International Journal of Engineering & Technology, 7 (4.10) (2018) 1015-1017

International Journal of Engineering & Technology

Website: www.sciencepubco.com/index.php/IJET

Research paper

A Note on Boolean Like Algebras

K.Pushpalatha[1*] V.M.L.Hima Bindu[2]

[1]*Department of mathematics, KLEF,Vaddeswaram*
[2]*Department of mathematics, KLEF,Vaddeswaram*
**Corresponding author E mail: kpushpamphil@gmail.com*

Abstract

In this paper we develop on abstract system: viz Boolean-like algebra and prove that every Boolean algebra is a Boolean-like algebra. A necessary and sufficient condition for a Boolean-like algebra to be a Boolean algebra has been obtained. As in the case of Boolean ring and Boolean algebra, it is established that under suitable binary operations the Boolean-like ring and Boolean-like algebra are equivalent abstract structures.

Keywords*: Boolean algebra; Boolean like algebra; Boolean like ring; Boolean ring;*

1. Preliminaries

Following A.L.Foster's, the concept of Boolean-like ring is as follows:

Definition 1.1: A Boolean-like ring B is a commutative ring with unity which satisfies the following conditions.

(1) $a + a = 0$, and

(2) $a (1+a) b (1+b) = 0$ for all $a,b \in B$.

we give some examples of Boolean-like rings.

Example 1.2: Every Boolean ring is a Boolean - like ring.

Proof: If B is a Boolean ring, then for all $a \in B$,

$(a+ a)^2 = a + a$, whence $a^2 + a^2 + a^2 + a^2 = a + a$

and so $a + a + a + a = a+ a$. Thus $a + a = 0$.

Further $a(1+ a) = a^2 + a = a + a = 0$.

Hence $a(1+a)b (1+b)= 0$, for all $a,b \in B$.

By a remark, B is a commutative ring with unity.

Thus B is a Boolean-like ring.

But the converse need not be true (For this refer example 1.4).

Example 1.3 : Let R be a ring with unity and characteristic 2. Let B be the set of all central idempotent of R . Then B is a Boolean subring of R. Further B × R is a Boolean - like ring with addition and multiplication defined as follows:

$(b_1, r_1) + (b_2, r_2) = (b_1 + b_2 , r_1 + r_2)$

$(b_1, r_1).(b_2 , r_2) = (b_1b_2 ,b_1r_2+ b_2r_1)$

for all $b_1, b_2 \in B$ and $r_1, r_2 \in R$

Proof: we first prove that B is a Boolean subring of R .

Let $b_1, b_2 \in B$.

We show that $b_1 - b_2 \in B$ and

$b_1 b_2 \in B$

$(b_1 - b_2)^2 = b_1 - b_2$ (Since R has characteristic 2)

For $a \in R$, $(b_1 - b_2) a = b_1a - b_2a$

$= ab_1 - ab_2 = a(b_1 - b_2)$

Hence $b_1 - b_2 \in B$.

Also, $(b_1 b_2)^2 = (b_1 b_2)(b_1 b_2) = b_1(b_2b_1)b_2 = b_1 (b_1 b_2) b_2$

$= b_1^2 b_2^2 = b_1 b_2$

Further $(b_1b_2) a = b_1(b_2 a) = b_1(ab_2) = (b_1 a) b_2 = a(b_1b_2)$.

Hence $b_1 , b_2 \in B$.

Trivially $1 \in B$ and $e^2 = e$ for all $e \in B$

Therefore B is a Boolean subring of R.

We now verify that B × R is a Boolean –like ring.

For $b_1 , b_2, b_3 \in B$ and $r_1, r_2, r_3 \in R$,

$[(b_1,r_1) + (b_2 ,r_2)] + (b_3 , r_3)$

$= (b_1 , r_1) + [(b_2, r_2) + (b_3, r_3)]$

Hence '+' is associative.

Now $(0, 0) \in B \times R$ and $(b_1, r_1)+(0, 0)$

$= (b_1+0 , r_1+0)$ s $= (b_1 , r_1)$

Therefore (0,0) is additive identity of B × R.

