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Research paper

Common Fixed Point Theorems in Bipolar Metric Spaces with Applications to Integral Equations

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Abstract

This paper establishes the existence of coincidence fixed-point and common fixed-point results for two mappings in a complete bipolar metric spaces. Some interesting consequences of our results is achieved. Finally, an illustration which presents the applicability of the results is achieved.

Keywords: *Bipolar metric space; common fixed point; completeness; coincidence point; Covariant and contravariant maps; weakly compatible.*

1. Introduction

In 1922, S. Banach, [4] made the introduction to the concept of Banach contraction principle. It is considered as the most fundamental tool in non-linear analysis. It explains that in complete metric spaces, each contractive mapping has a solitary fixed point. It has been extended and generalization of various types of metric spaces (see [6] - [9], [13]). Jungck [11] has introduced the concept of common fixed point in metric spaces for commuting mappings in 1966. Afterwards Jungck [12] initiated concept of compatibility and established some results. Subsequently to improve many authors have established common and coincidence fixed point results for mappings (see [3], [5], [10]) and reference therein.

Very recently, Mutlu and Gürdal [2] introduced notion of bipolar metric spaces in 2016. Also, they investigated some fixed and coupled fixed point results on this space (see, [1] [2]) and reference therein.

In this paper, we will continue to study fixed points in the frame of bipolar metric-spaces. More squarely, some common fixed-point results for two covariant and contravariant mappings under various contractive conditions will be established. We have illustrated the validity and effectiveness of the hypotheses of the results. The present results extends and improves the concepts in some of the recent literatures [2].

Definition 1.1: [2] Let U and V be a two non-empty sets. Suppose d: U×V →[0,∞) be a mapping satisfying the below properties:

(B_1) If d (u, v) = 0, then u=v for all (u, v) ∈ U×V,

(B_2) If u = v, then d (u, v) = 0, for all (u, v) ∈ U×V,

(B_3) If d (u, v) =d (v, u), for all u, v ∈ U∩V

(B_4) If d (u_1, v_2) ≤ d (u_1, v_1) + d (u_2, v_1) + d (u_2, v_2) for all u_1, u_2 ∈U, and v_1, v_2 ∈V.

Then the mapping d is termed as Bipolar-metric of the pair (U, V) and the triple (U, V, d) is termed as Bipolar-metric space.

Example 1.2 ([2]): Let U= (1, ∞) and V= [-1, 1]. Define d: U×V →[0,∞) as d (a, b) =$|a^2 - b^2|$, for all (a, b) ∈ U×V. Then the triple (U, V, d) is a disjoint Bipolar-metric space.

Definition 1.3: [2] Assume (U_1, V_1) and (U_2, V_2) as two pairs of sets and a function as F: $U_1 \cup V_1 \rightrightarrows U_2 \cup V_2$ is said to be a covariant map. If F $(U_1) \subseteq U_2$ and F $(V_1) \subseteq V_2$ and denote this with S: $(U_1, V_1) \rightrightarrows (U_2, V_2)$. And the mapping S: $U_1 \cup V_1 \rightleftarrows U_2 \cup V_2$ is said to be a contravariant map. If F $(U_1) \subseteq V_2$, and F $(V_1) \subseteq U_2$, and write F: $(U_1, V_1) \rightleftarrows (U_2, V_2)$. In particular, if d_1 and d_2 are bipolar metric on (U_1, V_1) and (U_2, V_2), respectively, we sometimes use the notation F: $(U_1, V_1, d_1) \rightrightarrows (U_2, V_2, d_2)$ and F: $(U_1, V_1, d_1) \rightleftarrows (U_2, V_2, d_2)$.

Definition 1.4: [2] Assume (U, V, d) as a bipolar metric space. A point v ∈U ∪V is termed as a left point if v ∈ U, a right point if v ∈ V and a central point if both. Similarly, a sequence $\{u_n\}$ on the set U and a sequence $\{v_n\}$ on the set V are called a left sequence and right sequence respectively. In a bipolar metric space, sequence is the simple term for a left or right sequence. A sequence $\{v_n\}$ is considered convergent to a point v, if and only if $\{v_n\}$ is the left sequence, v is the right point and $\lim_{n\to\infty} d(v_n, v) = 0$; or $\{v_n\}$ is a right sequence, v is a left point and $\lim_{n\to\infty} d(v, v_n) = 0$. A bi-sequence $(\{u_n\}, \{v_n\})$ on (U, V, d) is a sequence on the set U ×V. If the sequence $\{u_n\}$ and $\{v_n\}$ are convergent, then the bi-sequence $(\{u_n\}, \{v_n\})$ is said to be convergent. $(\{u_n\}, \{v_n\})$ is Cauchy sequence, if $\lim_{n\to\infty} d(u_n, v_n) = 0$. In a bipolar metric space, every convergent Cauchy bi-sequence is bi-convergent. A bipolar metric space is called complete, if every Cauchy bi-sequence is convergent hence bi-convergent.

Definition 1.5: [2] Let (U_1, V_1, d_1) and (U_2, V_2, d_2) be a bipolar metric spaces.

(i) A map F is called continuous, if it left continuous at each point u∈ U_1 and right continuous at each point v∈ V_1

(ii) A contravariant map F: $(U_1, V_1, d_1) \rightleftarrows (U_2, V_2, d_2)$ is continuous if and only if it is continuous as a covariant map F: $(U_1, V_1, d_1) \rightrightarrows (U_2, V_2, d_2)$.

It can be seen from the definition (1.4) that a covariant or a contravariant map F: $(\boldsymbol{U_1}, \boldsymbol{V_1}, \boldsymbol{d_1}) \rightrightarrows (\boldsymbol{U_2}, \boldsymbol{V_2}, \boldsymbol{d_2})$ is continuous if and only if $(\boldsymbol{u_n}) \to \boldsymbol{v}$ on $(\boldsymbol{U_1}, \boldsymbol{V_1}, \boldsymbol{d_1})$ implies F$((\boldsymbol{u_n})) \to \boldsymbol{F(v)}$ on $(\boldsymbol{U_2}, \boldsymbol{V_2}, \boldsymbol{d_2})$

2. Main Results

In this section, we will give some common fixed-point theorems for two covariant and contravariant mappings satisfying various contractive conditions in complete bipolar metric spaces.

Definition 2.1: The mappings F and G on bipolar metric-space (U, V, d) are said to be compatible, if for arbitrary bisequences $(\{a_n\}, \{b_n\}) \subseteq (U, V)$, such that $\lim_{n\to\infty} Fa_n = \lim_{n\to\infty} G\,b_n = v \in U \cup V$, then d $(GFa_n, FG\,b_n) \to 0$ as $n \to \infty$.

