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**Research paper**

# Crossing numbers of complete bipartite graphs and complete graphs

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## Abstract

The crossing number of a graph is the smallest number of two edge crossings over all planar representations of the graph. In this paper, we investigate the crossing numbers of complete bipartite and complete graphs. Further, we identify optimal drawings and present results on crossing numbers of these classes of graphs. In addition, Zarankiewicz's conjecture on complete bipartite graphs and Guy's conjecture on complete graphs are verified to be true.

***Keywords***: *Planar Graphs; Non-Planar Graphs; Crossing Number Mathematics Subject Classifications: 05C62; 68R10.*

## 1. Introduction

Graphs are structures with simplicity and visibility. They serve as excellent models of data representation. The concepts of graph theory aid in addressing real life problems. During World War II, Mathematician Pál Turán worked in a brick factory and pushed wagons containing bricks from the kilns to their respective storage sites. The wagons were pushed along tracks that were placed between every kiln and every storage site. The workers noticed that it was harder to push the wagons wherever the tracks intersected. This situation led Turán to think on how the tracks could be placed in such a manner that crossing of tracks is minimized. This situation came to be known as Turán's Brick Factory Problem. The concept of crossing number has been applied in Very Large Scale Integration (VLSI) and incidence geometry. VLSI is a process of constructing integrated circuits by merging thousands of transistors into a single chip. The area and size reduction of VLSI devices lowers the production costs and increases performance of chips. The study of crossing number plays a crucial role in analysing and minimising the layout area and the chip size while constructing VLSI devices [1]. Moreover, minimizing the crossings of the edges of graphs makes these graphs aesthetically pleasing which in turn makes them easier to analyse.

## 2. Preliminaries

The following terminologies are in reference to [2]. A graph G (V,E) is a collection of vertices and edges where V represents the set of vertices and E represents the set of edges. A complete bipartite graph is a graph in which the vertices are partitioned into two sets, set M and set N and every vertex of M is adjacent to every vertex of N. It is represented as $K_{m,n}$, where m is the number of vertices in M and n is the number of vertices in N. A complete graph is a graph in which every vertex is adjacent to all the other vertices. It is represented by $K_n$ where n is the number of vertices. A planar graph is a graph in which the edges do not cross each other. The crossing number of a graph is the smallest number of two edge crossings over all planar representations of the graph.

In 1954, Zarankiewicz proposed a formula for the crossing number of a complete bipartite graph, Cr $(K_{m,n}) = \left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$. Zarankiewicz has proved that this formula serves as an upper bound to the actual number. The original proof given by Zarankiewicz contained an error, which was corrected by Richard K Guy in 1969 [3]. It remains a conjecture, as the formula has not been proven as a lower bound as well. In the year 1969, Kleitman proved that this formula applies to $K_{5,n}$ for all n [4]. Later in the year 1993, Dr.Woodall published a result stating that this conjecture holds for $K_{7,7}$[5], showing that the formula holds for all $K_{m,n}$ with min (m, n) $\leq$ 8. The smallest unsettled cases for crossing numbers are known to be for $K_{7,11}$ and $K_{9,9}$.

In the year 1972, Richard Guy presented a conjecture stating that the crossing number of a complete graph of order n is given as $Cr(K_n)= \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$. This conjecture is yet to be proven or disproven for all values of n. Guy proved the conjecture for n $\leq$ 10 [6] which was later extended to n $\leq$ 12 by Pan and Richter in 2007 [7].

## 3. Main results

Complete Bipartite Graphs:

Consider $K_{m,n}(V_1,V_2,E)$ where $|V_1|=m$, $|V_2|=n$ and m $\leq$ n. Hence $K_{m,n}$, has m+n vertices and m×n edges.

In our drawing of $K_{m,n}$ we represent vertices of $V_1$ in dark and $V_2$ in a lighter shade. Place n vertices of $V_2$ vertically and two vertices of $V_1$ on either sides. These two vertices of $V_1$ can be made adjacent to all the n vertices of $V_2$ without causing any crossing of edges using 2n edges. Place the remaining m-2 vertices of $V_1$ alternatively between the n vertices of $V_2$. These m-2 vertices can be made adjacent to two vertices of $V_2$ each without causing any crossing of edges using 2 (m-2) edges.