For $(b_1 , r_1) \in B \times R$

There exists $(-b_1 , -r_1) \in B \times R$ such that

$(b_1, r_1) + (-b_1 ,-r_1 = (b_1 - b_1 , r_1 - r_1) = (0 , 0)$

Hence $(-b_1,-r_1)$ is the additive inverse of (b_1 , r_1)

$(b_1, r_1) + (b_1, r_2) = (b_2, r_2) + (b_1, r_1)$

Therefore '+' is commutative.

Thus (B×R, +) is an abelian group.

Now $[(b_1, r_1). (b_2, r_2)] . (b_3, r_3)$

$= (b_1b_2, b_1r_2 + b_2 r_1).(b_3, r_3)$

$= (b_1b_2b_3 , b_1b_2r_3 + b_3 (b_1r_2 + b_2r_1)$

$= (b_1,r_1)[(b_2, r_2) (b_3,r_3)]$

Hence '.' is associative.

Also $(1, 0) \in B \times R$ and $(b_1, r_1).(1,0) = (b_1, r_1)$

Further $(b_1, r_1).(b_2, r_2) =(b_1b_2 , b_1r_2+b_2r_1) = (b_2b_1, b_2r_1+b_1r_2)$

$= (b_2, r_2) . (b_1, r_1)$

To prove the distributive law ,

Consider $(b_1, r_1) [(b_2, r_2) + (b_3, r_3)]$

$= (b_1, r_1) [b_2+ b_3, r_2+ r_3]$

$=[b_1(b_2+b_3, b_1(r_2+r_3)+(b_2 + b_3)r_1]$

Furthermore, $(b_1, r_1) (b_2, r_2) + (b_1, r_1)(b_3, r_3) =$

$(b_1b_2 , b_1r_2 + b_2r_1) + (b_1b_3,b_1r_3+ b_3r_1)$

$= (b_1b_2+b_1b_3, b_1r_2+b_2r_1+b_1r_3+b_3r_1)$

Therefore $(B\times R ,+, \cdot)$ is a commutative ring with unity.

Suppose $(b_1, r_1) \in B \times R$.

Since R is a ring of characteristic 2,

$(b_1, r_1) + (b_1, r_1) = (b_1+ b_1 , r_1+r_1) = (0 , 0)$

Also, $(b_1, r_1) [(1,0) + (b_1, r_1)] (b_2, r_2)[(1,0)+ (b_2, r_2)]$

$= (0, r_1)(0, r_2) = (0, 0)$
Hence $B \times R$ is a Boolean – like ring.
As a particular case of example 1.3, we have the following

Example 1.4 : Let $Z_2 = \{0,1\}$ be the ring of integers modulo 2. Then Z_2 is a commutative ring with unity and its characteristic is 2. Obviously Z_2 is a Boolean ring. Hence $Z_2 \times Z_2$ is a Boolean-like ring under the operations of addition and multiplication defined as in example 1.3 above. This Boolean-like ring is denoted by H_4.
Write 0 = (0,0), 1 = (1,0), p = (0,1) and q = (1,1).
Thus $H_4 = \{0, 1, p, q\}$ and addition and multiplication tab les of H_4 are as follows

+	0	1	p	q
0	0	1	p	q
1	1	0	q	p
p	p	q	0	1
q	q	p	1	0

·	0	1	p	q
0	0	0	0	0
1	0	1	p	q
p	0	p	0	p
q	0	q	p	1

Obviously H_4 is not a Boolean ring.

Theorem 1.5: Each element 'a' of a Boolean–like ring B satisfies $a^4 = a^2$.

Proof: We have that $a(1+a)b(1+b) = 0$ -------- (i)
By taking a = b in (i), we get that $a(1+a)a(1+a) = 0$
$\Rightarrow a^4 = a^2$, since the characteristic of B is 2.