Definition 2.2: Assume that F and G are two covariant or contravariant mappings of the set U∪ V

(i) If v= Fv = Gv for some v∈U∪ V, then v is named as a common fixed point of F and G

(ii) If u=Fv=Gv for some u∈U∪ V, then v is considered as a coincidence point of F and G, and u is termed as the point of coincidence of F and G.

(iii) If F and G commute at all of their coincidence points; i.e. FGv = GFv for all v ∈{v∈U∪ V: Fv=Gv}, then F and G are called weakly Compatible.

In the metric-space, if the mapping F and G are compatible, then they are weakly compatible, while the converse becomes untrue ([12]). The same comes for the bipolar metric spaces.

Lemma 2.3: If the mapping F and G on the bipolar metric space (U, V, d) are compatible, then they are weakly compatible.

Proof: Let *F v = Gv* for some $v \in U \cup V$. It is sufficient to show that *FGv = GF v*.

Putting $\alpha_n \equiv v$ and $\beta_n \equiv v$ for every $n \in N$, we have

$\lim_{n\to\infty} F\alpha_n = \lim_{n\to\infty} G\beta_n$

and then, since *F* and *G* are compatible,

we have $d\,(GF\alpha_n; FG\beta_n) \to 0$ as n→ ∞.

Hence *d* (*GF v; FGv*) = 0, which means *GF v = FGv*.

But, the converse does not hold. For example, Let *U* = (0*;*∞) and *V* = [-1*;* 1]. Define $d: U \times V \to [0;\infty)$ as $d\,(v;\, u) = |v^2 - u^2|$, for all (*v; u*) ∈ (*U; V*). Then (*U; V; d*) is a Bipolar-metric space. Set

$$Fv = \begin{cases} v, & if\ v \in (0, \frac{3}{2}\,] \\ \frac{1}{3}, & if\ v \in (\frac{3}{2}, \infty) \end{cases} \quad \text{and}$$

$$Gv = \begin{cases} 1-2v, & if\ v \in [0, \frac{2}{5}\,] \\ \frac{v}{3}, & if\ v \in [-1, 0) \cup (\frac{2}{5}, 1] \end{cases}$$

Firstly, we can calculate that set of their coincidence point is singleton set $\{\frac{1}{3}\}$, and then we have *F* and *G* are commute at this point. Hence *F* and *G* are weakly compatible. However, we can prove they are not compatible. In this purpose, we construct a bisequence $(\{\alpha_n\}, (\{\beta_n\}) \subseteq$ (U, V) such that $\alpha_n = 1 - \frac{1}{n} \in U$ and $\beta_n = \frac{1}{n} \in V$ for n∈ *N* with n≥3. In this case, we have $F\alpha_n = 1 - \frac{1}{n}$ and $G\beta_n = 1 - \frac{2}{n}$. Then $\lim_{n\to\infty} F\alpha_n = \lim_{n\to\infty} G\beta_n = 1$ In fact we have

d $(F\alpha_n, 1) =$ d$(1-\frac{1}{n}, 1) = \left|(1-\frac{1}{n})^2 - 1^2\right| = \left|1 + \frac{1}{n^2} - \frac{2}{n} - 1\right| \to 0$ as n→ ∞. and

d$(1, G\beta_n) =$d$(1, 1-\frac{2}{n}) = \left|1^2 - (1-\frac{2}{n})^2\right| = \left|1 - 1 - \frac{4}{n^2} + \frac{4}{n}\right| \to 0$ as n→ ∞.

But d$(G\,F\alpha_n, F\,G\beta_n) = d\left(G\left(1-\frac{1}{n}\right), F(1-\frac{2}{n})\right)$

$= d\left(\frac{1-n}{n}, \frac{n-2}{n}\right) = \left|(\frac{n-1}{3n})^2 - (\frac{n-2}{n})^2\right|$

$= \left|\frac{1}{9n^2}(1+n^2-2n) - \frac{1}{n^2}(n^2+4-4n)\right| \to \frac{8}{9}$ as $n \to \infty$. Which means that d$(G\,F\alpha_n, F\,G\beta_n) \nrightarrow 0$.

Lemma 2.4: If the mappings F and G be weakly compatible mappings of a set U∪V. If F and G have a unique coincidence point, then F and G have a unique common fixed point.

Proof: Since *u = F v = Gv* for some *v; u* ∈ *U*∪ *V* and *F* and *G* are weakly compatible, we have *Fu = FGv = GF v = Gu* is a point of coincidence of *F* and *G*. But *u* is the only coincidence point of *F* and *G*, so *u = Fu = Gu*. Moreover, if $v' = Fv' = Gv'$, then v' is coincidence point of *F* and *G*, and hence $v = v'$ by the uniqueness. Thus *v* is a unique common fixed point of *F* and *G*.

2.1. Common fixed point theorems on covariant maps

Theorem 2.5: Assume (U, V, d) be a complete bipolar metric spaces and given contractions, F, G: (U, V, d) ⇉ (U, V, d) satisfies d(Fu, Gv)≤ μ d(u, v) for all (u, v)∈ U× V , where $\mu \in (0, 1)$. (1)

Then the mappings F, G: U∪ V →U∪V have a unique common fixed point.

Proof: Let $\alpha_0 \in$U and $\beta_0 \in$V and we construct a bisequences $(\{\alpha_n\}, \{\beta_n\}) \subseteq$ (U, V) by the way: F $\alpha_{2n} = \alpha_{2n+1}$, G $\alpha_{2n+1} = \alpha_{2n+2}$ and F$\beta_{2n} = \beta_{2n+1}$ G$\beta_{2n+1} = \beta_{2n+2}$, for all n∈ N.