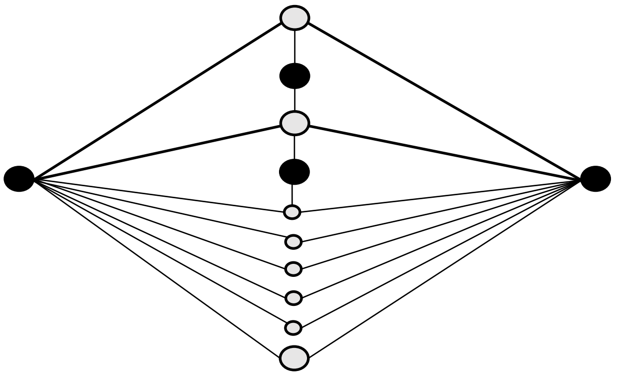

**Fig. 1:** Complete Bipartite Graph Km, N.

Hence, the number of edges that would cause crossings is,

$= mn - (2n + 2(m-2))$

$= mn - (2m + 2n - 4)$

The edges, which cause crossing, will now be strategically placed to minimise the number of crossings. We place edges in such a way that we attain maximum degree for the remaining m-2 vertices, sequentially. We continue in this manner until all mn – (2m+2n-4) edges are placed. We prove this result by considering drawings of $K_{3,n}$, where n = 3, 4, 5, …, determining the sequences of crossings, analysing and generalizing for all values of n.

**Theorem 1:** *The crossing number of $K_{3,n}$, $n \geq 3$ is given by,*

$$Cr(K_{3,n}) = \begin{cases} \frac{n(n-2)}{4}, & \text{n is even} \\ \frac{(n-1)^2}{4}, & \text{n is odd} \end{cases}$$

Proof:

Consider $K_{3,3}$. From the figure of $K_{3,3}$ [Fig.2], it can be noted that the drawing can be completed by placing $mn - (2m + 2n - 4) = 1$ edge which causes one crossing.

Hence Cr $(K_{3,3}) = 1$

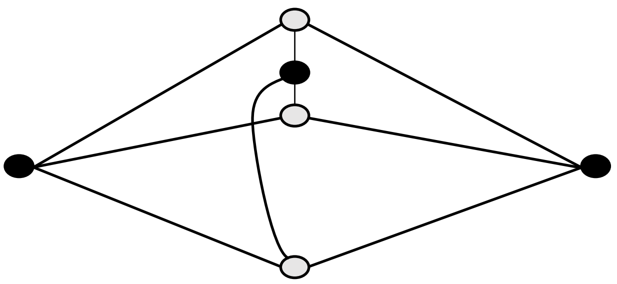

**Fig. 2:** Complete Bipartite Graph K3, 3.

Consider $K_{3,4}$. From the figure of $K_{3,4}$ [Fig.3], it can be noted that the drawing can be completed by placing $mn - (2m + 2n - 4) = 2$ edges. These two edges cause crossings in the sequence of 1, 1.

Hence, crossing number Cr $(K_{3,4}) = 1+1 = 2$

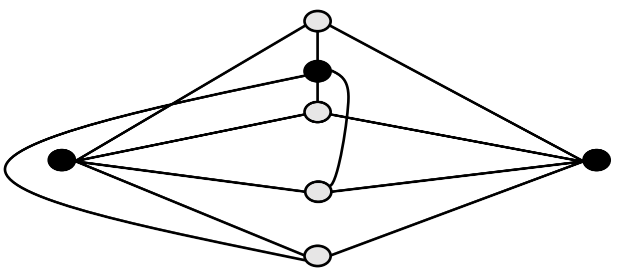

**Fig. 3:** Complete Bipartite Graph K3, 4.

Consider $K_{3,5}$. From the figure of $K_{3,5}$ [Fig.4], it can be noted that the drawing can be completed by placing $mn - (2m + 2n - 4) = 3$ edges. These three edges cause crossings in the sequence of 1, 2, 1.

