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Research paper

Movement of Fluid Inside the Sphere

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Abstract

The paper presents an exact analytical solution of the stationary problem of an incompressible ideal fluid flow inside a sphere under the action of an external potential mass force.

Keywords: *continuity equation; four-dimensional functions; generalized Cauchy - Riemann conditions.*

1. Introduction

Most applied problems in mechanics and theoretical physics lead to the need to define specific four-dimensional vector functions in the form of $V=[\vartheta_0(x_0,x_1,x_2,x_3), \vartheta_1(x_0,x_1,x_2,x_3), \vartheta_2(x_0,x_1,x_2,x_3), \vartheta_3(x_0,x_1,x_2,x_3)]$, whose components are real functions of four real variables: time x_0 and spatial variables x_1, x_2, x_3 [2-4]. The components $\vartheta_k(x_0,x_1,x_2,x_3)$, $k=\overline{0,3}$ of unknown four-dimensional vector are usually appeared in various systems of differential motion equations deduced from the laws of conservation in mechanics and physics. A special interest for practical purposes (engineering calculations) is finding classical solutions of such systems of equations in explicit form, when the components of desired four-dimensional vector are smooth functions in some domain of definition. However, existing mathematical apparatus often leads to the need to study one-dimensional or two-dimensional (simplified) models of these applied problems. In most cases, it is necessary to consider only stationary physical processes. Therefore, searching for a proper mathematical apparatus for finding classical solutions of the basic systems of motion equations in mechanics and theoretical physics is an actual problem. This paper describes an application of a perspective, in our opinion, approach for solving a rather difficult problem of hydrodynamics.

2. Definition of the Four-Dimensional Functions Space

Let $G \subset R^4$ is some four-dimensional domain.

Definition 1: *Image $U=(u_0(x_0, x_1, x_2, x_3), u_1(x_0, x_1, x_2, x_3), u_2(x_0, x_1, x_2, x_3), u_3(x_0, x_1, x_2, x_3))$ under continuous mapping $U:(x_0, x_1, x_2, x_3) \in G \to (u_0, u_1, u_2, u_3) \in R^4$ is called four-dimensional function and corresponding components of the function $u_k(x_0, x_1, x_2, x_3)$, $k=\overline{0,3}$ as the function components.*
It is easy to understand, that each component $u_k(x_0, x_1, x_2, x_3)$, $k=\overline{0,3}$ is a real continuous function of four real variables determined in domain G. In what follows, a set of all possible four-dimensional functions with continuous components will be denoted by C[M(G)].

Lemma 2.1. *With regard to the operation of componentwise addition and multiplication by a real scalar, the set C[M(G)] is a linear vector space over the field of real numbers.*

Proof. Let $\lambda, \mu \in R$ are arbitrary real numbers and $U=(u_0, u_1, u_2, u_3)$, $W=(w_0, w_1, w_2, w_3) \in C[M(G)]$ are arbitrary continuous four-dimensional functions. Then it is easy to understand that: $\lambda U+ \mu W=Q=(\lambda u_0+ \mu w_0, \lambda u_1+ \mu w_1, \lambda u_2+ \mu w_2, \lambda u_3+ \mu w_3) \in C[M(G)]$
Thus, the set of four-dimensional functions C[M(G)] is indeed a linear vector space over the field of real numbers. It is infinite-dimensional, which will be clear hereafter. Now focus readers' attention on the following key conclusion from the abovementioned.

Lemma 2.2. *Any four-dimensional vector of theoretical physics with continuous components $V=[\upsilon_0(x_0, x_1, x_2, x_3), \upsilon_1(x_0, x_1, x_2, x_3), \upsilon_2(x_0, x_1, x_2, x_3), \upsilon_3(x_0, x_1, x_2, x_3)]$, can be represented as an element of the C[M(G)] space.*

Indeed, let the required four-dimensional vector is being sought in some four-dimensional domain $G \subset R^4$ and has continuous components. Then it inevitably follows that $V \in C[M(G)]$, since by definition this space contains all possible four-dimensional functions in the given domain.