Let $\mu \in (0, 1)$, put K=d (α_0, β_0) + d(α_0, β_1) and $S_n = \frac{\mu^{n+1}}{1-\mu}$ K . Then for each positive integer n and l from (1) , we have

$d(\alpha_{2n+1}, \beta_{2n+2}) = d(F\alpha_{2n}, G\beta_{2n+1})$
$\leq \mu\, d(\alpha_{2n}, \beta_{2n+1})$
$\leq \mu^{2n+1}\, d\,(\alpha_0, \beta_1)$

and also

$d(\alpha_{2n+1}, \beta_{2n+1}) = d(F\alpha_{2n}, G\beta_{2n})$
$\leq \mu\, d(\alpha_{2n}, \beta_{2n})$

$\leq \mu^{2n+1}\, d\,(\alpha_0, \beta_0)$

Therefore,

$d(\alpha_{2n+1}, \beta_{2n+2}) + d(\alpha_{2n+1}, \beta_{2n+1})$
$\leq \mu^{2n+1}\,(d\,(\alpha_0, \beta_1) + d\,(\alpha_0, \beta_0)) \leq \mu^{2n+1}$ K

Now we can obtain that for any n ∈ N

$d(\alpha_n, \beta_{n+1}) + d(\alpha_n, \beta_n) \leq \mu^{n+1}\,(d\,(\alpha_0, \beta_1) + d\,(\alpha_0, \beta_0))$
$\leq \mu^{n+1}$ K

for all n, l ∈ *N* with n> l

$d(\alpha_{n+l}, \beta_n) \leq d(\alpha_{n+l}, \beta_{n+1}) + d(\alpha_n, \beta_{n+1}) + d(\alpha_n, \beta_n)$
$\leq d(\alpha_{n+l}, \beta_{n+1}) + \mu^{n+1}$ K
$\leq d(\alpha_{n+l}, \beta_{n+2}) + d(\alpha_{n+1}, \beta_{n+2}) + d(\alpha_{n+1}, \beta_{n+1})$
$+\mu^{n+1}$ K
$\leq d(\alpha_{n+l}, \beta_{n+2}) + (\mu^{n+1} + \mu^{n+2})$K
$\leq d(\alpha_{n+l}, \beta_{n+l}) + (\mu^{n+l} + \cdots \ldots + \mu^{n+2} + \mu^{n+1})$K
$\leq (\mu^{n+l+1} + \mu^{n+l} + \cdots \ldots + \mu^{n+2} + \mu^{n+1})$K
$= \frac{\mu^{n+l}}{l-\mu}$ K $= S_n$

And similarly, $d(\alpha_n, \beta_{n+l}) \leq S_n$.

Let $\epsilon > 0$ and $0 < \mu < l$, there exist a positive integer $n_0 \in N$ such that $S_{n_0} = \frac{\mu^{n_0+l}}{l-\mu}$ K$< \frac{\epsilon}{3}$ then

$d(\alpha_n, \beta_m) \leq d(\alpha_n, \beta_{n_0}) + d(\alpha_{n_l}, \beta_{n_0}) + d(\alpha_{n_l}, \beta_{n_0}) \leq 3S_{n_0} < \epsilon$ and hence $(\{\alpha_n\}, \{\beta_n\})$ *is* a Cauchy bisequence . (U, V, d) is complete, the bisequence $(\{\alpha_n\}, \{\beta_n\})$ converges, and thus biconverges to point v ∈ U∩ V such that $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = $ v.

Then there exist $n_l \in N$ with $d(\alpha_n, v) < \frac{\epsilon}{3}$ and $d(\beta_n, v) < \frac{\epsilon}{3}$ for all n ≥ n_l and $\epsilon > 0$. Since $(\{\alpha_n\}, \{\beta_n\})$ *is* a Cauchy bisequence, we get $d(\alpha_n, \beta_n) < \frac{\epsilon}{3}$. Now using the (B_4) and from (1), we have

$d(Fv, v) \leq d(Fv, \beta_{n+l}) + d(\alpha_{n+1}, \beta_{n+l}) + d(\alpha_{n+1}, v)$
$\leq d(Fv, G\beta_n) + d(\alpha_{n+1}, \beta_{n+l}) + d(\alpha_{n+1}, v)$
$\leq \mu\, d(v, \beta_n) + d(\alpha_{n+1}, \beta_{n+l}) + d(\alpha_{n+1}, v)$
$\leq \mu\frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon$.

For each n∈ N and $0 < \mu < l$. Then d(Fv, v)=0, and hence Fv= v.

Again, nothing that

d(v, Gv)≤ d(Fv, Gv)≤ μ d(v, v) < d(v, v) =0.

We have d(v, Gv)= 0, which implies that Gv= v.

Hence v is common fixed point of F and G.

