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Research paper

The Impulsive Neutral Integro-Differential Equations with Infinite Delay and Non-Instantaneous Impulses

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Abstract

In this manuscript, we work to accomplish the Krasnoselskii's fixed point theorem to analyze the existence results for an impulsive neutral integro-differential equations with infinite delay and non-instantaneous impulses in Banach spaces. By deploying the fixed point theorem with semigroup theory, we developed the coveted outcomes.

Keywords: *Neutral equations; Equations with impulses; Non-instantaneous impulse condition; Integro-differential equations; fixed point theorem.*

1. Introduction

In this paper, we consider the impulsive neutral integro-differential equation with infinite delay of the model

$$\frac{d}{dt}[w(t)\text{-}G(t, w_t, (H_1w)(t))] = Aw(t) + G_2(t, w_t, (H_2w)(t)) + G_3(t, w_t, (H_3w)(t)),\ t \in (s_i,t_{i+1}],\ i = 0,1,...N,\ I = [0,T] \quad (1)$$

$$w(t) = h_i(t, w_t),\ \text{for } t \in (t_i, s_i],\ i = 1,2,...N, \quad (2)$$

$$w_0 = \phi \in B_h, \quad (3)$$

where the operator A is the infinitesimal generator of a analytic semigroup $\{T(t)\}_{t\geq 0}$ in a Banach space X having norm $\|\cdot\|$ and M_1 is a positive constant to ensure that $\|T(t)\| \leq M_1$, $G_j : I \times B_h \times X \to X$, j = 1,2,3 are given X- valued functions, H_j, j = 1,2,3 are described as $(H_j w)(t) = \int_0^t e_j(t, s, w_t)ds$ where $e_j : D \times B_h \to X$, j = 1,2,3; $D = \{(t, s) \in I \times I : 0 \leq s \leq t \leq T\}$ are suitable functions, and B_h is a phase space characterized in preliminaries. Here $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$. For any continuous function w indicated on (-∞, T] and any t ≥ 0, we represent by w_t the part of B_h specified by $w_t(\theta)=w(t+\theta)$ for $\theta \in (-\infty, 0]$. The following $w_t(\cdot)$ denote the history of the state from time -∞, up to the current time t.

The study of abstract differential equations with non-instantaneous impulses was initiated recently by Hernández and O'Regan in [1]. Lately, M. Pierri, D. O'Regan and V. Rolnik [3] study the existence results of some abstract differential equations with non-instantaneous impulses with the help of fractional powers of operators and semigroup theory.

Our objective here is to give existence results for the given problem by using semigroup theory. In section 2, we recall some preliminary results and definitions which will be utilized throughout this manuscript. In section 3, we present and prove the existence of solutions for the given problem. Our approach here is based on Krasnoselskii's fixed point theorem. Finally in section 4, an application is given to demonstrate the gained results.

2. Preliminaries

Let X be a Banach space provided with the norm $\|\cdot\|$. Consider the analytic semigroup $\{T(t)\}_{t\geq 0}$ bounded linear operators in X. Let A : D(A) → X be the infinitesimal generator of $\{T(t)\}_{t\geq 0}$. Then it is possible to determine the fractional power A^α for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(A)^\alpha$, being dense in X. If X^α represents the space $D(A)^\alpha$ endowed with the norm

$$\|x\|^\alpha = \|A^\alpha x\|,\ x \in D(A^\alpha).$$

Then the following properties are well known. With this discussion, we recall fundamental properties of fractional powers A^α from Pazy [2]. Now we consider the space

$B_h' = PC((-\infty, T), X) = \{w: (-\infty, T] \to X$ such that $w_k \in C(I_k, X)$ and there exists $w(t_k^+)$ and $w(t_k^-)$, $w_0 = \emptyset \in B_h,\ k = 0,1,2,\dots m\}$

Where w_k is the restriction of w to $I_k = (t_k, t_{k+1}]$, set $\|\cdot\|_{B_h'}$ be the seminorm in B_h' defined by $\|w\|_{B_h'} = \|\phi\|_{B_h} + \sup\{|w(s)|: S \in [0,T]\}$, $w \in B_h'$.