Hence, crossing number Cr $(K_{3,5}) = 1+2+1 = 4$

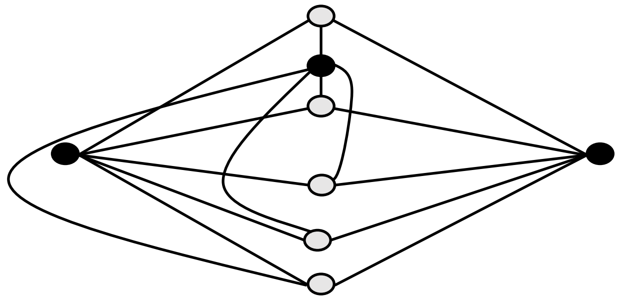

**Fig. 4:** Complete Bipartite Graph K3, 5.

It can be verified that Cr $(K_{3,6}) = 1+2+2+1 = 6$

Proceeding in this manner, the sequence of crossings in $K_{3,n}$, where n is even is given by,

$1, 2, 3, \ldots, \frac{n-4}{2}, \frac{n-2}{2}, \frac{n-2}{2}, \frac{n-4}{2}, \ldots, 3, 2, 1$

Also, the sequence of crossings in $K_{3,n}$, where n is odd is given by,

$1, 2, 3, \ldots, \frac{n-3}{2}, \frac{n-1}{2}, \frac{n-3}{2}, \ldots, 3, 2, 1$

$$(K_{3,n}) = \begin{cases} (1 + 2 + \ldots + \frac{n-4}{2} + \frac{n-2}{2} + \frac{n-2}{2} + \frac{n-4}{2} + \ldots + 2 + 1), \\ (1+2+\ldots+\frac{n-3}{2} + \frac{n-1}{2} + \frac{n-3}{2} + 2 + 1), & \text{when n is odd} \end{cases}$$

$$Cr(K_m) = \begin{cases} 2\,(\sum_{i=1}^{\frac{n-2}{2}} i, & \text{when n is even} \\ ((2\sum_{i=1}^{\frac{n-3}{2}} i) + \frac{n-1}{2}), & \text{when n is odd} \end{cases}$$

$$Cr(K_{3,n}) = \begin{cases} \frac{n(n-2)}{4}, & \text{n is even} \\ \frac{(n-1)^2}{4}, & \text{n is odd} \end{cases}$$

**Theorem 2:** *The crossing number of $K_{m,n}$, $m,n \geq 4$ is given by,*

$$Cr(K_{m,n}) = \begin{cases} \frac{m(m-2)}{4} \times \frac{n(n-2)}{4}, & \text{when m is even, n is even.} \\ \frac{m(m-2)}{4} \times \frac{(n-1)^2}{4}, & \text{when m is even, n is odd.} \\ \frac{(m-1)^2}{4} \times \frac{n(n-2)}{4}, & \text{when m is odd, n is even.} \\ \frac{(m-1)^2}{4} \times \frac{(n-1)^2}{4}, & \text{when m is odd, n is odd.} \end{cases}$$

Proof:

Consider $K_{4,4}$. From the figure of $K_{4,4}$ [Fig.5], it can be noted that the drawing can be completed by placing $mn - (2m + 2n - 4) = 4$ edges. These four edges cause crossings in the sequence of 1, 1, 1, 1. From the result of $K_{3,n}$, it can be noted that the sum of the sequence of crossings in $K_{4,4}$ is twice the sum of the sequence of crossings in $K_{3,4}$.

Hence Cr $(K_{4,4}) = 2 \times$ Cr $(K_{3,4}) = 4$

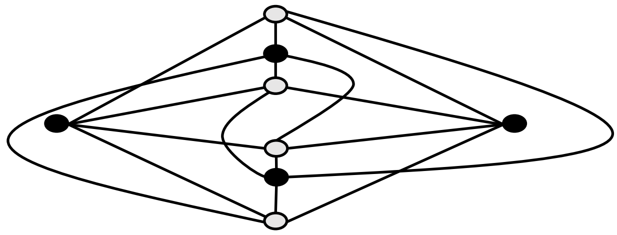

**Fig. 5:** Complete Bipartite Graph K4, 4.

Consider $K_{4,5}$. From the figure of $K_{4,5}$ [Fig.6], it can be noted that the drawing can be completed by placing $mn - (2m + 2n - 4)$= 6 edges. These six edges cause crossings in the sequence of 1, 2, 1, 1, 2, 1. From the result of $K_{3,n}$ it can be noted that the sum of the sequence of crossings in $K_{4,5}$ is twice the sum of the sequence of crossings in $K_{3,5}$.