In the following, we will prove the uniqueness of common fixed point in U ∪V. For this purpose, let $v' \in$U ∪V be another fixed point of F and G such that F v'=G $v' = v'$ implies $v' \in$U ∩V.
From (1), we have
$d(v, v') = d(Fv, Gv')$
$\leq \mu\, d(v, v') < d(v, v')$
Thus, it's holds only when $d(v, v')=0$ which gives that $v = v'$. Hence F and G have a unique common fixed point in U ∪V.
Remark 2.6: In theorem 2.5, if F=G, (1) becomes
d(Fu, Fv) $\leq \mu$ d(u, v) for all (u, v)∈ U× V , where $\mu \in(0, 1)$. (2)
In this case, we have the following corollary, which can also be found in [2].
Corollary 1: Assume (U, V, d) be a complete bipolar metric spaces and given contractions, F: (U, V, d) ⇉ (U, V, d) satisfies (2).
Then the mappings F: U∪ V →U∪V has a unique fixed point.
Theorem 2.7: Assume (U, V, d) be a complete bipolar metric spaces and given contractions, F, G: (U, V, d) ⇉ (U, V, d) satisfies d(Fu, Fv)≤ μ d(Gu, Gv) for all (u, v)∈ U× V , where $\mu \in(0, 1)$. (3)
If R(F)⊆R(G) and R(G) is complete in U∪V. Then F and G have a unique point of coincidence in U∪V. Furthermore, if F and G are weakly compatible, then the mappings F, G: U∪ V →U∪V have a unique common fixed point.
Proof: Let $\alpha_0 \in$U and $\beta_0 \in$V and choose $\alpha_1 \in$U and $\beta_1 \in$V such that F $\alpha_0 = G\alpha_1$ and F$\beta_0 = G\beta_1$ which can be done R(F)⊆R(G). Let $\alpha_2 \in$U and $\beta_2 \in$V such that F $\alpha_1 = G\alpha_2$ and F$\beta_1 = G\beta_2$, Repeating the process, we get a bi-sequences $(\{\alpha_n\}, \{\beta_n\}) \subseteq$ (U, V) satisfying F $\alpha_{n-1} = G\alpha_n$ and F$\beta_{n-1} = G\beta_n$, for all n∈N.
Let $\mu \in(0, 1)$, put K=d $(G\alpha_0, G\beta_0)$ + d$(G\alpha_0, G\beta_1)$ and $S_n = \frac{\mu^n}{1-\mu}$ K .
Then for each positive integer n and l from (3) , we have
$d(G\alpha_n, G\beta_n) = d(F\alpha_{n-1}, F\beta_{n-1})$
$\leq \mu\, d(G\alpha_{n-1}, G\beta_{n-1})$
$\leq \mu^n\, d(G\alpha_0, G\beta_0)$
and also
$d(G\alpha_n, G\beta_{n+1}) = d(F\alpha_{n-1}, F\beta_n)$
$\leq \mu\, d(G\alpha_{n-1}, G\beta_n)$
$\leq \mu^n\, d(G\alpha_0, G\beta_1)$
Therefore,
$d(G\alpha_n, G\beta_{n+1}) + d(G\alpha_n, G\beta_n)$
$\leq \mu^n (d(G\alpha_0, G\beta_0) + d(G\alpha_0, G\beta_1)) \leq \mu^n K$
for all n, l $\in N$ with n> l
$d(G\alpha_{n+l}, G\beta_n) \leq d(G\alpha_{n+l}, G\beta_{n+1}) + d(G\alpha_n, G\beta_{n+1})$
$+ d(G\alpha_n, G\beta_n)$
$\leq d(G\alpha_{n+l}, G\beta_{n+1}) + \mu^n K$
$\leq d(G\alpha_{n+l}, G\beta_{n+2}) + d(G\alpha_{n+1}, G\beta_{n+2})$
$+ d(G\alpha_{n+1}, G\beta_{n+1}) + \mu^n K$
$\leq d(G\alpha_{n+l}, G\beta_{n+2}) + (\mu^{n+1} + \mu^n)K$
$\leq d(G\alpha_{n+l}, G\beta_{n+l})$
$+(\mu^{n+l} + \cdots \ldots + \mu^{n+1} + \mu^n)K$
$\leq (\mu^{n+l} + \mu^{n+l-1} + \cdots \ldots \ldots + \mu^{n+1} + \mu^n)K$
$= \frac{\mu^n}{1-\mu}$ K $= S_n$.
And similarly, $d(G\alpha_n, G\beta_{n+l}) \leq S_n$.
Let $\epsilon > 0$ and 0 <μ < 1, there exist a positive integer $n_0 \in N$ such that $S_{n_0} = \frac{\mu^{n_0}}{1-\mu}$ K< $\frac{\epsilon}{3}$, then
$d(G\alpha_n, G\beta_m) \leq d(G\alpha_n, G\beta_{n_0}) + d(G\alpha_{n_1}, G\beta_{n_0}) + d(G\alpha_{n_1}, G\beta_{n_0})$
$\leq 3S_{n_0} < \epsilon$
and hence $(\{G\alpha_n\}, \{G\beta_n\})$ *is* a Cauchy bisequence in R(G). Since R(G) is complete in U∪V, so the bisequence $(\{G\alpha_n\}, \{G\beta_n\})$ converges, and thus biconverges to point v ∈ U∩ V such that $\lim_{n\to\infty} G\,\alpha_n = \lim_{n\to\infty} G\,\beta_n$=G v.
Then there exist $n_1 \in N$ with $d(G\alpha_n, Gv) < \frac{\epsilon}{3}$ and $d(v, G\beta_n) < \frac{\epsilon}{3}$ for all n ≥ n_1 and $\epsilon > 0$. Since $(\{G\alpha_n\}, \{G\beta_n\})$ *is* a Cauchy bisequence, we get $d(G\alpha_n, G\beta_n) < \frac{\epsilon}{3}$. Now using the (B_4) and from (3), we have
$d(G\alpha_n, Fv) = d(F\alpha_{n-1}, Fv)$
$\leq \mu\, d(G\alpha_{n-1}, Gv)$
$\leq \mu\frac{\epsilon}{3} < \epsilon$.

For each n∈ N and 0 <μ < 1. Then $\lim_{n\to\infty} G\,\alpha_n$= Fv.
Since $\lim_{n\to\infty} G\,\alpha_n$= Gv. Then it follows that Gv=Fv. Hence F and G have a unique point of coincidence in U ∪V. It follows from Lemma (2.4) that F and G have unique common fixed point.

Example 2.8: In Theorem 2.7, the condition that R(G) is complete in U∪ V is essential. For example, let U = {$\mathbf{U_m}$(R)/$\mathbf{U_m}$(R) is upper triangular matrices over R} and V = {$\mathbf{L_m}$(R)/$\mathbf{L_m}$(R) is lower triangular matrices over R}. Define d: $\mathbf{U_m}$(R)× $\mathbf{L_m}$(R) → [0, ∞) by d (P,Q) = $\sum_{\mathbf{I,j=1}}^{\mathbf{m}} |\mathbf{p_{ij}} - \boldsymbol{q}_{\mathbf{ij}}|$ for all P = $(\mathbf{p_{ij}})_{\mathbf{m\times m}} \in \mathbf{U_m}$(R) and Q = $(\mathbf{q_{ij}})_{\mathbf{m\times m}} \in \mathbf{L_m}$(R). Then obviously, (U, V, d) is a complete bipolar metric space. Define two mappings F, G: $(\boldsymbol{U}, \boldsymbol{V})$ ⇉ (U, V) by the following way:

$$\text{F (P)} = \begin{cases} \frac{\mathbf{1}}{\mathbf{4}}\mathbf{P_{m\times m}}, & \mathbf{0} \neq (\mathbf{p_{ij}})_{\mathbf{m\times m}} \in \mathbf{U_m(R)} \cup \mathbf{L_m(R)} \\ \mathbf{I_{m\times m}}, & (\mathbf{p_{ij}})_{\mathbf{m\times m}} = \mathbf{0} \end{cases}$$

$$\text{G (P)} = \begin{cases} \mathbf{P_{m\times m}}, & \mathbf{0} \neq (\mathbf{p_{ij}})_{\mathbf{m\times m}} \in \mathbf{U_m(R)} \cup \mathbf{L_m(R)} \\ \mathbf{2I_{m\times m}}, & (\mathbf{p_{ij}})_{\mathbf{m\times m}} = \mathbf{0}. \end{cases}$$

Then we have

d(F P, , F Q) = d $(\frac{\mathbf{1}}{\mathbf{4}} (\boldsymbol{p_{ij}})_{\boldsymbol{m\times m}}\ \frac{\mathbf{1}}{\mathbf{4}} (\boldsymbol{q_{ij}})_{\boldsymbol{m\times m}})$

$= \frac{1}{4}\sum_{i,j=1}^{m} |p_{ij} - q_{ij}| \leq \frac{1}{2}\sum_{i,j=1}^{m} |p_{ij} - q_{ij}|$
$\leq \mu$d(GP, GQ).
Where μ= $\frac{1}{2}$ ∈(0,1) and R(F)⊆ R(G) but R(G) is not complete in U∪V. We can compute that F and G don't have a point of coincidence in U ∪ V.