Next, we study one of the key subspaces of the linear space C[M(G)]. It is the elements of this subspace that are directly used in solving applied problems of mechanics and theoretical physics.

Definition 2. *A four-dimensional function $U \in C[M(G)]$ is called regular, if its components everywhere in domain G satisfy generalized Cauchy-Riemann conditions (D'Alembert-Euler) of the form:*

$$\frac{\partial u_0}{\partial x_0} = \frac{\partial u_1}{\partial x_1} = \frac{\partial u_2}{\partial x_2} = \frac{\partial u_3}{\partial x_3} \quad (1)$$

$$\frac{\partial u_0}{\partial x_1}=\frac{\partial u_1}{\partial x_0}=\frac{\partial u_2}{\partial x_3}=\frac{\partial u_3}{\partial x_2} \quad (2)$$

$$\frac{\partial u_0}{\partial x_2}=\frac{\partial u_1}{\partial x_3}=-\frac{\partial u_2}{\partial x_0}=-\frac{\partial u_3}{\partial x_1} \quad (3)$$

$$\frac{\partial u_0}{\partial x_3}=\frac{\partial u_1}{\partial x_2}=-\frac{\partial u_2}{\partial x_1}=-\frac{\partial u_3}{\partial x_0} \quad (4)$$

The whole set of regular functions in G is denoted by $M_A(G)$. It is obvious that any constant four-dimensional vector, an element of R^4, is certainly a constant regular function in any G domain.

It was shown in reference [5], that $M_A(G)\subset C[M(G)]$ is a subspace of C[M(G)]. This subspace is infinite-dimensional, since it contains a countable set (sequence) of linearly independent elements (four-dimensional functions) of the form: E, X, X^2, X^3, …, X^n, …,.., where:

$$E= X^0=(0,1, 0, 0);\ X^n=(w_{n0}, w_{n1}, w_{n2}, w_{n3}),\ n\in N \quad (5)$$

In (2.5), each component is determined on the basis of the following recurrence formulas:

$$w_{n0}=\frac{1}{2}\Big([(x_1+x_0)^2+(x_2+x_3)^2]^{\frac{n}{2}}\cos\left(n\arctan\frac{x_2+x_3}{x_1+x_0}\right)-$$
$$-\left[(x_1-x_0)^2+(x_2-x_3)^2\right]^{\frac{n}{2}}\cos\left(n\arctan\frac{x_2-x_3}{x_1-x_0}\right)\Big) \quad (6)$$

$$w_{n1}=\frac{1}{2}\Big([(x_1+x_0)^2+(x_2+x_3)^2]^{\frac{n}{2}}\cos\left(n\arctan\frac{x_2+x_3}{x_1+x_0}\right)+$$
$$+\left[(x_1-x_0)^2+(x_2-x_3)^2\right]^{\frac{n}{2}}\cos\left(n\arctan\frac{x_2-x_3}{x_1-x_0}\right)\Big) \quad (7)$$

$$w_{n2}=\frac{1}{2}\Big([(x_1+x_0)^2+(x_2+x_3)^2]^{\frac{n}{2}}\sin\left(n\arctan\frac{x_2+x_3}{x_1+x_0}\right)+$$
$$+\left[(x_1-x_0)^2+(x_2-x_3)^2\right]^{\frac{n}{2}}\sin\left(n\arctan\frac{x_2-x_3}{x_1-x_0}\right)\Big) \quad (8)$$

$$w_{n3}=\frac{1}{2}\Big([(x_1+x_0)^2+(x_2+x_3)^2]^{\frac{n}{2}}\sin\left(n\arctan\frac{x_2+x_3}{x_1+x_0}\right)-$$
$$-\left[(x_1-x_0)^2+(x_2-x_3)^2\right]^{\frac{n}{2}}\sin\left(n\arctan\frac{x_2-x_3}{x_1-x_0}\right)\Big) \quad (9)$$

