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Research paper

A study of an extension of the exponential distribution using logistic-x family of distributions

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Abstract

Compound probability models have played important roles in modeling real life events; their ability to withstand skewed datasets has been attributed to the extra shape parameters they possess. This paper focused on exploring a two-parameter compound distribution; Logistic-X Exponential distribution. The basic mathematical properties of the model were obtained and established. The maximum likelihood method of estimation was adopted in estimating the model parameters. The application and potentials of the Logistic-X Exponential distribution were illustrated with the aid of two real data sets; its performance was also compared with the Logistic distribution and Exponential distribution. A simulation study was performed and the behavior of the model parameters was investigated.

Keywords: *Exponential Distribution; Generalized Model; Logistic Distribution; Mathematical Statistics; Simulation; Statistical Properties.*

1. Introduction

A number of authors have in recent time introduced several compound distributions by inducing well-known standard theoretical distributions with additional shape parameter(s). According to [1], these extra parameter(s) help(s) to explore the tail properties and to also improve the goodness of fit of the generator family. Various families of generalized distributions have been introduced in the literature, they involve one or more extra shape parameters and most of them were developed from the logit of random variables. Example of these families include the Beta-G family of distributions [2], Kumaraswamy-G family of distribution [3] and many more as listed in [1], [4 - 6] and their references. This development has been demonstrated in the literature to be very helpful in deriving probability models with high modeling capability.
However, this paper attempts to explore the work of [1] who developed the logistic-X family of distributions. The reason for selecting this particular family in this research is because of its simplicity as it contains only one extra shape parameter. In addition, the logistic distribution itself has been found to be very useful both in theory and practice of statistics. Its shape is similar to that of Normal distribution but it has higher kurtosis. Its application can be found in logistic regression which is used to model categorical dependent variables. For a random variable X, the densities of the logistic distribution are:

$$F(x)=\frac{1}{(1+e^{-\lambda x})}\ ;\ -\infty<x<\infty \tag{1}$$

And

$$f(x)=\frac{\lambda e^{-\lambda x}}{(1+e^{-\lambda x})^2}\ ;\ -\infty<x<\infty \tag{2}$$

respectively, where $F(x)$ and $f(x)$ are the cdf (cumulative distribution function) and pdf (probability density function) respectively.
The logistic distribution was used to develop the Logistic family of distributions in [1] and it has been used to derive the Logistic Frechet distribution, Logistic Uniform distribution, Logistic logistic distribution, Logistic Burr XII distribution, Logistic Weibull distribution and Logistic Pareto distribution. For the sake of illustration, the Logistic Frechet distribution was fitted to two real data sets, it was discovered that it performs better than the other competing distributions.
Similarly, the Logistic-X Exponential distribution is defined in this paper, its statistical properties are explored, simulation studies and real life applications are also provided in sections 2, 3, 4 and 5 respectively.

2. The logistic-X exponential (LoE) distribution

Let $g(x)$, $G(x)$ and $\bar{G}(x)$ be the pdf, cdf and survival function of the baseline distribution, the densities (cdf and pdf) of the Logistic-X family of distributions are:

$$F(x)=\left[1+\left\{-\log\left[\bar{G}(x)\right]\right\}^{-\lambda}\right]^{-1} \tag{3}$$

And

$$f(x)=\frac{\lambda g(x)}{\bar{G}(x)}\left\{-\log\left[\bar{G}(x)\right]\right\}^{-(\lambda+1)}\left[1+\left\{-\log\left[\bar{G}(x)\right]\right\}^{-\lambda}\right]^{-2} \tag{4}$$

Respectively, where λ is an additional shape parameter.

In this context, $g(x)$, $G(x)$ and $\bar{G}(x)$ are the pdf, cdf and survival function of the exponential distribution (the baseline distribution in this research) with parameter α. Mathematically,

$$g(x) = \alpha e^{-\alpha x} \tag{5}$$

$$G(x) = 1 - e^{-\alpha x} \tag{6}$$

$$\bar{G}(x) = e^{-\alpha x} \tag{7}$$

To obtain the cdf of the LoE distribution, the expression in Equation (7) is substituted into that of Equation (3) as:

$$F(x) = \left[1 + (\alpha x)^{-\lambda}\right]^{-1} \tag{8}$$

Its pdf is obtained by simply substituting the expressions in Equations (5) and (7) into that of Equation (4) as:

$$f(x) = \frac{\lambda}{\alpha^{\lambda}} x^{-(\lambda+1)} \left[1 + (\alpha x)^{-\lambda}\right]^{-2}; \ x > 0, \alpha > 0, \lambda > 0 \tag{9}$$

Where α and λ are the scale and shape parameters respectively. It can be observed from Equation (9) that $\lim_{x \to \infty} F(x) = 1$.