2.2. Common fixed point theorems on contravariant maps

Theorem 2.9: Let (U, V, d) be a complete bipolar metric spaces and given contravariant contractions, F, G: (U, V, d) ⇄ (U, V, d) satisfies
d(Fv, Gu)≤ μ d(u, v) for all (u, v)∈ U× V , where $\mu \in(0, 1)$. (4)
Then the mappings F, G: U∪ V →U ∪V have a unique common fixed point.
Proof: Let $\alpha_0 \in$U and $\beta_0 \in$V and we construct a bisequences $(\{\alpha_n\}, \{\beta_n\}) \subseteq$ (U, V) by the way: F $\alpha_{2n} = \beta_{2n}$, G $\alpha_{2n+1} = \beta_{2n+1}$ and F$\beta_{2n} = \alpha_{2n+1}$ G$\beta_{2n+1} = \alpha_{2n+2}$, for all n∈ N.
Let $\mu \in(0, 1)$, put $S_n = \frac{\mu^{2n-1}}{1-\mu}$. Then for each positive integer n and l from (4), we have
$d(\alpha_{2n+1}, \beta_{2n+1}) = d(F\beta_{2n}, G\alpha_{2n+1})$
$\leq \mu\, d(\alpha_{2n+1}, \beta_{2n})$
$\leq \mu\, d(F\beta_{2n}, G\alpha_{2n})$
$\leq \mu^2\, d(\alpha_{2n}, \beta_{2n})$
$\leq \mu^{4n+1}\, d(\alpha_0, \beta_0)$
and also
$d(\alpha_{2n+1}, \beta_{2n}) = d(F\beta_{2n}, G\alpha_{2n})$
$\leq \mu^{4n}\, d(\alpha_0, \beta_0)$
Therefore,
$d(\alpha_{2n+1}, \beta_{2n+1}) + d(\alpha_{2n+1}, \beta_{2n}) \leq (\mu^{4n+1} + \mu^{4n})\, d(\alpha_0, \beta_0)$
Now we can get that for any n ∈ N
$d(\alpha_{n+1}, \beta_{n+1}) + d(\alpha_{n+1}, \beta_n) \leq (\mu^{2n+1} + \mu^{2n})\, d(\alpha_0, \beta_0)$
for all n, l $\in N$ with n> l we have
$d(\alpha_{n+l}, \beta_n) \leq d(\alpha_{n+l}, \beta_{n+1}) + d(\alpha_{n+1}, \beta_{n+1}) + d(\alpha_{n+1}, \beta_n)$
$\leq d(\alpha_{n+l}, \beta_{n+1}) + (\mu^{2n+1} + \mu^{2n})\, d(\alpha_0, \beta_0)$
$\leq d(\alpha_{n+l}, \beta_{n+2}) + d(\alpha_{n+2}, \beta_{n+2}) + d(\alpha_{n+2}, \beta_{n+1})$

$+(\mu^{2n+1} + \mu^{2n})\, d(\alpha_0, \beta_0)$
$\leq d(\alpha_{n+l}, \beta_{n+2})$
$+(\mu^{2n+3} + \mu^{2n+2} + \mu^{2n+1} + \mu^{2n})\, d(\alpha_0, \beta_0)$
$\leq (\mu^{2n+2l} + \mu^{2n+2l-1} + \cdots + \mu^{2n+1} + \mu^{2n})\, d(\alpha_0, \beta_0)$

$\leq \mu^{2n} \sum_{k=0}^{\infty} \mu^k\, d(\alpha_0, \beta_0)$
$= \mu S_n < S_n$
Now
$d(\alpha_n, \beta_{n+l}) \leq d(\alpha_n, \beta_n) + d(\alpha_{n+1}, \beta_n)) + d(\alpha_{n+1}, \beta_{n+l})$
$\leq (\mu^{2n-1} + \mu^{2n})\, d(\alpha_0, \beta_0) + d(\alpha_{n+1}, \beta_{n+l})$
$\leq (\mu^{2n-1} + \mu^{2n} + \mu^{2n+1} + \mu^{2n+2})\, d(\alpha_0, \beta_0)$
$+d(\alpha_{n+2}, \beta_{n+l})$

$\leq (\mu^{2n-1} + \mu^{2n} + \dots\dots\dots + \mu^{2n+2l-2})\, d(\alpha_0, \beta_0)$
$+d(\alpha_{n+l}, \beta_{n+l})$
$\leq (\mu^{2n-1} + \mu^{2n+2l-2} + \cdots\dots + \mu^{2n+2l-1})\, d(\alpha_0, \beta_0)$
$\leq \mu^{2n-1} \sum_{k=0}^{\infty} \mu^k\, d(\alpha_0, \beta_0)$
$< S_n$

Let $\epsilon > 0$ and $0 < \mu < 1$, there exists $n_0 \in N$ such that $S_{n_0} = \frac{\mu^{2n_0-1}}{1-\mu}\, d(\alpha_0, \beta_0) < \frac{\epsilon}{3}$ hence
$d(\alpha_n, \beta_m) \leq d(\alpha_n, \beta_{n_0}) + d(\alpha_{n_l}, \beta_{n_0}) + d(\alpha_{n_l}, \beta_{n_0})$
$\leq 3S_{n_0} < \epsilon$
and hence $(\{\alpha_n\}, \{\beta_n\})$ *is* a Cauchy bisequence . Since (U, V, d) is complete, the bisequence $(\{\alpha_n\}, \{\beta_n\})$ converges, and thus bi-converges to point v ∈ U∩ V such that $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = v$.
Then there exist $n_1 \in N$ with $d(\alpha_n, v) < \frac{\epsilon}{3}$ and $d(\beta_n, v) < \frac{\epsilon}{3}$ for all $n \geq n_1$ and $\epsilon > 0$. Since $(\{\alpha_n\}, \{\beta_n\})$ *is* a Cauchy bisequence, we get $d(\alpha_n, \beta_n) < \frac{\epsilon}{3}$. Now using the (B_4) and from (4), we have

$d(G\mathrm{v}, \mathrm{v}) \leq d(G\mathrm{v}, \beta_{n+l}) + d(\alpha_{n+1}, \beta_{n+l}) + d(\alpha_{n+1}, \mathrm{v})$
$\leq d(G\mathrm{v}, F\alpha_{n+1}) + d(\alpha_{n+1}, \beta_{n+l}) + d(\alpha_{n+1}, \mathrm{v})$
$\leq \mu\, d(\alpha_{n+1}, \mathrm{v}) + d(\alpha_{n+1}, \beta_{n+l}) + d(\alpha_{n+1}, \mathrm{v})$
$\leq \mu\frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon$