It is easy to verify that these components satisfy the regularity conditions (2.1) - (2.4). Further, in [5] an explicit form of the basic elementary functions $U(X)\in M_A(G)$ of the complex variable $X=(x_0, x_1, x_3, x_4)\in G$. Let us give an explicit form of some elementary functions:

$X = (x_0, x_1, x_2, x_3)$.
$X^2=(2x_1x_0-2x_2x_3;\ x_1^2-x_2^2-x_3^2+x_0^2;\ 2x_1x_2+2x_0x_3;\ 2x_1x_3+2x_0x_2)$.
$\exp(X)=(u_0(x_0, x_1, x_2, x_3), u_1, u_2, u_3)$, где
$u_0(x_0, x_1, x_2, x_3)=\exp(x_1+x_0)\cos(x_2+x_3)-\exp(x_1-x_0)\cos(x_2-x_3)$,
$u_1(x_0, x_1, x_2, x_3)=\exp(x_1+x_0)\cos(x_2+x_3)+\exp(x_1-x_0)\cos(x_2-x_3)$,
$u_2(x_0, x_1, x_2, x_3)=\exp(x_1+x_0)\sin(x_2+x_3)+\exp(x_1-x_0)\sin(x_2-x_3)$,
$u_3(x_0, x_1, x_2, x_3)=\exp(x_1+x_0)\sin(x_2+x_3)-\exp(x_1-x_0)\sin(x_2-x_3)$.

It is easy to verify that the components of each of these four-dimensional functions satisfy the Cauchy-Riemann conditions (1) - (4).

The material presented leads to the following conclusions:

1. The linear vector space C[M(G)] of continuous four-dimensional functions is infinite-dimensional, since it contains a countable number of linearly independent elements of the form (5).
2. All the elements of its linear subspace $M_A(G)\subset C[M(G)]$ can be considered as four-dimensional generalizations of one-dimensional or two-dimensional functions from real or complex analysis. Indeed, setting in the formula of any regular function $x_0= x_2= x_3=0$, we get a typical function of one real variable x_1. If you set $x_0= x_3=$ in the formula, you can get a typical function of the complex variable $z= x_1+ix_2$. More details about this are given in [5].

3. Valuation and Completeness of the $C[M(G)]$ Space

Let a domain $G\subset R^4$ is compact. Then for any element of the space $U=(u_0, u_1, u_2, u_3)\in C[M(G)]$, we can introduce a notion of norm by the following formula:

$$\|U\| = \sup_{X\in G}\sum_{k=0}^{3}|u_k| \quad (10)$$

It is easy to verify that, with respect to such a uniform norm, C[M(G)] is a complete normed space, if we mean the componentwise convergence of a sequence of four-dimensional continuous functions. Further, for the application, the most important conclusions following from Stone's well-known theorem [1] are:

1. The subspace $M_A(G)$ is an everywhere dense subset of C[M(G)].
2. Any element V=C[M(G)] can be approximated with any given accuracy by a finite sum of regular four-dimensional functions.
3. A finite sum of regular functions is also a regular function. Therefore, when solving applied problems of mechanics and theoretical physics, the mathematical apparatus of the theory of four-dimensional regular functions can be used.

All the above-mentioned, we will demonstrate below, based on an actual example of solving one problem of hydrodynamics.

3. Applications

In this section, we obtain an exact analytic solution of one, rather complicated, problem of hydrodynamics. Let suppose, that an ideal incompressible fluid filled with a spherical vessel of radius R flows with a characteristic velocity c, under the action of a stationary potential force. Need to find the hydrodynamic characteristics of the fluid moving inside the sphere. Let $\vec{V}=(V_1(x, y, z), V_2(x, y, z), V_3(x, y, z))$ is required velocity vector of the moving fluid, P(x, y, z) is required pressure function, $\rho>0$ is known fluid density. Then, as is known from [2], the mathematical setting up the problem is formulated as follows:
To find solution of the Euler equations system in the domain
$D : x^2+ y^2+ z^2< R^2$ with boundary $S : x^2+ y^2+ z^2= R^2$