2. Boolean Like Algebras

We now give the following definition:

Definition 2.1: An algebraic structure
$(A, \wedge, \vee, ^1, 0, 1)$ where $\wedge$ and $\vee$ are binary operations, ^{1} is an unary operation and
0 and 1 are elements of A, is called a Boolean-like algebra if the following conditions are satisfied

(1) $\wedge, \vee$ are associative and commutative
(2) $(a \vee b)^1 = a^1 \wedge b^1$; $(a^1)^1 = a$; $0^1 = 1$
(3) $a \wedge 0 = 0$; $a \wedge 1 = a$
(4) $b \wedge c = 0 \Rightarrow a \wedge (b \vee c)$
$= (a \wedge b) \vee (a \wedge c)$
(5) $a \wedge a^1 \wedge b \wedge b^1 = 0$
(6) $(a \wedge a^1) \vee (a \wedge a^1) = 0$
(7) $(a^1 \vee b) \wedge (a \vee b^1) = (a \wedge b) \vee (a^1 \wedge b^1)$
(8) $[(a \wedge b) \wedge (a \wedge c)^1] \vee [(a \wedge c) \wedge (a \wedge b)^1]$
$= [a \wedge b \wedge c^1] \vee [a \wedge b^1 \wedge c]$, for all $a, b, c \in A$

The following result gives the most important elementary properties of elements in a Boolean-like algebra.

Lemma 2.2: In any Boolean–like algebra A, we have the following

(i) $a \vee 0 = a$ (ii) $0 = 1^1$
(iii) $a \vee 1 = 1$ (iv) $(a \wedge b)^1 = a^1 \vee b^1$
(v) $(a \vee a) \wedge (a^1 \vee a^1) = 0$ (vi) $(a \wedge a) \vee (a^1 \wedge a^1) = 1$
(vii) $(a \vee a^1) \wedge (a \vee a^1) = 1$

Proof: (i) By (2) and (3) of definition 2.1
$(a \vee 0)^1 = a^1 \wedge 0^1 = a^1 \wedge 1 = a^1$. Hence $a \vee 0 = [(a \vee 0)^1]^1$
$= (a^1)^1 = a$
By (2) of definition 2.1, we have that
(ii) $0 = (0^1)^1 = 1^1$
(iii) $a \vee 1 = [(a \vee 1)^1]^1 = (a^1 \wedge 0)^1 = 0^1 = 1$
(iv) $(a \wedge b)^1 = ((a^1 \vee b^1)^1)^1 = a^1 \vee b^1$
(v) By taking $b = a^1$ in (7), we get that
$(a^1 \vee a^1) \wedge (a \vee a) = (a \wedge a^1) \vee (a^1 \wedge a) = 0$
[vi] By (v) we have that $(a \vee a) \wedge (a^1 \vee a^1) = 0$
Therefore $1 = 0^1 = [(a \vee a) \wedge (a^1 \vee a^1)]^1$
$= (a \vee a)^1 \vee (a^1 \vee a^1)^1 = (a^1 \wedge a^1) \vee (a \wedge a)$
(vii) $(a \vee a^1) \wedge (a \vee a^1) = (a \wedge a) \vee (a^1 \wedge a^1) = 1$, follows from (7) and (vi).

Remark 2.3: Every complemented distributive lattice is a Boolean like algebra.

Proof: Let $(L, \wedge, \vee, ^1, 0, 1)$ be a complemented distributive lattice.
By the definition of a complemented distributive lattice the conditions (1)to (6) of a Boolean- like algebra are satisfied.
(7) $(a^1 \vee b) \wedge (a \vee b^1)$
$= [(a^1 \vee b) \wedge a] \vee [(a^1 \vee b) \wedge b^1]$
$= 0 \vee (a \wedge b) \vee (a^1 \wedge b^1) \vee 0$
$= (a \wedge b) \vee (a^1 \wedge b^1)$
(8) $[(a \wedge b) \wedge (a \wedge c)^1] \vee [(a \wedge c) \wedge (a \wedge b)^1]$
$= [(a \wedge b) \wedge (a^1 \vee c^1)] \vee [(a \wedge c) \wedge (a^1 \vee b^1)]$
$= (a \wedge b \wedge a^1) \vee (a \wedge b \wedge c^1) \vee (a \wedge c \wedge a^1) \vee (a \wedge c \wedge b^1)$
$= (a \wedge b \wedge c^1) \vee (a \wedge c \wedge b^1)$.
Therefore L is a Boolean-like algebra.
By a theorem and remark 2.3, we get that every Boolean algebra is a Boolean- like algebra.

Theorem 2.4: A Boolean-like algebra $(A, \wedge, \vee, ^1, 0, 1)$ is a Boolean algebra if and only if $a \wedge a = a$ for all $a \in A$.