Various plots representing the pdf of the LoE distribution are presented in Figure 1.

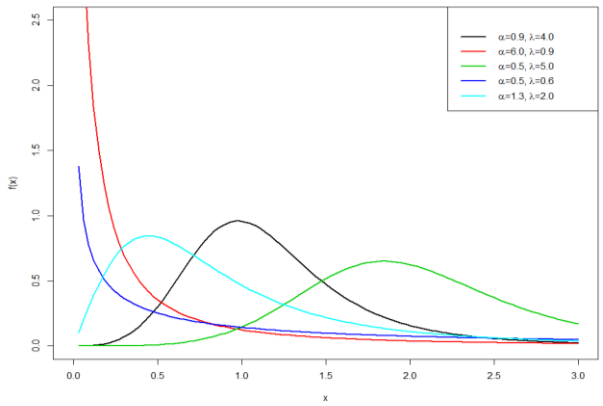

Fig. 1: PDF Plot for the LoE Distribution.

Figure 1 shows decreasing curves and unimodal curves. Hence, the shape of the LoE distribution could be inverted bathtub or decreasing depending on the parameter values.

Various plots representing the cdf of the LoE distribution are presented in Figure 2.

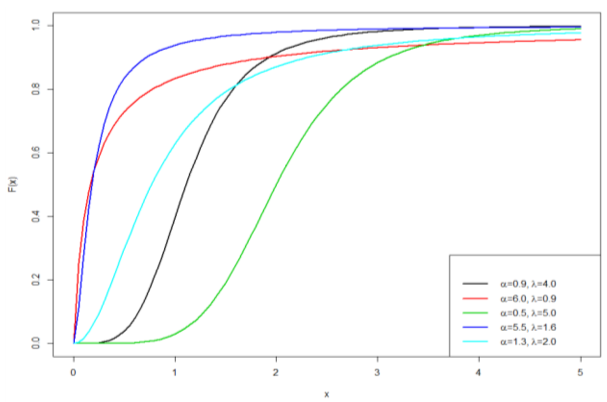

Fig. 2: CDF Plot for the LoE Distribution.

3. Mathematical properties of the LoE distribution

Explored in this section are some of the basic structural and mathematical properties of the LoE distribution.

Survival Function:

The mathematical representation of the survival/reliability function is given by:

$$\bar{F}(x) = 1 - F(x)$$

Thus, the reliability function of the LoE distribution is:

$$\bar{F}(x) = 1 - \left[1 + (\alpha x)^{-\lambda}\right]^{-1} \tag{10}$$

However, the expression in Equation (10) can also be:

$$\bar{F}(x) = (\alpha x)^{-\lambda} \times \left[1 + (\alpha x)^{-\lambda}\right]^{-1} \tag{11}$$

Plots for the survival function are presented in Figure 3.

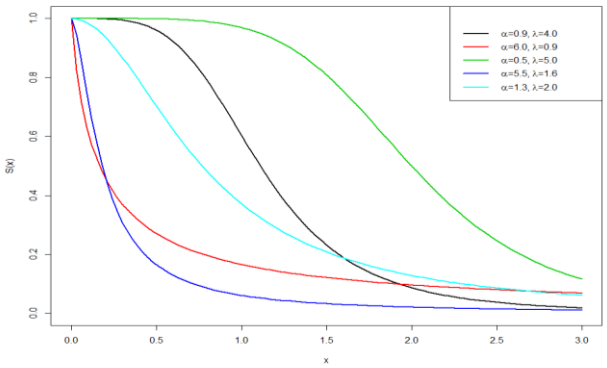

Fig. 3: Survival Functions of the LoE Distribution.

Hazard Function

The mathematical representation of hazard rate function is given by the division of the pdf by the survival/reliability function. So, the hazard rate function of the LoE distribution is obtained as:

$$h(x) = \frac{\frac{\lambda}{\alpha^{\lambda}} x^{-(\lambda+1)} \left[1 + (\alpha x)^{-\lambda}\right]^{-2}}{1 - \left[1 + (\alpha x)^{-\lambda}\right]^{-1}} \tag{12}$$

Various plots representing the hazard function of the LoE distribution are displayed in Figure 4.