For n∈ N and $0 < \mu < 1$. Then $d(G\mathrm{v}, \mathrm{v}) = 0$, and hence $G\mathrm{v} = \mathrm{v}$.
Again, nothing that
$d(\mathrm{v}, F\mathrm{v}) \leq d(G\mathrm{v}, F\mathrm{v}) \leq \mu\, d(\mathrm{v}, \mathrm{v}) < d(\mathrm{v}, \mathrm{v}) = 0$.
We have $d(\mathrm{v}, F\mathrm{v}) = 0$, which implies that $F\mathrm{v} = \mathrm{v}$.
Hence v is common fixed point of F and G.
In the following we will prove the uniqueness of common fixed point in U ∪V. For this purpose, let $\mathrm{v}' \in U \cup V$ be another fixed point of F and G such that $F\mathrm{v}' = G\mathrm{v}' = \mathrm{v}'$ implies $\mathrm{v}' \in U \cap V$.
From (3), we have
$d(\mathrm{v}, \mathrm{v}') = d(G\mathrm{v}, F\mathrm{v}')$
$\leq \mu\, d(\mathrm{v}, \mathrm{v}') < d(\mathrm{v}, \mathrm{v}')$
Thus, it's holds only when $d(\mathrm{v}, \mathrm{v}') = 0$ which gives that $\mathrm{v} = \mathrm{v}'$.
Hence F and G have a unique common fixed point in U ∪V.

Remark 2.10: In theorem 2.9, if F=G, (4) becomes
$d(Fv, Fu) \leq \mu\, d(u, v)$ for all (u, v)∈ U× V , where $\mu \in (0, 1)$. (5)
In this case, we have the following corollary, which can also be found in [2].

Corollary 2: Assume (U, V, d) be a complete bipolar metric spaces and contravariant contractions, F: (U, V, d)⇄ (U, V,d) satisfies (5). Then the mappings F: U∪ V →U ∪V has a unique fixed point.

Theorem 2.11: Assume (U, V, d) be a complete bipolar metric spaces and given contractions, F, G: (U, V, d) ⇄ (U, V, d) satisfies $d(Fv, Fu) \leq \mu\, d(Gu, Gv)$ for all (u, v)∈ U× V , where $\mu \in (0, 1)$. (6)
If R(F)⊆R(G) and R(G) is complete in U∪V. Then F and G have a unique point of coincidence in U∪V. Furthermore, if F and G are weakly compatible, then the mappings F, G: U∪ V →U∪V have a unique common fixed point.

Theorem 2.12: Assume (U, V, d) be a complete bipolar metric spaces and given covariant contractions, F, G: (U, V, d) ⇄ (U, V, d) satisfies $d(Fv, Fu) \leq \mu(d(Gu, Fu) + d(Fv, Gv))$ (7)
for all (u, v)∈ U× V , where $\mu \in (0, \frac{1}{2})$. If R(F)⊆R(G) and R(G) is complete in U∪V. Then F and G have a unique point of coincidence in U∪V. Furthermore, if F and G are weakly compatible, then the mappings F, G: U∪ V →U∪V have a unique common fixed point.

Proof: Let $\alpha_0 \in U$ and $\beta_0 \in V$, for each nonnegative integer n, we construct a bi-sequences $(\{\alpha_n\}, \{\beta_n\}) \subseteq (U, V)$ as $F\alpha_n = G\beta_n$ and $F\beta_n = G\alpha_{n+1}$, for all n∈N.
Then for each positive integer n and from (7), we have
$d(G\alpha_n, G\beta_n) = d(F\beta_{n-1}, F\alpha_n)$
$\leq (d(G\alpha_n, F\alpha_n) + d(F\beta_{n-1}, G\beta_{n-1}))$
$\leq (d(G\alpha_n, G\beta_n) + d(G\alpha_n, G\beta_{n-1}))$

For all integers n≥ 1, we have
$d(G\alpha_n, G\beta_n) \leq \frac{\mu}{1-\mu}\, d(G\alpha_n, G\beta_{n-1})$
and also
$d(G\alpha_n, G\beta_{n-1}) = d(F\beta_{n-1}, F\alpha_{n-1})$
$\leq \mu(d(G\alpha_{n-1}, F\alpha_{n-1}) + d(F\beta_{n-1}, G\beta_{n-1}))$
$\leq \mu(d(G\alpha_{n-1}, F\beta_{n-1}) + d(G\alpha_n, G\beta_{n-1}))$