3. Existence results

Definition 3.1: A function w: (-∞,T]⟶X is called mild solution of model (1)-(3) if $\boldsymbol{w_0} = \boldsymbol{\phi} \in Bh$ on (-∞,0] $w(t) = \boldsymbol{h_i(t, w_t)}$ for $t \in (ti, si]$ and for each i = 0,1,...N, the constraint of w(.) to interval (0,T] – $\{\boldsymbol{t_1, t_2, t_3, t_4, \dots t_m}\}$ is continuous and for each s∈[0,T) the function $AT(t\text{-}s)\boldsymbol{G_1}(s, \boldsymbol{w_s}, \int_0^t \boldsymbol{e_1(s, \tau, w_\tau)d\tau})$ is integrable and sub-sequent impulsive integral equation is

$$\mathbf{W(t)} = \begin{cases} \mathbf{T(t)[\phi\ (0) - G_1(0,\phi\ ,0) + \xi_1 + \int_0^t A\ \xi_2 ds} \\ \quad + \mathbf{\int_0^t \xi_3 ds + \int_0^t \xi_4 ds\ , t \in [0, t_1]} \\ \mathbf{h_i(t, w_t), t \in (t_i, s_i]\ for\ i = 1, 2, 3, \ldots, N.} \\ \mathbf{T(t - s_i)[h(s_i, w_i) - \xi_5] + \xi_1 + \int_{s_i}^t A\ \xi_2 ds} \\ \quad + \mathbf{\int_{s_i}^t \xi_3 ds + \int_{s_i}^t \xi_4 ds\ \ t \in (s_i, t_{i+1},]} \end{cases}$$

Where $\mathbf{\xi_j} = \mathbf{T(t-s)G_k}(s,\mathbf{w_s}\ ,\int_0^s \mathbf{e_k\,(t,s,w_s))ds}$ for j =2,3,4 and k= 1,2,3, $\mathbf{\xi_1}$ $=\mathbf{G_1}(t,\mathbf{w_t},\int_0^t \mathbf{e_1\,(t,s,w_s)ds})$ and $\mathbf{\xi_5}=\mathbf{G_1(s_i,w_{s_i}},\int_0^{s_i} \mathbf{e_1\,(s_i,\tau,w_\tau))d\tau}$.

In order to study the existence results for the problem (1)- (3) we need to list the following hypotheses:

(H1) For the infinitesimal generator A of a compact analytic semi-group 0∈ $\boldsymbol{\varrho}$ (A) there exist conststants $\mathbf{M_1\ and\ M_0}$ such that $\|\mathbf{T(t)}\|_{\mathbf{L(X)}} \leq \mathbf{M_1\ for\ all\ t \geq 0, M_0} = \|\mathbf{A^{-\beta}}\|$,
and $\|\mathbf{(A)^{1-\beta}\ T(t,s)}\| \leq \frac{\mathbf{M_{1-\beta}}}{\mathbf{(t-s)^{1-\beta}}}$, 0≤ $\mathbf{t \leq T}$.

(H2) For the continuous function $\mathbf{G_1}$:I× $\mathbf{B_h \times X \to X}$ there exist a positive constant $\mathbf{\beta\epsilon(0,1)}$, $\mathbf{K_{G_1} > 0}, \widehat{\mathbf{K_G}} > \mathbf{0}$ and $\mathbf{K^*_{G_1} > 0}$ such that $\mathbf{G_1}$ is $\mathbf{X_\beta}$- valued and for all (t, $\boldsymbol{\varphi_j}$) ∈ I× $\mathbf{B_h}$, x,y∈X, j=1,2;
(i)$\|\mathbf{A^\beta G_1(t,\varphi_1,x) - A^\beta G_1(t,\varphi_2,y)}\|_{\mathbf{x}} \leq \mathbf{K_{G_1}}\|\boldsymbol{\varphi_1 - \varphi_2}\|_{\mathbf{B_h}} + \widehat{\mathbf{K_G}}\|\mathbf{x-y}\|_{\mathbf{X}}$,x,y∈X;
(ii) $\|\mathbf{A^\beta G_1(t,\varphi,0)}\|_{\mathbf{x}} \leq \mathbf{K_{G_1}}\|\boldsymbol{\varphi}\|_{\mathbf{B_h}}+\mathbf{K^*_{G_1}\ and\ K^*_{G_1}} = \mathbf{max_{t\in I}}\|\mathbf{A^\beta G_1(t,0,0)}\|_{\mathbf{x}}$.