Hence Cr ($K_{4,5}$) = 2 × Cr ($K_{3,5}$) = 8

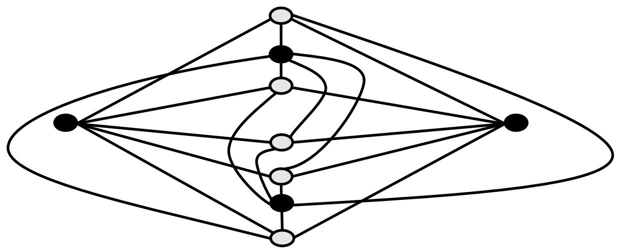

**Fig. 6:** Complete Bipartite Graph K4, 5.

Proceeding in this manner, crossing number of $K_{4,n}$ where $n \geq 4$ is twice the crossing number of $K_{3,n}$.

Cr ($K_{4,n}$) = 2 × Cr ($K_{3,n}$)

Consider $K_{5,5}$. From the figure of $K_{5,5}$ [Fig.7], it can be noted that the drawing can be completed by placing $mn - (2m + 2n - 4)$ = 9 edges. These nine edges cause crossings in the sequence of 1, 2, 1, 1, 2, 1, 2, 4, 2. This sequence can be rearranged as 1, 2, 1, 1, 2, 1, 1, 2, 1, 1, 2, 1. From theresult of $K_{3,n}$, it can be noted that the sum of the sequence of $K_{5,5}$ is four times the sum of the sequence of $K_{3,5}$. Hence Cr ($K_{5,5}$) = 4 x Cr ($K_{3,5}$) = 16.

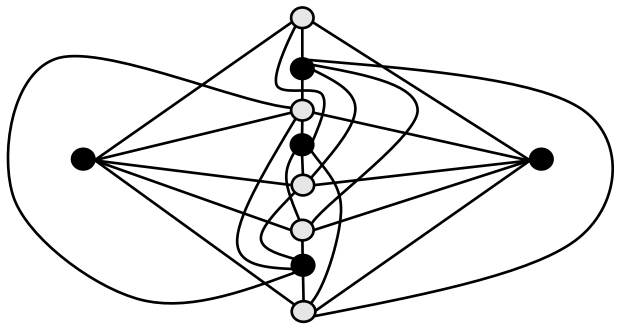

**Fig. 7:** Complete Bipartite Graph K5,5.

Consider $K_{5,6}$. From the figure of $K_{5,6}$ [Fig.8], it can be noted that the drawing can be completed by placing $mn - (2m + 2n - 4)$= 12 edges. These 12 edges cause crossings in the sequence of 1, 2, 2, 1, 1, 2, 2, 1, 2, 4, 4, 2. This sequence can be rearranged as 1, 2, 2, 1, 1, 2, 2, 1, 1, 2, 2, 1, 1, 2, 2, 1. From the result of $K_{3,n}$, it can be noted that the sum of the sequence of $K_{5,6}$ is four times the sum of the sequence of $K_{3,6}$. Hence Cr ($K_{5,6}$) = 4 x Cr ($K_{3,6}$) = 24

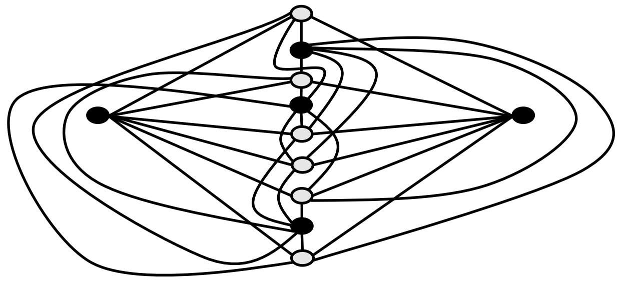

**Fig. 8:** Complete Bipartite Graph K5,6.

Proceeding in this manner, it can be noted that the crossing number of $K_{5,n}$ where $n \geq 5$ is four times the crossing number of $K_{3,n}$.

