \lambda_1, \lambda_2 = const, \lambda_1 \neq 0$
3. $G = \lambda_1\rho^{m_1} + \lambda_2\rho,\ \lambda_1, \lambda_2, m_1 = const, \lambda_1 \neq 0, m \neq 0, 1.$

Consider each of the cases obtained.

When G is an arbitrary function, with (28) we get the following conditions:

1. $N = 0,\ c_1 = c_2 = c_4 = 0, \gamma = c_6, \varphi = 0;$
2. $N = 0, l_n = k_n = 0, C_{in} = 0, c_1 = c_2 = c_4 = 0, \gamma = c_6, \varphi = 0;$

after substitution of which in (4.23) we obtain the coordinates of the infinitesimal operator $\xi^0, \xi^1, \eta^1, \eta^2, \eta^0$ defining algebras A in Theorems 3 and 4, respectively.

Let $G = \lambda_1\rho\ln\rho + \lambda_2\rho, \lambda_1 \neq 0.$ When $\lambda_1 \neq 0$ system (21) reduce to replacement $\rho = e^{\lambda_1 x_0}\rho$ to system (5), whose symmetry properties are investigated in Theorems 1 2.

After substituting the function G into equation (28), we have the following conditions:

$$Nl_n = Nk_n = 0, NC_{in} = 0, \gamma = c_6, c_1 = c_4 = c_2 = 0. \tag{29}$$

Solving. (29) depending on the value of N, we obtain two cases:

1. $c_1 = c_2 = c_4 = 0, \gamma = c_6, \varphi = 0$, if $N = 0;$
2. $c_1 = c_2 = c_4 = 0 = l_n = k_{n=}\varphi = 0, \gamma = c_6, C_{in} = 0$, if $N \neq 0.$

After substitution of which into (23), we obtain the coordinates of the infinitesimal operator $\xi^0, \xi^1, \eta^1, \eta^2, \eta^0$, defining algebras in Theorems 3 and 4, respectively.

Consider the case when $G = \lambda_1 \rho \ln \rho + \lambda_2 \rho, \lambda_1 \neq 0, m_1 \neq 0, 1.$
At $\lambda_1 = 0$ and $m_1 = 0,$ or $m_1 = 1$ system (21) is locally equivalent to system (5), whose symmetrical properties are investigated in Theorems 1, 2.
Substituting the function G in equation (28), we get:

$$
\begin{aligned}
& Nl_n = Nk_n = 0, NC_{in}(1-m_1) = 0, \gamma = c_6, \\
& Nl_n(1-m_1) = 0, \ Nk_n(1-m_1) = 0, \\
& \varphi = \frac{2}{m}(-2c_1x_0 - c_2 - c_4), \\
& c_1(m_1 - m - 1) = 0, \lambda_2 = 0, \\
& 2(m_1 - m - 1)c_2 + (2m_1 - m - 2)c_4 = 0, \ c_1\left(n + N - \frac{2}{m}\right) = 0 .
\end{aligned} \tag{30}
$$

When solving (30), there are two different cases:

$$
\begin{aligned}
& c_1 = 0, Nl_n = Nk_n = 0, \ NC_{in}(1-m_1) = 0, \gamma = c_6, \ Nl_n(1-m_1) = 0, \\
& Nk_n(1-m_1) = 0, \ \varphi = \frac{2}{m}(-c_2 - c_4), \lambda_2 = 0, \\
& 2(m_1 - m - 1)c_2 + (2m_1 - m - 2)c_4 = 0
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
& m = \frac{2}{n+N}, \ m_1 = m+1, \lambda_2 = 0, c_1 = 0, \\
& Nl_n = Nk_n = 0, \ NC_{in} = 0, \gamma = c_6, \ Nl_n = 0, \ \ Nk_n = 0, \\
& \varphi = (n+N)(= 2c_1x_0 - c_2 - c_4).
\end{aligned} \tag{32}
$$

When solving equations (31) and (32) depending on the values N, m_1 and m, three more ramifications arise:

1. If $m_1 = m+1$.

$$c_1 = 0, \gamma = c_6, \varphi - \frac{2}{m}(-c_2 - c_4), \lambda_2 = 0, \ c_4, \ \text{ when } \ N = 0; \tag{33}$$

$$c_1 = 0, l_n = k_n = 0, C_{in} = 0, \gamma = c_6, \ \varphi = \frac{2}{m}(-c_2 - c_4), \lambda_2 = 0, \tag{34}$$