(H3) For the continuous function $\mathbf{G_2}$:I × $\mathbf{B_h \times X \to X}$ is strongly measurable and we can find $\mathbf{m_{G_2}}$:[0,T] → $\mathbf{(0,\infty)}$ and a non-decreasing function $\mathbf{\Omega_{G_2}}$: [0, ∞)→ $\mathbf{(0,\infty)}$ such that for all (t, $\boldsymbol{\varphi_j}$) ∈ I× $\mathbf{B_h}$, j=1,2; x in X, $\|\mathbf{G_2(t,\varphi,x)}\|_{\mathbf{x}} \leq \mathbf{m_{G_2}(t)\Omega_{G_2}}(\|\boldsymbol{\varphi}\| + \|\mathbf{x}\|)_{\mathbf{B_h}}$.

(H4) There exists positive constants $\mathbf{K_{G_3} > 0}, \widehat{\mathbf{K_{G_3}}} > \mathbf{0}$ and $\mathbf{K^*_{G_3} > 0}$ and the continuous function $\mathbf{G_3}$: I× $\mathbf{B_h \times X \to X}$ for all (t, $\boldsymbol{\varphi_j}$) ∈ I× $\mathbf{B_h}$, j=1,2;such that (i) $\|\mathbf{G_3(t,\varphi_1,x) - G_3(t,\varphi_2,y)}\|_{\mathbf{x}} \leq \mathbf{K_{G_3}}\|\boldsymbol{\varphi_1 - \varphi_2}\|_{\mathbf{B_h}}+\widehat{\mathbf{K_{G_3}}}\|\mathbf{x - y}\|_{\mathbf{X}}$, x,y ∈ X;
(ii) $\|\mathbf{A^\beta G_3(t,\varphi,0)}\|_{\mathbf{x}} \leq \mathbf{K_{G_3}}\|\boldsymbol{\varphi}\|_{\mathbf{B_h}}+\mathbf{K^*_{G1}\ and\ K^*_{G_3}} = \mathbf{max_{t\in I}}\|\mathbf{G_1(t,0,0)}\|_{\mathbf{x}}$.

(H5) For $\mathbf{K_{h_i} > 0}$, $\mathbf{which\ is\ considered\ as\ a\ constant\ and\ I} = \mathbf{1,2,\ldots,N}$, such that $\|\mathbf{h_i(t,\varphi_1) - h_i(t,\varphi_2)}\|_{\mathbf{x}} \leq \mathbf{K_{h_I}}\|\boldsymbol{\varphi_1 - \varphi_2}\|_{\mathbf{B_h}}$, for each t ∈ $(\mathbf{t_I}\ ,\mathbf{s_i}]$ and all $\boldsymbol{\varphi_1, \varphi_2} \in \mathbf{B_h}$.

(H6) The functions $\mathbf{e_j}$: $\mathfrak{D} \times \mathbf{B_h} \to \mathbf{X}$ are continuous and there exist constants $\mathbf{K_{e_j} > 0}$, $\mathbf{K^*_{e_j} > 0}$ *t*o ensure that $\|\mathbf{e_j(t,s,\varphi) - e_j(t,s,\Psi)}\|_{\mathbf{x}} \leq \mathbf{K_{e_j}}\|\boldsymbol{\varphi - \Psi}\|_{\mathbf{B_h}}$,(t,s)∈ $\mathfrak{D}$, $\boldsymbol{\varphi, \Psi} \in \mathbf{B^2_h}$; and $\mathbf{K^*_{e_j}} = \mathbf{max_{t\in I}}\|\mathbf{e_j(t,s,0)}\|_{\mathbf{x}}$, j=1,2,3.