Cr ($K_{5,n}$) = 4×Cr($K_{3,n}$)

On constructing diagrams of $K_{m,n}$ for $4 \leq m \leq 7$ it can be verified that,

Cr ($K_{4,n}$)= 2 ×Cr( $K_{3,n}$ ) = ( 1+1 ) × Cr ($K_{3,n}$)

Cr ($K_{5,n}$)= 4 ×Cr( $K_{3,n}$ ) = ( 1+2+1 ) × Cr ($K_{3,n}$)

Cr ($K_{6,n}$)= 6 ×Cr($K_{3,n}$) = ( 1+2+2+1 ) × Cr ($K_{3,n}$)

Cr ($K_{7,n}$)= 9× Cr($K_{3,n}$) = ( 1+2+3+2+1 ) × Cr ($K_{3,n}$)

Hence, the crossing number of $K_{m,n}$ is given by,

$$Cr(K_{m,n}) = \begin{cases} \left(1+2+\ldots+\frac{m-4}{2}+\frac{m-2}{2}+\frac{m-2}{2}+\frac{m-4}{2}+\ldots+2+1\right) \times Cr(K_{3,n}), & \text{when m is even} \\ \left(1+2+\ldots+\frac{m-3}{2}+\frac{m-1}{2}+\frac{m-3}{2}+2+1\right) \times Cr(K_{3,n}), & \text{when m is odd} \end{cases}$$

$$Cr(K_{m,n}) = \begin{cases} 2\left(\sum_{i=1}^{\frac{m-2}{2}} i\right) \times Cr(K_{3,n}), & \text{when m is even} \\ \left(\left(2\sum_{i=1}^{\frac{m-3}{2}} i\right) + \frac{m-1}{2}\right) \times Cr(K_{3,n}), & \text{when m is odd} \end{cases}$$

$$Cr(K_{m,n}) = \begin{cases} \frac{m}{2} \times \frac{m-2}{2} \times Cr(K_{3,n}), & \text{when m is even} \\ \frac{(m-1)^2}{4} \times Cr(K_{3,n}), & \text{when m is odd} \end{cases}$$

$$Cr(K_{m,n}) = \begin{cases} \frac{m(m-2)}{4} \times \frac{n(n-2)}{4}, & \text{when m is even, n is even} \\ \frac{m(m-2)}{4} \times \frac{(n-1)^2}{4}, & \text{when m is even, n is odd.} \\ \frac{(m-1)^2}{4} \times \frac{n(n-2)}{4}, & \text{when m is odd, n is even.} \\ \frac{(m-1)^2}{4} \times \frac{(n-1)^2}{4}, & \text{when m is odd, n is odd.} \end{cases}$$

Hence, the above theorem proves that crossing number of $K_{m,n}$, $m,n \geq 4$ is a multiple of crossing number of $K_{3,n}$, $n \geq 3$.

Complete Graphs

Consider a complete graph $K_n$ with n vertices and ${}^nC_2$ edges.

In our drawing of $K_n$, [Fig.9], we place n-2 vertices vertically and two vertices on either sides. Each of these vertices can be made adjacent to all n-2 vertices and each other without causing any crossing of edges using 2(n-2) +1 edges.

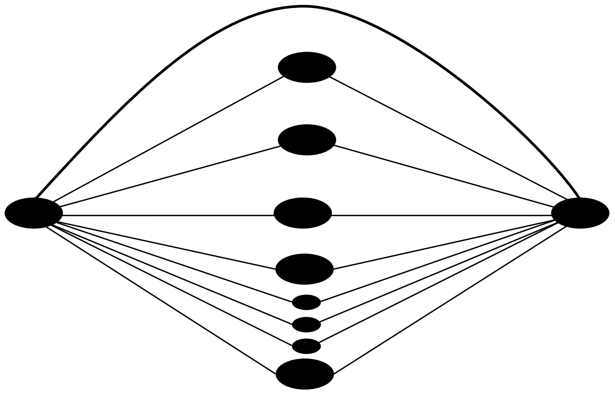

**Fig. 9:** Stage 1 of Constructing a Complete Graph.

Also, as seen in Fig.10, n-3 edges can be placed to form a path consisting of n-2 vertices without any crossings.