So that we have
$d(G\alpha_n, G\beta_{n-1}) \leq \frac{\mu}{1-\mu}\, d(G\alpha_{n-1}, G\beta_{n-1})$
If we say $\lambda = \frac{\mu}{1-\mu}$ then we have $\lambda \in (0, 1)$ and since $\mu \in (0, \frac{1}{2})$
Therefore,
$d(G\alpha_n, G\beta_n) \leq \lambda^{2n}\, d(G\alpha_0, G\beta_0)$
and $d(G\alpha_n, G\beta_{n-1}) \leq \lambda^{2n-1}\, d(G\alpha_0, G\beta_0)$.
Hence
$d(G\alpha_n, G\beta_n) + d(G\alpha_n, G\beta_{n-1})) \leq (\lambda^{2n} + \lambda^{2n-1})d(G\alpha_0, G\beta_0)$
Now, for all n, m ∈ N with m> n
$d(G\alpha_n, G\beta_m) \leq d(G\alpha_n, G\beta_n) + d(G\alpha_{n+1}, G\beta_n) + d(G\alpha_{n+1}, G\beta_m)$
$\leq (\lambda^{2n} + \lambda^{2n-1})d(G\alpha_0, G\beta_0) + d(G\alpha_{n+1}, G\beta_m)$
$\leq (\lambda^{2n} + \lambda^{2n+1} + \cdots + \lambda^{2m})\, d(G\alpha_0, G\beta_0)$
And if for all n, m ∈ N with m< n,
$d(G\alpha_n, G\beta_m) \leq d(G\alpha_{m+1}, G\beta_m) + d(G\alpha_{m+1}, G\beta_{m+1})$
$+ d(G\alpha_n, G\beta_{m+1})$
$\leq (\lambda^{2m+1} + \lambda^{2m+2})d(G\alpha_0, G\beta_0) + d(G\alpha_n, G\beta_{m+1})$
$\leq (\lambda^{2m+1} + \lambda^{2n+1} + \lambda^{2n+1})\, d(G\alpha_0, G\beta_0)$
Since λ∈ (0, 1), this gives $d(G\alpha_n, G\beta_m)$ can be made arbitrarily small by larger *m* and *n*, hence $(\{G\alpha_n\}, \{G\beta_n\})$ *is* a Cauchy bisequence in R(G). Since R(G) is complete in U∪V, so the bisequence $(\{G\alpha_n\}, \{G\beta_n\})$ converges, and thus biconverges to point v ∈ U∩ V such that $\lim_{n\to\infty} G\,\alpha_n = \lim_{n\to\infty} G\,\beta_n = G\,v$.
Then there exist $n_1 \in N$ with $d(G\alpha_n, Gv) < \frac{\epsilon}{3}$ and $d(v, G\beta_n) < \frac{\epsilon}{3}$ for all $n \geq n_1$ and $\epsilon > 0$. Since $(\{G\alpha_n\}, \{G\beta_n\})$ *is* a Cauchy bisequence, we get $d(G\alpha_n, G\beta_n) < \frac{\epsilon}{3}$. Now using the (B_4) and from (7), we have
$d(F\mathrm{v}, G\beta_m) = d(F\mathrm{v}, F\alpha_m,)$
$\leq \mu\, (d(G\alpha_m, F\alpha_m) + d(Fv, G\mathrm{v}))$
$\leq \mu\, (d(G\alpha_m, G\beta_m) + d(Fv, G\mathrm{v}))$
Therefore, $d(F\mathrm{v}, G\beta_m) \leq \frac{\mu}{1-\mu}\, d(G\alpha_m, G\beta_m)$
$\leq \lambda\, d(G\alpha_m, G\beta_m) < \frac{\epsilon}{3}$.
For each m∈ N and $0 < \lambda < 1$. Then $\lim_{n\to\infty} G\,\beta_m = Fv$.
Since $\lim_{n\to\infty} G\,\beta_m = Gv$. Then it follows that Gv=Fv. Hence F and G have a unique point of coincidence in U ∪V. If there is a another point $\kappa \in U \cup V$ such that Fκ=Gκ implies $\kappa \in U \cap V$.
From (7), we have
$d(G\kappa, Gv) = d(F\kappa, Fv)$
$\leq \mu(d(Gv, Fv) + d(F\kappa, G\kappa))$
$\leq \mu(d(Gv, Gv) + d(F\kappa, F\kappa) = 0$.
Thus, consequently Gκ= Gν. Hence *F* and *G* have a unique point of coincidence in U∪ V, it follows from Lemma (2.4) that F and G have unique common fixed point.

Example 2.13: In Theorem 2.12, the condition that R(G) is complete in U∪ V is essential. For example,
let U = {$\mathbf{U_m}$(R)/$\mathbf{U_m}$(R) is upper triangular matrices over R} and V = {$\mathbf{L_m}$(R)/$\mathbf{L_m}$(R) is lower triangular matrices over R}. Define d: $\mathbf{U_m}$(R)× $\mathbf{L_m}$(R) → [0, ∞) by d (P,Q) = $\sum_{\mathbf{I,j=1}}^{\mathbf{m}} |\mathbf{p_{ij}}\text{-}\boldsymbol{q}_{\mathbf{ij}}|$ for all P = $(\mathbf{p_{ij}})_{\mathbf{m\times m}} \in \mathbf{U_m}$(R) and Q = $(\mathbf{q_{ij}})_{\mathbf{m\times m}} \in \mathbf{L_m}$(R). Then obviously, (U, V, d) is a complete bipolar metric space. Define two mappings F, G: ($\boldsymbol{U}$, $\boldsymbol{V}$) ⇉ (U, V) by the following way:

$$\text{F (P)} = \begin{cases} \frac{1}{3}\mathbf{P_{m\times m}}, & \mathbf{0} \neq (\mathbf{p_{ij}})_{\mathbf{m\times m}} \in \mathbf{U_m(R)} \cup \mathbf{L_m(R)} \\ \mathbf{I_{m\times m}}, & (\mathbf{p_{ij}})_{\mathbf{m\times m}} = \mathbf{0} \end{cases}$$

$$\text{G (P)} = \begin{cases} \mathbf{2P_{m\times m}}, & \mathbf{0} \neq (\mathbf{p_{ij}})_{\mathbf{m\times m}} \in \mathbf{U_m(R)} \cup \mathbf{L_m(R)} \\ \mathbf{3I_{m\times m}}, & (\mathbf{p_{ij}})_{\mathbf{m\times m}} = \mathbf{0} \end{cases}$$

Then we have

$$\text{d(F Q, , F P)} = \text{d }(\tfrac{1}{3}(\boldsymbol{q_{ij}})_{\boldsymbol{m\times m}} \tfrac{1}{3}(\boldsymbol{p_{ij}})_{\boldsymbol{m\times m}})$$
$$= \tfrac{1}{5}\textstyle\sum_{\mathrm{I,j=1}}^{\mathrm{m}} \tfrac{5}{3}|\mathrm{q_{ij}}\text{-}p_{\mathrm{ij}}|$$
$$\leq \tfrac{1}{2}\left(\textstyle\sum_{\mathrm{i,j=1}}^{\mathrm{m}} |2\mathrm{q_{ij}}\text{-}\tfrac{1}{3}q_{\mathrm{ij}}| + \textstyle\sum_{\mathrm{i,j=1}}^{\mathrm{m}} |2\mathrm{p_{ij}}\text{-}\tfrac{1}{3}p_{\mathrm{ij}}|\right)$$
$$\leq \mu(d(\mathrm{GP,FP}) + \mathrm{d(FQ,GQ)})$$

Where μ= $\frac{1}{2}$ ∈(0,1) and R(F)⊆ R(G) but R(G) is not complete in U∪V. We can compute that F and G do not have a point of coincidence in U ∪ V.