$c_4 = 0,$ when $N \neq 0.$

When to substitute (33) and (34) into (23), we obtain the coordinates of the infinitesimal operators that define algebras A_5 in Theorems 3 and 4, respectively;

2. If $m_1 \neq m+1$. When $c_1 = 0, \gamma = c_6,$

$$
\begin{aligned}
& \varphi = \frac{2}{m}(-c_2 - c_4), \lambda_2 = 0, \\
& c_2 = \frac{m + 2 - 2m_1}{2(m_1 - m - 1)}c_4, \ \text{ at } \ N = 0
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
& c_1 = 0, \ l_n = k_n = 0, \ \ C_{in} = 0, \ \gamma = c_6, \\
& \varphi = \frac{2}{m}(-c_2 - c_4), \ \lambda_2 = 0, \\
& c_2 = \frac{2m_1 - m - 2}{2(m_1 - m - 1)}c_4, \ \text{ at } \ N \neq 0
\end{aligned} \tag{36}
$$

Then, after substituting (35) and (36) into (23), we obtain the coordinates of the infinitesimal operator, which define the algebras A_6 in Theorems 3 and 4, respectively:

3. At $m = \dfrac{2}{n+N}, m_1 = m+1$ we get: (37)

$\lambda_2 = 0, c_4 = 0, \gamma = c_6, \ \varphi = n(-2c_1x_0 - c_2 - c_4),$ if $N = 0$

and

$$
\begin{aligned}
& \lambda_2 = 0, \ \ c_4 = 0, \ \ l_n = k_n = 0, \ \ C_{in} = 0, \ \gamma = c_6, \\
& \varphi = (n+N)(-2c_1x_0 - c_2 - c_4), \text{if } \ N \neq 0 .
\end{aligned} \tag{38}
$$

Substituting (37) and. (38) in (23) we obtain the coordinates of the infinitesimal operator that define the algebras in Theorems 3 and 4, respectively.
Theorems 3, 4 are proved.

Remarks. The system of equations (21) under the condition $N = 0$ is invariant with respect to the generalized Galilean algebra, if it has the form:

$$
\begin{aligned}
& u_0 + \frac{1}{2}(\vec{\nabla} u)^2 = \lambda_1 \rho^{\frac{2}{n}}, \\
& v_0 + \vec{\nabla} u \vec{\nabla} v = 0, \\
& \rho_0 + \vec{\nabla} u \vec{\nabla} \rho + \rho \Delta u = \lambda_2 \rho^{\frac{2}{n}+1},
\end{aligned} \tag{39}
$$

where λ_1, λ_2 - arbitrary constants.
In the case when $N \neq 0$, system (21) is invariant with respect to a generalized Galilean algebra $AG_2(1, n-1)$ with an arbitrary power nonlinearity:

$$
\begin{aligned}
& u_0 + \frac{1}{2}(\vec{\nabla} u)^2 = \lambda_1 \rho^m, \\
& v_0 + \vec{\nabla} u \vec{\nabla} v = 0, \\
& \rho_0 + \vec{\nabla} u \vec{\nabla} \rho + \rho\left(\Delta u + \frac{N}{x_n}u_n\right) = \lambda_2 \rho^{m+1},
\end{aligned} \tag{40}
$$

where $\lambda_1, \ \lambda_2, \ m, \ N -$ arbitrary constants, where $N = \dfrac{2}{m} - n,$ but the dimension of this algebra is one less than the dimension of the invariance algebra of system (39).

3. Conclusion

Systems of equations (6) and (21) describe the adiabatic motion of an inviscid compressible fluid and are widely used in penetration theory. In this paper, for the first time, the maximal algebra of invariance of systems that describe adiabatic motion in the absence of mass forces, as well as in their presence, is found. Nonlinearities are found at which these systems are invariant with respect to the generalized Galilean algebra in cases with and without axial symmetry.
Note that the results obtained echo the known results for the classical equations of mathematical physics. Namely, for example: [2, 11] nonlinear wave equations $\Box u = F(x,u)$, [3,12] nonlinear Schrödinger equations $i\psi_0 + \Delta\psi = F(x,\psi)$. These equations contain an arbitrary power nonlinearity, but their maximal invariance algebras have a dimension one less than the equations themselves. In our work, it is shown that a similar situation is observed for systems (6) and (21).

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