(H7) The following inequalities holds: Let $M_1[\,K_{h_i}(D^*_1 q + c_n) + K^*_{h_i}] + (M_0(1+M_1) + \frac{M_1-\beta T^\beta}{\beta}[(K_{G_1}+\widehat{K_{G_1}}TK_{e_1})\ D^*_1 q +\ p_1 M_1 T m_{G_2}(s)\Omega_{G_2}[(1+TK_{e_2})\ D^*_1 q + p_2] + M_0 M_1 T[(\,K_{G_3} + \widehat{K_{G_3}}T\ K_{e_3})\ D^*_1 q + p_3] \leq q, t \in [0, T]$ where

$$\mathbf{p_1}=[(\mathbf{K_{G_1}}+\widehat{\mathbf{K_{G_1}}}\mathrm{T}\mathbf{K_{e_1}})\ \mathbf{c_n}+\widehat{\mathbf{K_{G_1}}}\mathrm{T}\mathbf{K^*_{e_1}}+\mathbf{K^*_{G_1}}],$$

$$\mathbf{p_2}=\mathbf{c_n}+\mathbf{TK_{e_2}c_n}+\ \mathrm{T}\mathbf{K^*_{e_2}},$$

$$\mathbf{p_3}=[(\mathbf{K_{G_3}}+\widehat{\mathbf{K_{G_3}}}\mathrm{T}\mathbf{K_{e_3}})\ \mathbf{c_n}+\widehat{\mathbf{K_{G_3}}}\mathrm{T}\mathbf{K^*_{e_3}}+\mathbf{K^*_{G_3}}].$$

Theorem 3.1. Suppose that the hypotheses (H1)-(H7) hold. Then the structure (1)-(3) has a unique solution on I. Then

$$A^* = D^*_1\left\{M_1 K_{h_i} + \left((1+M_1)M_0 + \frac{M_{1-\beta}T^\beta}{\beta}\right)\left[K_{G_1} + \widetilde{K}_{G_1}TK_{e_1}\right] + M_1 D^*_1\left[K_{G_3} + \widetilde{K}_{G_3}TK_{e_3}\right]\right\} < 1. \tag{4}$$

Proof:
The problems (1)-(3) will be transformed into a fixed point problem. Consider the operator $\Phi: B'_h \to B'_h$ by

$$(\Phi\omega)(t) = \begin{cases} T(t)[\varphi(0) - G_1(0,\varphi,0)] + \xi_1 + \int_0^t A\xi_2 ds + \int_0^t \xi_3 ds \\ \quad + \int_0^t \xi_4 ds\ \ , \text{for } t \in [0,t_1], \\ h_i(t,\omega_t), t \in (t_i, s_i], \text{for } i = 1,2,\ldots,N, \\ T(t-s_i)\left[h_i(s_i,\omega_{s_i}) - \xi_5\right] + \xi_1 + \int_{s_i}^t A\xi_2 ds + \int_{s_i}^t \xi_3 ds \\ + \int_{s_i}^t \xi_4 ds\ \ , \text{for } t \in [s_i, t_{i+1}]. \end{cases} \tag{5}$$

Obviously the fixed points of the operator Φ are mild solutions of the model (1)-(3).

If ω(.) fulfills (5), we can easily split it as $\omega(t) = u(t) + v(t)$, for all $t \in I$, this means $\omega_t = u_t + v_t$. Let $B''_h = \{u \in B'_h : u_0 = 0 \in B_h\}$. Let $\|\,.\,\|_{B''_h}$ be the seminorm in B''_h described by $\|u\|_{B''_h} = \sup_{t\in I}\|u(t)\|_X + \|u_0\|_{B_h} = \sup_{t\in I}\|u(t)\|_X$, $u \in B''_h$ as a result $(B''_h, \|\,.\,\|_{B''_h})$ is a banach space.

Consider $B_q = \{u \in B''_h : \|u\|_X \leq q\}$ for some $q \geq 0$; then for each q, $B_q \subset B''_h$ is clearly bounded closed convex set. For $u \in B_q$, $\|u_s + v_s\|_{B_h} \leq D^*_1\|u\|_X + (D^*_1 M_1 + D^*_2)\|\varphi\|_{B_h}$.

In the event that $\|u\|_X < q$, $q > 0$ then

$$\|u_s + v_s\|_{B_h} \leq D^*_1 + c_n \tag{6}$$

where $c_n = (D^*_1 M_1 + D^*_2)\|\varphi\|_{B_h}$. We introduce the operator $\overline{\Phi}: B''_h \to B''_h$ where $\overline{\Phi}$ maps $B_q(0,B''_h)$ into $B_q(0,B''_h)$. For any $u(.) \in B''_h$,

$$(\overline{\Phi}u)(t) = \begin{cases} -T(t)G_1(0,\varphi,0) + \xi_1 + \int_0^t A\xi_2 ds + \int_0^t \xi_3 ds + \\ \int_0^t \xi_4 ds\ \ , \text{for } t \in [0,t_1], \\ h_i(t,u_t+v_t), t \in (t_i,s_i], \text{fori} = 1,2,\ldots,N, \\ T(t-s_i)\left[h_i(s_i,u_{s_i}+v_{s_i}) - \xi_5\right] + \xi_1 + \int_{s_i}^t A\xi_2 ds + \\ \int_{s_i}^t \xi_3 ds + \int_{s_i}^t \xi_4 ds\ \ , \text{for } t \in [s_i,t_{i+1}], \end{cases} \tag{7}$$