Proof: Suppose $a \wedge a = a$ for all $a \in A$ .Then $(A, \wedge)$ is a semilattice, By (5) of definition 2.1 $x \wedge x^1 = 0$, for all $x \in A$. Also, By (iv) of lemma 2.2,
$1 = 0^1 = (x \wedge x^1)^1 = x^1 \vee x$. If $a \wedge b^1 = 0$, for some $a, b \in A$,
Then $a = a \wedge 1 = a \wedge (b \vee b^1) = (a \wedge b) \vee (a \wedge b^1) = a \wedge b$,
by (4) of def 2.1.
Conversely, if $a \wedge b = a$, then $a \wedge b^1 = a \wedge b \wedge b^1 = a \wedge 0 = 0$. Thus, $(A, \wedge, ^1, 0)$ is a Boolean algebra. Conversely, if $(A, \wedge, ^1, 0)$ is a Boolean algebra, then
$a \wedge a = a$ for all $a \in A$, follows from the fact that $(A, \wedge)$ is a semilattice.

Corollary 2.5: A Boolean-like algebra is a complemented distributive lattice $\Leftrightarrow a \wedge a = a$, for all a.

Proof: Let B be a Boolean-like algebra. If B is a complemented distributive lattice, then evidently,
$a \wedge a = a$ for all $a \in B$.
Conversely suppose that $a \wedge a = a$ for all $a \in B$.
By the above theorem B is a Boolean algebra. Then, by the theorem [1], B is a complemented distributive lattice.
We now prove that every Boolean-like algebra is a Boolean-like ring under some binary operations.

Theorem 2.6: Let $(A, \wedge, \vee, ^1, 0, 1)$ be a Boolean-like algebra. Define binary operations + , · by $a + b = (a \wedge b^1) \vee (a^1 \wedge b)$; $a.b = a \wedge b$ for all $a, b \in A$. Then $(A, +, \cdot, 0, 1)$ is a Boolean-like ring.

Proof: In order to prove that $(A, +, \cdot, 0, 1)$is a Boolean-like ring,
We have to prove that
1) $(A, +)$ is an abelian group with identity 0
2) $(A, \cdot)$ is a commutative semi group with identity 1.
3) Distributive law $a(b + c) = ab + ac$
for all $a, b, c \in A$
4) $a + a = 0$ for all $a \in A$, and
5) $a(1+a)b(1+b) = 0$ for all $a, b \in A$

Now, $a+b = (a \wedge b^1) \vee (a^1 \wedge b) = (b \wedge a^1) \vee (b^1 \wedge a) = b + a$
Therefore '+' is commutative.
$(a + b) + c =$
$[\,[(a \wedge b^1) \vee (a^1 \wedge b)] \wedge c^1] \vee [[(a \wedge b^1) \vee (a^1 \wedge b)]^1 \wedge c$
$= (a \wedge b^1 \wedge c^1) \vee (a^1 \wedge b \wedge c^1) \vee (a \wedge b \wedge c) \vee (a^1 \wedge b^1 \wedge c)$ -- (A)
$a + (b + c) =$
$[a \wedge ((b \wedge c^1) \vee (b^1 \wedge c))^1] \vee [a^1 \wedge ((b \wedge c^1) \vee (b^1 \wedge c))]$
$= (a \wedge b \wedge c) \vee (a \wedge b^1 \wedge c^1) \vee (a^1 \wedge b \wedge c^1) \vee (a^1 \wedge b^1 \wedge c$---- (B)
From (A) and (B), $(a + b) + c = a + (b + c)$. Further
$a + 0 = (a \wedge 0^1) \vee (a^1 \wedge 0) = (a \wedge 1) \vee 0 = a \wedge 1 = a$.
Therefore 0 is the additive identity in A.
Also $a+ a = (a \wedge a^1) \vee (a^1 \wedge a) = 0$
Thus inverse of a is itself.
Therefore, (A,+) is an abelian group with identity 0. Further
$a(b .c) = a \wedge (b \wedge c) = (a \wedge b) \wedge c = (a.b).c$ and $a.1 = a \wedge 1 = a$
Also, $a.b = a \wedge b = b \wedge a = b.a$
Therefore, (A,.) is a semigroup with identity 1.
Distributive law:
$a.(b + c) = a \wedge [(b \wedge c^1) \vee (b^1 \wedge c)] = (a \wedge b \wedge c^1) \vee (a \wedge b^1 \wedge c)$,
by (4) of def 2.1
$ab + ac = (a \wedge b) + (a \wedge c)$
$= [(a \wedge b) \wedge a \wedge c)^1] \vee [(a \wedge b)^1 \wedge (a \wedge c)]$
$= (a \wedge b \wedge c^1) \vee (a \wedge b^1 \wedge c)$ by (8) of def 3.1
Hence $a(b + c) = ab + ac$.
Observe that $a + a = 0$ for all a is already proved.
Finally, $1+ a = (1 \wedge a^1) \vee (1^1 \wedge a) = a^1 \vee 0 = a^1$
Therefore $a (1+ a) b (1 + b) = aa^1bb^1 = a \wedge a^1 \wedge b \wedge b^1 = 0$
by (5) of def 2.1.
Hence A is a Boolean-like ring.