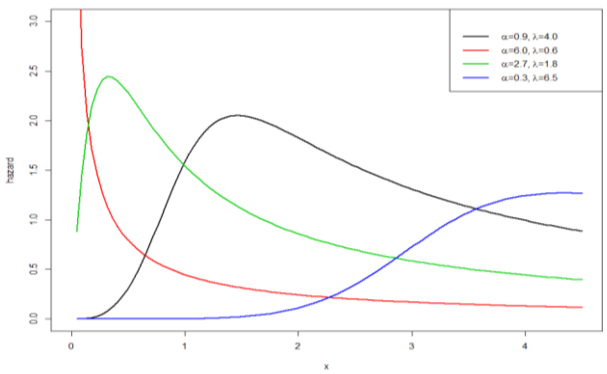

Fig. 4: Hazard Function of the LoE Distribution.

Figure 4 shows that the shapes of the hazard rate function of LoE distribution could be decreasing or unimodal (inverted bathtub). This suggests that the model can be helpful in describing real life phenomena that have decreasing and inverted bathtub failure rates.

Reversed Hazard Function (RHF)
The mathematical representation of the RHF is given by the ratio of the pdf to the cdf. Therefore, the expression for the RHF of the LoE distribution is:

$$r(x)=\frac{\frac{\lambda}{\alpha^{\lambda}}x^{-(\lambda+1)}\left[1+(\alpha x)^{-\lambda}\right]^{-2}}{\left[1+(\alpha x)^{-\lambda}\right]^{-1}} \quad (13)$$

Odds Function (OF)
Odds function is given by the ratio of the cdf to the survival function. Thus, the odds function of the LoE distribution is obtained as:

$$O(x)=\frac{\left[1+(\alpha x)^{-\lambda}\right]^{-1}}{1-\left[1+(\alpha x)^{-\lambda}\right]^{-1}} \quad (14)$$

The Quantile Function (QF) and Median
QF is usually obtained as the inverse of the cumulative distribution function. Mathematically, it is given as:

$$Q(u)=F^{-1}(u)$$

For the LoE distribution, the quantile function is derived as:

$$Q(u)=\alpha^{-1}\left(\frac{u}{1-u}\right)^{1/\lambda} \quad (15)$$

Where $u \sim Uniform(0,1)$
As a consequence, random samples can be generated for the LoE distribution using:

$$x=\alpha^{-1}\left(\frac{u}{1-u}\right)^{1/\lambda}$$

Also, the median can be obtained using the appropriate substitution of $u=0.5$ in Equation (15) as:

$$Median=\frac{1}{\alpha} \quad (16)$$

Other quantiles can simply be obtained as well by substituting the appropriate value of u in Equation (15).
Order Statistics
Given the $F(x)$ and $f(x)$ as in Equations (8) and (9), the pdf of the p^{th} order statistics for a random sample of size n is obtained as follows:

$$f_{p:n}(x)=\frac{n!}{(p-1)!(n-p)!}f(x)\left[F(x)\right]^{p-1}\left[1-F(x)\right]^{n-p} \quad (17)$$

$$f_{p:n}(x)=\frac{n!}{(p-1)!(n-p)!}\left[\frac{\lambda}{\alpha^{\lambda}}x^{-(\lambda+1)}\left[1+(\alpha x)^{-\lambda}\right]^{-2}\right]\times\left\{\left[1+(\alpha x)^{-\lambda}\right]^{-1}\right\}^{p-1}\left\{1-\left[1+(\alpha x)^{-\lambda}\right]^{-1}\right\}^{n-p} \quad (18)$$

$X_{(1)}$; the pdf of the minimum order statistics is:

$$f_{X_{(1)}}(x)=n\times\left[\frac{\lambda}{\alpha^{\lambda}}x^{-(\lambda+1)}\left[1+(\alpha x)^{-\lambda}\right]^{-2}\right]\times\left\{1-\left[1+(\alpha x)^{-\lambda}\right]^{-1}\right\}^{n-1} \quad (19)$$