3. Applications

3.1. Application to the existence of solutions of integral equations

Theorem 3.1: Let us consider the integral equation

$\boldsymbol{\gamma}$(κ)=f(κ)+$\int \boldsymbol{S_1}(\boldsymbol{\kappa},\boldsymbol{\nu},\boldsymbol{\gamma}(\boldsymbol{\nu}))$dν, κ∈ $\boldsymbol{E_1} \cup \boldsymbol{E_2}$

$\boldsymbol{\gamma}$(κ)=f(κ)+$\int \boldsymbol{S_2}(\boldsymbol{\kappa},\boldsymbol{\nu},\boldsymbol{\gamma}(\boldsymbol{\nu}))$dν, κ∈ $\boldsymbol{E_1} \cup \boldsymbol{E_2}$

Where $\mathbf{E_1} \cup \mathbf{E_2}$ is Lebesgue measurable set with m($\mathbf{E_1} \cup \mathbf{E_2}$) $< \infty$. Suppose that

($\boldsymbol{a}$) $\boldsymbol{S_1}$: $(\boldsymbol{E_1}^2 \cup \boldsymbol{E_2}^2) \times [\mathbf{0}, +\infty)$ → [0, +∞) and $\boldsymbol{S_2}$: $(\boldsymbol{E_1}^2 \cup \boldsymbol{E_2}^2) \times [\mathbf{0}, +\infty)$ → [0, +∞), f∈ $\boldsymbol{L}^\infty(\boldsymbol{E_1})$∪ $\boldsymbol{L}^\infty(\boldsymbol{E_2})$

There is a continuous function

Γ: $E_1{}^2 \cup E_2{}^2$ →[0, ∞) and μ∈ (0, 1)such that for all (κ, ν)∈ $E_1{}^2 \cup E_2{}^2$

$$|S_1(\kappa,\nu,\gamma(\nu)) - S_2(\kappa,\nu,\beta(\nu))| \leq \mu\Gamma(\kappa,\nu)|\gamma(\nu) - \beta(\nu)| .$$

$$\|\textstyle\int \Gamma(\kappa,\nu)\mathrm{d}\nu\| \leq 1 \quad \text{i.e} \sup_{\kappa\in E_1\cup E_2} \int|\Gamma(\kappa,\nu)\mathrm{d}\nu| \leq 1.$$

Then, the equation has unique solution in $L^\infty(E_1)$∪ $L^\infty(E_2)$.

Proof: Let U=$\boldsymbol{L}^\infty(\boldsymbol{E_1})$ and V=$\boldsymbol{L}^\infty(\boldsymbol{E_2})$ be two normed linear spaces, where $\boldsymbol{E_1}, \boldsymbol{E_2}$ are two Lebesgue measurable sets with m($\boldsymbol{E_1} \cup \boldsymbol{E_2}$) $< \infty$. Consider d: U×V→ $[\mathbf{0}, \infty)$ be defined by d(f, g)=$\|\boldsymbol{f} - \boldsymbol{g}\|_\infty$ for all (f, g)∈ U×V. Then (U, V, d) is complete bipolar metric spaces. Define covariant map

F, G: $\boldsymbol{L}^\infty(\boldsymbol{E_1}) \cup \boldsymbol{L}^\infty(\boldsymbol{E_2}) \to \boldsymbol{L}^\infty(\boldsymbol{E_1})$∪ $\boldsymbol{L}^\infty(\boldsymbol{E_2})$ by

F($\boldsymbol{\gamma}$(κ))= $\int \boldsymbol{S_1}(\boldsymbol{\kappa},\boldsymbol{\nu},\boldsymbol{\gamma}(\boldsymbol{\nu}))$dν+ f(κ) κ∈ $\boldsymbol{E_1} \cup \boldsymbol{E_2}$.

G($\boldsymbol{\gamma}$(κ))= $\int \boldsymbol{S_2}(\boldsymbol{\kappa},\boldsymbol{\nu},\boldsymbol{\gamma}(\boldsymbol{\nu}))$dν+ f(κ) κ∈ $\boldsymbol{E_1} \cup \boldsymbol{E_2}$.

Notice that

$$\text{d(F }\boldsymbol{\gamma}(\boldsymbol{\nu}), \boldsymbol{G\beta}(\boldsymbol{\nu})) = \|\mathbf{F}\,\boldsymbol{\gamma}(\boldsymbol{\nu}) - \boldsymbol{G\beta}(\boldsymbol{\nu})\|$$
$$= \left\|\textstyle\int \boldsymbol{S_1}(\boldsymbol{\kappa},\boldsymbol{\nu},\boldsymbol{\gamma}(\boldsymbol{\nu}))\,\mathbf{d\nu} + \mathbf{f(\kappa)} - \int \boldsymbol{S_2}(\boldsymbol{\kappa},\boldsymbol{\nu},\boldsymbol{\beta}(\boldsymbol{\nu}))\,\mathbf{d\nu} - \mathbf{f(\kappa)}\right\|$$
$$= |\textstyle\int \boldsymbol{S_1}(\boldsymbol{\kappa},\boldsymbol{\nu},\boldsymbol{\gamma}(\boldsymbol{\nu}))\,\mathbf{d\nu} - \int \boldsymbol{S_2}(\boldsymbol{\kappa},\boldsymbol{\nu},\boldsymbol{\beta}(\boldsymbol{\nu}))\,\mathbf{d\nu}|$$
$$\leq \textstyle\int |\boldsymbol{S_1}(\boldsymbol{\kappa},\boldsymbol{\nu},\boldsymbol{\gamma}(\boldsymbol{\nu})) - \boldsymbol{S_2}(\boldsymbol{\kappa},\boldsymbol{\nu},\boldsymbol{\beta}(\boldsymbol{\nu}))|\mathbf{d\nu}$$
$$\leq \boldsymbol{\mu}\textstyle\int \boldsymbol{\Gamma}(\boldsymbol{\kappa},\boldsymbol{\nu})\,|\boldsymbol{\gamma}(\boldsymbol{\nu}) - \boldsymbol{\beta}(\boldsymbol{\nu})|\mathbf{d\nu}$$
$$\leq \boldsymbol{\mu}\|\boldsymbol{\gamma}(\boldsymbol{\nu}) - \boldsymbol{\beta}(\boldsymbol{\nu})\|_\infty \textstyle\int \|\boldsymbol{\Gamma}(\boldsymbol{\kappa},\boldsymbol{\nu})\|\mathbf{d\nu}$$

Then

$$\text{d(F }\gamma(\nu), G\beta(\nu)) \leq \mu\|\,\gamma(\nu) - \beta(\nu)\| \sup_{\kappa\in E_1\cup E_2} \int|\Gamma(\kappa,\nu)\mathrm{d}\nu|.$$
$$\leq \boldsymbol{\mu}\|\boldsymbol{\gamma}(\boldsymbol{\nu}) - \boldsymbol{\beta}(\boldsymbol{\nu})\|_\infty$$

Hence d(F $\boldsymbol{\gamma}$, $\boldsymbol{G\beta}$)≤ $\boldsymbol{\mu}$ d($\boldsymbol{\gamma}$, $\boldsymbol{\beta}$)

Thus, it is verified that the functions *F* and *G* satisfy all the conditions of Theorem 2.5, and then *F* and *G* have a unique common fixed point in $U \cup V$.

4. Conclusion

In the present research, we have presented unique common fixed point results on various contractive conditions defined on bipolar metric spaces, suitable examples that supports our main results. Also, applications to integral equations are provided.

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