Where $\xi_1 = G_1\left(t, u_t + v_t, \int_0^t e_1\,(t,s,u_s+v_s)ds\right)$,

$\xi_2 = AT(t-s)G_1\left(s, u_s + v_s \int_0^s e_1\,(s,\tau,u_\tau+v_\tau)d\tau\right)$,

$\xi_3 = T(t-s)\ G_2(s, u_s + v_s, \int_0^s e_1(s,\tau,u_\tau + v_\tau)d\tau)$,
$\xi_4 = T(t\text{-}s)\ G_3(s, u_s + v_s, \int_0^s e_3(s,\tau, u_\tau + v_\tau)\, d\tau)$ and
$\xi_5 = G_1(s_i, u_{s_i} + v_{s_i}, \int_0^{s_i} e_1(s_i, \tau, u_\tau + v_\tau)\, d\tau)$. From this, it is understood that the operator Φ has a fixed point if and only if $\bar{\Phi}$ has a fixed point. Give us a chance to demonstrate that $\bar{\Phi}$ has a fixed point. Now, we enter the main proof of the theorem. To apply Krasnoselskii's fixed point theorem, we introduce the decomposition
$\bar{\Phi} = \sum_{i=0}^N \Phi_i^1 + \sum_{i=0}^N \Phi_i^2$

$$(\Phi_i^1 u)(t) = \begin{cases} -T(t)\{G_1(0,\Phi,0) + \xi_1 + \int_0^t \xi_2\, ds + \int_0^t \xi_4\, ds, & \text{for } t \in [0,t_1], \\ 0, & \text{for } t \in [t_i,t_{i+1}], \quad i \geq 0 \\ h_i(t, u_t + v_t), & \text{for } t \in (t_i,s_i], i = 1,2,\dots,N, \\ T(t-s_i)[h_i(s_i, u_{s_i} + v_{s_i}) - \xi_5] + \xi_1 + \int_{s_i}^t \xi_2\, ds + \int_{s_i}^t \xi_4\, ds, & \text{for } t \in [s_i,t_{i+1}], i = 1,2,\dots,N. \end{cases}$$

$$(\Phi_i^2 u)(t) = \begin{cases} \int_0^t \xi_3\, ds, & t \in [0,t_1], \\ 0, & \text{for } t \in (t_i,s_i], \\ \int_{s_i}^t \xi_3\, ds, & \text{for } t \in [s_i,t_{i+1}], i = 1,2,\dots,N. \end{cases}$$