While $X_{(n)}$; the pdf of the maximum order statistics is:

$$f_{X_{(n)}}(x)=n\times\left[\frac{\lambda}{\alpha^{\lambda}}x^{-(\lambda+1)}\left[1+(\alpha x)^{-\lambda}\right]^{-2}\right]\times\left\{\left[1+(\alpha x)^{-\lambda}\right]^{-1}\right\}^{n-1} \quad (20)$$

Estimation
Let $x_1, x_2, ..., x_n$ denote random samples from the LoE distribution having parameters α and λ as presented in Equation (9). By the method of MLE, the likelihood function is:

$$f(x_1,x_2,...,x_n;\alpha,\lambda)=\prod_{i=1}^{n}\left[\frac{\lambda}{\alpha^{\lambda}}x_i^{-(\lambda+1)}\left[1+(\alpha x_i)^{-\lambda}\right]^{-2}\right]$$

If $'l'$ represent the log-likelihood function, then:

$$l=n\log\lambda-n\lambda\log\alpha-(\lambda+1)\sum_{i=1}^{n}\log x_i-2\sum_{i=1}^{n}\log\left[1+(\alpha x_i)^{-\lambda}\right] \quad (21)$$

The log-likelihood function is further differentiated partially with respect to the two parameters then the results are equated to zero as follows:

$$\frac{\partial l}{\partial\alpha}=-\frac{n\lambda}{\alpha}-2\sum_{i=1}^{n}\left[-\frac{(\alpha x_i)^{-\lambda}\lambda}{\alpha\left[1+(\alpha x_i)^{-\lambda}\right]}\right]=0 \quad (22)$$

$$\frac{\partial l}{\partial\lambda}=\frac{n}{\lambda}-n\ln(\alpha)-\sum_{i=1}^{n}\ln(x_i)-2\sum_{i=1}^{n}\left[-\frac{(\alpha x_i)^{-\lambda}\ln(\alpha x_i)}{1+(\alpha x_i)^{-\lambda}}\right]=0 \quad (23)$$

When Equations (22) and (23) are solved simultaneously, the results are maximum likelihood estimates of the parameters. However, this can be obtained with the aid of software like R, MAPLE, e.t.c when data sets are available since the solution could not be obtained in closed form.

4. Simulation

The performance of the parameters of the LoE distribution is investigated by means of simulation studies using R software. Data sets were generated from the LoE distribution with a replication number of $m=1{,}000$ from which random samples of sizes $n=20,50,100,200$ and 500 were selected. For the purpose of this simulation study, three different cases with various true parameter values were considered. The true parameter values considered for the three cases are $\alpha=0.9,\lambda=1.5$, $\alpha=1.0,\lambda=2.5$ and $\alpha=1.01,\lambda=0.5$ respectively. The mean estimates of the true parameter values were obtained with the bias and mean square error (MSE). Tables 1 to 3 present the results.

Table 1: Result of the Simulation Study when $\alpha=0.9,\lambda=1.5$

n	Parameters	Means	Bias	MSE
20	$\alpha=0.9$	1.1627	0.2627	0.1597
	$\lambda=1.5$	1.6207	0.1207	0.1287
50	$\alpha=0.9$	1.1375	0.2375	0.0905
	$\lambda=1.5$	1.5466	0.0466	0.0408
100	$\alpha=0.9$	1.1217	0.2217	0.0654
	$\lambda=1.5$	1.5236	0.0236	0.0178
200	$\alpha=0.9$	1.1164	0.2164	0.0551
	$\lambda=1.5$	1.5105	0.0105	0.0085
500	$\alpha=0.9$	1.1127	0.2127	0.0483
	$\lambda=1.5$	1.5022	0.0022	0.0031

Table 2: Simulation Study at $\alpha=1.0,\lambda=2.5$

n	Parameters	Means	Bias	MSE
20	$\alpha=1.0$	1.0196	0.0196	0.0251
	$\lambda=2.5$	2.7006	0.2006	0.3560
50	$\alpha=1.0$	1.0110	0.0110	0.0098
	$\lambda=2.5$	2.5776	0.0776	0.1128
100	$\alpha=1.0$	1.0041	0.0041	0.0047
	$\lambda=2.5$	2.5393	0.0393	0.0494
200	$\alpha=1.0$	1.0021	0.0021	0.0024
	$\lambda=2.5$	2.5175	0.0175	0.0236