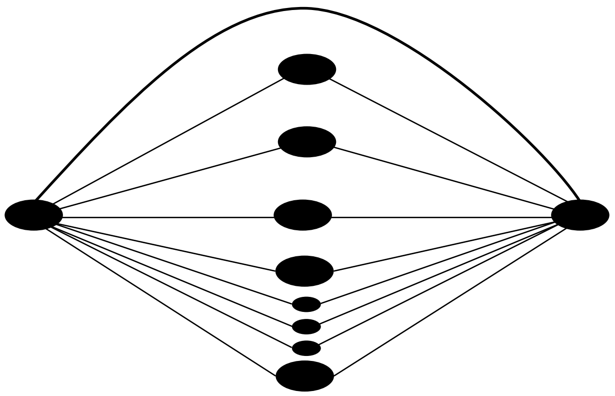

**Fig. 10:** Stage 2 of Constructing a Complete Graph.

Hence, the number of edges that would cause crossings is,

= nC2– $((2(n-2)+1)+n-3)$

= nC2 $-(3n-6)$

= $\frac{n^2-7n+12}{2}$

The edges, which cause crossing, will now be strategically placed to minimise the number of crossings. We place the maximum possible number of edges that intersect with only one edge each. Next, we place the maximum possible number of edges that intersect with only two edges each. We continue in this manner until all $\frac{n^2-7n+12}{2}$ edges are placed.

Theorem 3

The crossing number of $K_n$ is given by,

$$Cr(K_N) = \begin{cases} \frac{2k^2+k(n-2)}{2}, \text{ where k is the sum of first } \frac{(n-4)}{2} \text{ terms; when n is even} \\ k^2, \text{ where k is the sum of first } \left\lfloor\frac{n-2}{2}\right\rfloor \text{ terms; when n is odd.} \end{cases}$$

Proof:

Consider $K_5$. From the figure of $K_5$ [Fig.11], it can be noted that the drawing can be completed by placing $\frac{n^2-7n+12}{2}$ = 1 edge which causes one crossing.

Hence, crossing number Cr ($K_5$) = 1

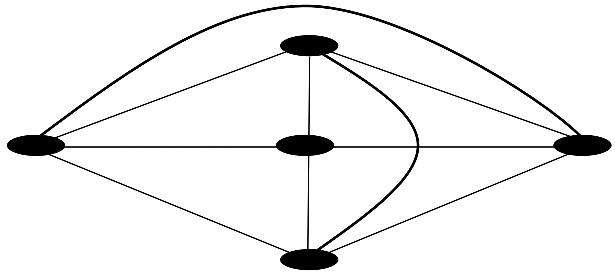

**Fig. 11:** Complete Graph K5.

Consider $K_6$. From the figure of $K_6$ [Fig.12], it can be noted that the drawing can be completed by placing $\frac{n^2-7n+12}{2}$ = 3 edges. These three edges cause crossings in the sequence of 1, 1, 1.

Hence, crossing number Cr ($K_6$) = 1+1+1 = 3

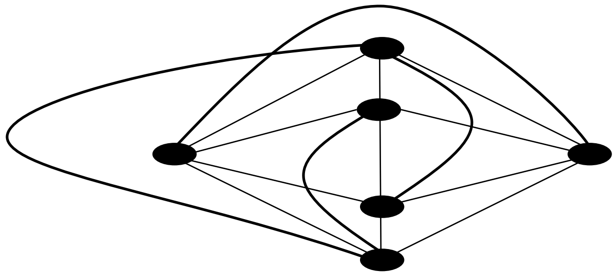

**Fig. 12:** Complete Graph K6.

Consider $K_7$. From the figure of $K_7$ [Fig.13], it can be noted that the drawing can be completed by placing $\frac{n^2-7n+12}{2}$ = 6 edges. These six edges cause crossings in the sequence of 1, 1, 1, 1, 2, 3. This sequence can be rearranged as1, 2, 3, 2, 1 whose sum is 2(1+2) + 3.