For better readability, we divide our results into four steps.
Step 1: First we show that $\Phi_i^1 u(t) + \Phi_i^2 u(t) \in B_q$, whenever $u \in B_q$. For all $u \in B_q$, we have
(i) $\| \Phi_i^1 u(t) + \Phi_i^2 u(t)\| \leq M_1 M_0 [K_{G_1}\|\varphi\|_{B_h} + K_{G_1}^*] + \left[M_0 + \frac{M_{1-\beta}T^\beta}{\beta}\right][(K_{G_1} + \widetilde{K}_{G_1} T\, Ke_1) D_1^* q + p_1] + M_0 M_1 T [(K_{G_3} + \widetilde{K}_{G_3} T\, Ke_3) D_1^* q + p_3] + M_1 T m_{G_2}(s)\Omega_{G_2} [(1 + TKe_2)D_1^* q + p_2] \leq q, t \in [0,t_1]$,
(ii) $\| \Phi_i^1 u(t) + \Phi_i^2 u(t)\| \leq K_{h_i}(D_1^* q + c_n) + K_{h_i}^* \leq q$, for $t \in (t_i,s_i]$.
(iii) $\| \Phi_i^1 u(t) + \Phi_i^2 u(t)\| \leq M_1[K_{h_i}(D_1^* q + c_n) + K_{h_i}^*] + \left(M_0(1+M_1) + \frac{M_{1-\beta}(t_{i+1}-s_i)^\beta}{\beta}\right)[(K_{G_1} + \widetilde{K}_{G_1} TKe_1)D_1^* q + p_1] + M_1(t_{i+1} - s_i)m_{G_2}(s)\Omega_{G_2}[(1 + TKe_2)D_1^* q + p_2] + M_0 M_1(t_{i+1} - s_i)[(K_{G_3} + \widetilde{K}_{G_3} TKe_3)D_1^* q + p_3] \leq q$, for $t \in (s_i, t_{i+1}]$.
(iv) $\| \Phi_i^1 u(t) + \Phi_i^2 u(t)\| \leq M_1[K_{h_i}(D_1^* q + c_n) + K_{h_i}^*] + \left(M_0(1+M_1) + \frac{M_{1-\beta}T^\beta}{\beta}\right)[(K_{G_1} + \widetilde{K}_{G_1} TKe_1)D_1^* q + p_1] M_1 T m_{G_2}(s)\Omega_{G_2}[(1 + TKe_2)D_1^* q + p_2] + M_0 M_1 T[(K_{G_3} + \widetilde{K}_{G_3} TKe_3)D_1^* q + p_3] \leq q$, $t \in [0,T]$, where p_1, p_2, p_3 are independent of q. dividing both sides by q, we have $\Phi_i^1 u(t) + \Phi_i^2 u(t) \in B_q$.
Step 2: Next we will show that $\Phi^1 = \sum_{i=0}^N \Phi_i^1$ is a contraction. From the definition of $\phi^1 u(t)$, $\Phi_i^1 u(t)$ and the assumption of (H_2), (H_4) and (H_6), we get
(i) $\| \Phi_i^1 u(t) + \Phi_i^2 u(t)\| \leq D_1^*[\left(M_0 + \frac{M_{1-\beta}T^\beta}{\beta}\right)(K_{G_1} + \widetilde{K}_{G_1} TKe_1) + M_0 M_1 T[K_{G_3} + \widetilde{K}_{G_3} TKe_3]]\|u - \bar{u}\|_{B_h''}, t \in [0,t_1]$.
(ii) $\| (\Phi_i^1 u)(t) - (\Phi_i^1 \bar{u})(t)\| \leq D_1^* K_{h_i}\|u - \bar{u}\|_{B_h''}, t \in (t_i,s_i]$,
(iii) $\| (\Phi_i^1 u)(t) - (\Phi_i^1 \bar{u})(t)\| \leq D_1^*[\{M_1 K_{h_i} + \left(M_0(1+M_1) + \frac{M_{1-\beta}(t_{i+1}-s_i)^\beta}{\beta}\right)[K_{G_1} + \widetilde{K}_{G_1} TKe_1] + M_0 M_1 (t_{i+1} - s_i) D_1^*[K_{G_3} + \widetilde{K}_{G_3} TKe_3]\} \times \|u - \bar{u}\|_{B_h''}, t \in (s_i, t_{i+1}]$
(iv) $\| (\Phi_i^1 u)(t) - (\Phi_i^1 \bar{u})(t)\| \leq D_1^*\{M_1 K_{h_i} + \left(M_0(1+M_1) + \frac{M_{1-\beta}T^\beta}{\beta}\right)[K_{G_1} + \widetilde{K}_{G_1} TKe_1] + M_0 M_1 D_1^*[K_{G_3} + \widetilde{K}_{G_3} TKe_3]\}\|u - \bar{u}\|_{B_h''}, t \in (0,T] \leq \Lambda^*\|u - \bar{u}\|_{B_h''}$, where $\Lambda^* = D_1^*\{M_1 K_{h_i} + \left(M_0(1+M_1) + \frac{M_{1-\beta}T^\beta}{\beta}\right)[K_{G_1} + \widetilde{K}_{G_1} TKe_1] + M_0 M_1 D_1^*[K_{G_3} + \widetilde{K}_{G_3} TKe_3]\} < 1$. $\phi^1 u(t)$ is a contraction.
Step 3: Next we will prove that Φ_i^2 is compact and continuous. We split the proof into three parts