We now prove that every Boolean-like ring becomes a Boolean –like algebra.

Theorem 2.7: Let $(A, +, \cdot ,0,1)$ be a Boolean-like ring. Define the binary operations $\wedge$ and $\vee$ and complementation ^{1} by $a \vee b = a+b+ab$; $a \wedge b = a.b$ and $a^1 = 1+a$ for all $a, b \in A$. Then the algebraic system $(A,\wedge, \vee, ^1 ,0, 1)$ is a Boolean like algebra.

Proof: In order to prove that A is a Boolean like algebra, we need to verify the following.
(1) $\vee$ and $\wedge$ are associative and commutative.
Now $a \vee b = a + b + ab = b + a + ba = b \vee a$,
and $a \wedge b = a.b = b.a = b \wedge a$.
Also, $a \vee (b \vee c) = a + (b + c + bc) + a(b + c + bc)$
$= a+ b + c + bc + ab + ac + abc$
$= (a + b + ab) + c + (a + b + ab)c = (a \vee b) \vee c$
Further, $(a \wedge b) \wedge c = (ab)c = a(bc) = a \wedge (b \wedge c)$
Therefore $\vee$ and $\wedge$ are associative and commutative
(2) $(a \vee b)^1 = a^1 \wedge b^1$; $(a^1)^1 = a$; $0^1 = 1$
Now $(a \vee b)^1 = 1+ (a + b + ab) = 1+ a + b + ab$
$= (1+ a)(1+ b) = a^1b^1 = a^1 \wedge b^1$
Also $(a^1)^1 = (1+ a)^1 = 1 + 1 + a = a$.
Trivially $0^1= 1$
(3) $a \wedge 0 = 0$; $a \wedge 1 = a$.
Trivially, $a \wedge 0 = a.0 = 0$ and $a \wedge 1 = a.1 = a$
(4) $b \wedge c = 0 \Rightarrow a \wedge (b \vee c)=(a \wedge b) \vee (a \wedge c)$
Let $b \wedge c = 0$.
Then $a \wedge (b \vee c) = a (b + c + bc) = ab + ac + abc$
$= (a \wedge b) \vee (a \wedge c)$
(5) $a \wedge a^1 \wedge b \wedge b^1 = a (1+ a) b (1+ b) = 0$
(6) $(a \wedge a^1) \vee (a \wedge a^1)$
$= a (1+ a) + a (1+ a) + aa (1+ a) (1+ a) = 0$
[by (1) & (2) conditions of Boolean-like ring]
(7) $(a^1 \vee b) \wedge (a \vee b^1) = (a^1+ b + a^1b) (a+ b^1+ ab^1)$
$= a^1a + a^1b^1 + a^1a b^1 + ba + bb^1 + bab^1 +$
$a^1ba + a^1bb^1 + a^1bab^1$
$= bb^1(1 + a^1 + a) + aa^1(1 + b + b^1) + a^1b^1+ ba$
(since $a^1bab^1= 0$)
$= bb^1(0) + aa^1(0) + (1+a)(1+b) + ba = 1 + a + b$---(i)
$(a \wedge b) \vee (a^1 \wedge b^1) = ab + a^1 b^1 + aba^1b^1 = ab + a^1b^1$
$= ab + (1+a)(1+b) = 1+ a + b$ ----(ii)
From (i) & (ii) , (7) is satisfied.
(8)$(a \wedge b \wedge c^1) \vee (a \wedge b^1 \wedge c) = abc^1+ab^1c + abc^1ab^1c$
$= abc^1 + ab^1c = ab(1+ c)+a(1+b)c=$
$ab + abc + ac + abc = ab+ac$ ------ (iii)
$[(a \wedge b) \wedge (a \wedge c)^1] \vee [(a \wedge c) \wedge (a \wedge b)^1]$
$= ab (ac)^1 + ac(ab)^1 + (ab)(ac)^1(ac)(ab)^1$
$= ab + ac$ ------ (iv)
From (iii) & (iv) , (8) is satisfied
Thus $(A, \wedge, \vee, ^1,0,1)$ is a Boolean-like algebra.
As in the case of Boolean ring and Boolean algebra, we now show that the Boolean –like ring and Boolean-like algebra are equivalent structures.