Hence, crossing number Cr ($K_7$) = 9

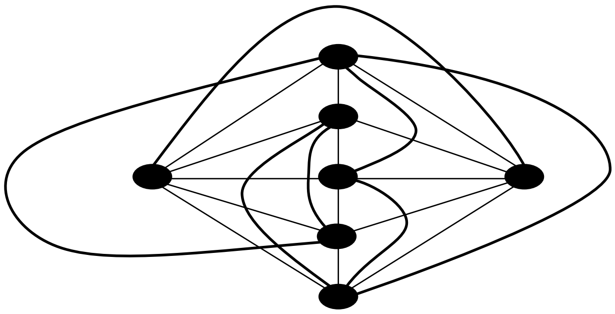

**Fig. 13:** Complete Graph K7.

Consider $K_8$. From the figure of $K_8$ [Fig.14], it can be noted that the drawing can be completed by placing $\frac{n^2-7n+12}{2}$ = 10 edges. These ten edges cause crossings in the sequence of 1, 1, 1, 1, 1, 2, 2, 3, 3, 3. This sequence can be rearranged as1, 2, 3, 3, 3, 3, 2, 1 whose sum is 2(1+2) + 3 + 3 + 3 + 3.

Hence, crossing number Cr ($K_8$) = 18

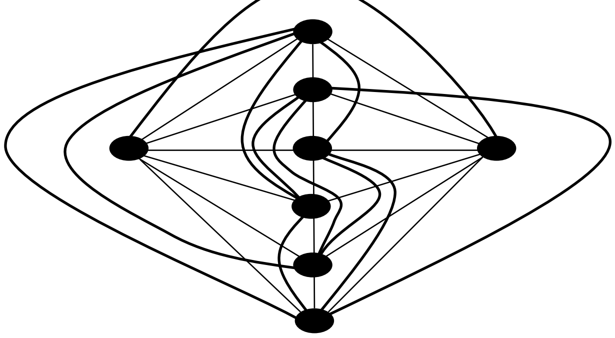

**Fig. 14:** Complete Graph K8.

Proceeding in this manner, the sequence of crossings in $K_n$, where n is even is given by,
1, 2, 3, …, k-1, (k, …$\frac{n}{2}$ times), k-1, …, 3, 2, 1 where k is the sum of first $\frac{n-4}{2}$ terms.
Also, the sequence of crossings in $K_n$, where n is odd is given by,
1, 2, 3, …, k-1, k, k-1 …, 3, 2, 1 where k is the sum of first $\left\lfloor\frac{n-2}{2}\right\rfloor$ terms.
Hence, the crossing number of $K_n$ is given by,

$$Cr(K_n) = \begin{cases} 2(1+2+\cdots+k-1) + k \times \frac{n}{2}, \text{ where } k = \sum_{i=1}^{\frac{n-4}{2}} i, \text{ when n is even} \\ 2(1+2+\cdots+k-1) + k, \text{ where } k = \sum_{i=1}^{\left\lfloor\frac{n-2}{2}\right\rfloor} i, \text{ when n is odd.} \end{cases}$$

$$= \begin{cases} 2\left(\frac{(k-1)(k)}{2}\right) + k \times \frac{n}{2}, \text{ where k is the sum of first } \frac{(n-4)}{2} \text{ terms; when n is even} \\ 2\left(\frac{(k-1)(k)}{2}\right) + k, \text{ where k is the sum of first } \left\lfloor\frac{n-2}{2}\right\rfloor \text{ terms; when n is odd.} \end{cases}$$

$$= \begin{cases} (k-1)(k) + k \times \frac{n}{2}, \text{ where k is the sum of first } \frac{(n-4)}{2} \text{ terms; when n is even} \\ (k-1)(k) + k, \text{ where k is the sum of first } \left\lfloor\frac{n-2}{2}\right\rfloor \text{ terms; when n is odd.} \end{cases}$$

$$Cr(K_n) = \begin{cases} \frac{2k^2+k(n-2)}{2}, \text{ where k is the sum of first } \frac{(n-4)}{2} \text{ terms; when n is even} \\ k^2, \text{ where k is the sum of first } \left\lfloor\frac{n-2}{2}\right\rfloor \text{ terms; when n is odd.} \end{cases}$$

## 4. Conclusion

In this paper, we investigate the crossing numbers of complete bipartite and complete graphs. Based on our results, the conjectures are verified to be true. Further, the focus of our research would be on developing theories on related parameters and identifying the applications of the concept of crossing number.

## Acknowledgements

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