500	$\alpha = 1.0$ $\lambda = 2.5$	1.0006 2.5037	0.0006 0.0037	0.0009 0.0087

Table 3: Simulation Study at $\alpha = 1.01, \lambda = 0.5$

n	Parameters	Means	Bias	MSE
20	$\alpha = 1.01$ $\lambda = 0.5$	1.3421 0.5401	0.3321 0.0401	1.4324 0.0143
50	$\alpha = 1.01$ $\lambda = 0.5$	1.1357 0.5155	0.1257 0.0155	0.3279 0.0045
100	$\alpha = 1.01$ $\lambda = 0.5$	1.0538 0.5078	0.0438 0.0078	0.1259 0.0020
200	$\alpha = 1.01$ $\lambda = 0.5$	1.0234 0.5035	0.0134 0.0035	0.0627 0.0009
500	$\alpha = 1.01$ $\lambda = 0.5$	1.0014 0.5007	-0.0086 0.0007	0.0221 0.0003

Remark: It can be seen from Tables 1, 2 and 3 that as the sample size increases, the biasedness and the mean square error reduces for all the cases considered. It can also be affirmed that as the sample size increases, the means of the estimates approaches (or tend towards) the true parameter values.

5. Applications

Two real data sets are used for illustration purposes in this section; comparison was made between the LoE distribution, Logistic distribution and the Exponential distribution. The distribution with the best fit was selected based on the log-likelihood and the Akaike Information Criteria (AIC) values obtained by these distributions using the maxLik function [7] in R software.
First Data Set: The dataset used in [8] and [9] was adopted here. This dataset has 76 observations; it is summarized and presented in Table 4.

Table 4: Summary of the First Data Set

n	Min	Max	Mean	Variance	Skewness	Kurtosis
76	0.0251	9.0960	1.9590	2.4774	1.9796	8.1608

The performance of the three competing distributions based on the first data set is presented in Table 5.

Table 5: Performance Rating Based on the First Data Set (with Standard Error in Parenthesis)

Distributions	Estimates	Log-likelihood	AIC	Rank
Logistic Exponential	$\hat{\alpha} = 0.65415(0.06137)$ $\hat{\lambda} = 2.07876(0.20300)$	-124.2725	252.545	1
Logistic	$\hat{\lambda} = 0.72782(0.06928)$	-176.0827	354.1654	3
Exponential	$\hat{\lambda} = 0.51040(0.05855)$	-127.1143	256.2287	2

Based on the results in Table 5, it can be seen that the LoE distribution has the highest log-likelihood value and the lowest AIC value. Hence, it can be selected as the best distribution out of the three distributions.
Second Data Set: The dataset used by [6, 10, 11] and [12] is adopted here. This dataset consists of 63 observations and it relates to the strength of 1.5cm glass fibres. The dataset is summarized and presented in Table 6.

Table 6: Summary of the Second Data Set

n	Min	Max	Mean	Variance	Skewness	Kurtosis
63	0.550	2.240	1.507	0.1051	-0.8999	3.9238

The performance of the three competing distributions based on the second data set is presented in Table 7.

Table 7: Performance Rating Based on the Second Data Set (Standard Error in Parenthesis)

Distributions	Estimates	Log-likelihood	AIC	Rank
Logistic Exponential	$\hat{\alpha} = 0.65523(0.01753)$ $\hat{\lambda} = 7.92596(0.90077)$	-22.78996	49.57993	1
Logistic	$\hat{\lambda} = 1.01272(0.09894)$	-121.1714	244.3428	3
Exponential	$\hat{\lambda} = 0.66365(0.08361)$	-88.83032	179.6606	2

Based on the results in Table 7, it can be seen that the LoE distribution poses the highest log-likelihood value and has the lowest AIC value. Hence, it can be selected as being better than the Logistic and the Exponential distributions.

6. Conclusion

The Logistic-X Exponential distribution has been successfully defined and explored. Its shape could either be unimodal or decreasing (depending on the parameter values). Mathematical properties which include the survival function, hazard function, reversed hazard function, quantile function, odds function, median and distribution of order statistics have been derived and established. The method of MLE was used to estimate the unknown parameters. The simulation studies reveal that the biasedness and MSE of the parameter estimates reduces as the sample size increases. Applications to real data sets show that the LoE distribution is more suitable for modeling than the duo of Logistic distribution and the Exponential distribution based on the selection criteria used.

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