Theorem 2.8: The following abstract structures are equivalent
(i) Boolean-like ring and (ii) Boolean-like algebra.

Proof: Let $(A,+,\cdot,0,1)$ be a Boolean-like ring. By theorem 2.7,we get a Boolean-like algebra $(A,\wedge,\vee,^1,0,1)$ in which the binary operations $\wedge$, $\vee$ are defined by $a \vee b = a + b + ab$; $a \wedge b = ab$ and the complementation ^{1} is defined by
$a^1 = 1+ a$ for all $a,b \in A$
by theorem 2.6,
we obtain a Boolean-like ring out of this Boolean-like algebra. where new operations $+^1$, '$\cdot^1$' in A are defined by
$a +^1 b = (a \wedge b^1) \vee (a^1 \wedge b)$; $a \cdot^1 b = a \wedge b$ and $1=1$; $0=0$.
Then $a +^1 b = (a \wedge b^1) \vee (a^1 \wedge b) = a + ab + b + ab = a + b$
and $a \cdot^1 b = a \wedge b = a \cdot b$. Therefore the newly obtained Boolean-like ring is same as the originally given one.
On the otherhand , let $(A,\wedge,\vee,^1,0,1)$ be a Boolean-like algebra. By theorem 2.6, we obtain a Boolean-like ring $(A,+,\cdot,0,1)$ where + and $\cdot$ are defined by
$a + b = (a \wedge b^1) \vee (a^1 \wedge b)$; $a \cdot b = a \wedge b$ for all a , b in A.
As in theorem 2.7, to construct a Boolean-like algebra out of this Boolean-like ring
we introduce new binary operations $\wedge^1$ and $\vee^1$ as
$a \vee^1 b = a + b + ab$; $a \wedge^1 b = ab$ for all a,b in A
$a^1 = 1+a$ and $0=0$; $1=1$.
Now $a \vee^1 b = a + b + ab = 1+ (1+ a + b + ab)$
$= 1+ (1+ a)(1+ b) = (a^1b^1)^1 = a \vee b$
$a \wedge^1 b = ab = a{\cdot}b = a \wedge b$.
Therefore, $\wedge^1 = \wedge$ and $\vee^1 = \vee$.
This completes the proof.
Thus, the newly obtained Boolean-like algebra is same as the originally given Boolean-like algebra.

Acknowledgement

The author extends her gratitude to Prof.Y.V.Reddy, Retd. Prof, Department of Mathematics, ANU and also the management of Koneru Lakshmaiah Educational Foundation, Vaddeswaram.

References:

[1] Bhattacharya P.B., (1995) etc Basic Abstract Algebra, 2nd edition, Cambridge University Press
[2] Foster, A.L. (1951), p^k–rings and ring-logics, Ann.sci.Norm Pisa 5,279-300.
[3] George F. Simmons (1963), Introduction to Topology and Modern Analysis, McGraw – Hill Book, Company, Inc
[4] Jachim Lambek (1986) , Lectures on Rings and Modules, 3rd edition Chelsea pub. Company, New York, N.Y.
[5] Swaminathan, V (1982), Boolean-like rings Ph.D thesis, A.U.
[6] Thomas W. Hungerford(1974), Algebra, Holt, Rinehart and Winston, INC., New York