$$(\vec{V}\cdot\nabla)\vec{V}=-\frac{1}{\rho}\nabla P \quad (11)$$

with continuity condition:

$$\operatorname{div}\vec{V}=0 \quad (12)$$

and the boundary condition:

$$\vec{V}_n\big|_s=0 \quad (13)$$

In (11), the potential mass force is included in the pressure gradient in advance. Further, in the reference [6] a formula of a general solution of (12) in a class of smooth functions is given in the form:

$$V_1(x, y, z)=c\alpha u_1(0, \tfrac{x}{R}, \tfrac{\beta y}{R}, \tfrac{\gamma z}{R}) \quad (14)$$

$$V_2(x, y, z)=-c\beta_1 u_2(0, \tfrac{x}{R}, \tfrac{\beta y}{R}, \tfrac{\gamma z}{R}) \quad (15)$$

$$V_3(x, y, z)=-c\gamma_1 u_3(0, \tfrac{x}{R}, \tfrac{\beta y}{R}, \tfrac{\gamma z}{R}) \quad (16)$$

Here: $\alpha, \beta, \gamma, \beta_1, \gamma_1$ are arbitrary scalars (real or complex) connected by the condition $\alpha- \beta\beta_1- \gamma\gamma_1=0$, c is some speed typical for given flow, R is the characteristic dimension of the flow, $u_k(0, x_1, x_2, x_3)$, $k=\overline{1, 3}$ are components of an arbitrary regular function calculated for $x_0=0$.

Solution of the original problem (11)-(13) will be sought in the class of vector-functions with components of the form (14) - (16). In this case, from the symmetry considerations (the domain, where the solution is sought, is spherically symmetric) we choose in the formulas (4.4) - (4.6) the corresponding components of the four-dimensional function $U= X^2-1$ (they are indicated above), i.e:.

$$u_1(0, x_1, x_2, x_3)= x_1^2- x_2^2- x_3^2-1 \quad (17)$$

$$u_2(0, x_1, x_2, x_3)=2x_1x_2 \quad (18)$$

$$u_3(0, x_1, x_2, x_3)=2x_1x_3 \quad (19)$$

In view of the foregoing, (14) - (16) can be rewritten as:

$$V_1(x, y, z)= \frac{\alpha c}{R^2}(x^2- \beta^2y^2- \gamma^2z^2- R^2) \quad (20)$$

$$V_2(x, y, z)= -\frac{2c\beta_1\beta xy}{R^2} \quad (21)$$

$$V_3(x, y, z)= -\frac{2c\gamma_1\gamma xz}{R^2} \quad (22)$$

There are five unknown coefficients $\alpha, \beta, \gamma, \beta_1, \gamma_1$ in the formulas (20) - (22). At the same time, for their definition we also have five relations (21) - (23). Using these relations, we find:

$$\alpha=1;\ \beta= \gamma= \sqrt{2}i;\ \beta_1= \gamma_1= -\frac{\sqrt{2}i}{4}$$

Finally, an exact solution of the original problem is written as:

$$V_1(x, y, z)= \frac{c}{R^2}(x^2+2y^2+2z^2- R^2) \quad (23)$$

$$V_2(x, y, z)= -\frac{cxy}{R^2} \quad (24)$$

$$V_3(x, y, z)= -\frac{cxz}{R^2} \quad (25)$$

$$P(x, y, z)= -\frac{\rho c^2}{2R^4}\left[x^2\left(x^2-2R^2\right)+\left(y^2+ z^2\right)\left(R^2- y^2- z^2\right)\right]+C \quad (26)$$

An arbitrary constant C is determined from the condition $\int_D P(x, y, z)dxdydz=0$. By direct substitution it is easy to verify that the functions (23) - (26) give an exact solution of the original problem.

4. Conclusion

The exact analytical solution of the problem of fluid flowing inside a sphere with Neumann boundary conditions using the theory of four-dimensional regular functions is obtained. The proposed approach can be applied for solving other similar hydrodynamic problems.

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