Let the sequence u_n such that $u_n \to u$ in B_h''. Then for all $t \in I$ by the definition of $\Phi^2 u(t)$, $\Phi_i^2 u(t)$ and by assumptions H(3) and H(6)
$\|(\Phi_i^2 u^n)(t) - (\Phi_i^2 u)(t)\| \leq M_1 t_1 \|G_2(s, u_s^n + v_s, \int_0^s e_2(s,\tau, u_\tau^n + v_\tau)d\tau - G_2(s, u_s + v_s, \int_0^s e_2(s,\tau, u_\tau + v_\tau)d\tau\|$, $t \in [0,t_1]$
$\|(\Phi_i^2 u^n)(t) - (\Phi_i^2 u)(t)\| \leq M_1(t_{i+1} - s_i)\|G_2(s, u_s^n + v_s, \int_0^s e_2(s,\tau, u_\tau^n + v_\tau)d\tau - G_2(s, u_s + v_s, \int_0^s e_2(s,\tau, u_\tau + v_\tau)d\tau\|$, $t \in [s_i, t_{i+1}]$ and G_2, G_3 are continuous, we know that $\|(\Phi_i^2 u^n)(t) - (\Phi_i^2 u)(t)\| \to 0$ as $n \to \infty, u_n \to u$ which shows that Φ^2 is continuous.
(b) Φ^2 maps bounded sets into bounded sets in B_h''
It is sufficient to demonstrate that for any $R > 0$, there exists $R' > 0$ such that for each $u \in B_q = u \in B_h'' : \|u\|_{PC} \leq R$, we have $\|\Phi^2 u\|_{B_h''} \leq R'$. From the definition of $\Phi^2 u(t)$,
$\|(\Phi^2{}_i u)(t)\| \leq M_1 t_1 m_{G_2}(s)\Omega_{G_2}[(1+K_{e_2}) D_{1^*} Q + p_2]$, $t \in [o, t_1]$,
$\|(\Phi^2{}_i u)(t)\| \leq M_1(t_{i+1} - s_i)m_{G_2}(s)\Omega_{G_2}[(1+K_{e_2})D_{1^*} Q + p_2]$, $t \in [s_i, t_{i+1}]$,
$\|(\Phi^2{}_i u)(t)\| \leq M_1 T m_{G_2}(s)\Omega_{G_2} [(1+K_{e_2})D_{1^*} Q + p_2] \leq R, t \in [0,T]$.
Then we conclude that Φ^2 maps bounded sets into bounded sets.
(c)Finally, we exhibit that Φ^2 maps bounded sets into equicontinuous set.
For, interval $t \in [s_i, t_{i+1}]$, $s_i \leq l_1 \leq l_2 \leq t_{i+1}$, $i = 1,\dots,N$, for every $u(t) \in B_q$, by definition of $(\Phi^2{}_i u)(t)$ and hypotheses H(3) and H(6),
$\|(\Phi^2{}_i u)(l_2) - \Phi^2{}_i u)(l_1)\| \leq M_1(l_2 - l_1)\, m_{G_2}(s)\Omega_{G_2}[(1 + Ke_2 D_1^* q + p_2] + (l_1 - s_i)\|T(l_2 - s) - T(l_1 - s)\|\, m_{G_2}(s)\Omega_{G_2} + [(1+K_{e_2}) D_1^* q + p_2], t \in [s_i, t_{i+1}]$, as $l_1 \to l_2$ the right hand side tends to zero is equicontinuous.
Step 4: Φ_i^2 maps B_q into a precompact set in B_h''.
Now, we shall prove that Φ_i^2 is relatively compact in Φ_i^2. Obiviously Φ_i^2 is relatively compact in B_h'' for t = 0,0< $\varepsilon < t$, for $u \in B_q$. We define
$(\Phi_i^{2,\varepsilon} u)(t) = T(\varepsilon) \int_0^{t-\varepsilon} T(t-s-\varepsilon)\ G_2(s, u_s + v_s, \int_0^s e_2(s,\tau, u_\tau + v_\tau)d\tau\, ds$.
For the reason that T(t) is compact operator, $V_\varepsilon(t) = \{(\Phi_i^{2,\varepsilon} u)(t) : u \in B_q\}$ is relatively compact in X for every is relatively compact in X for every ε, for every $0 < \varepsilon < t$, for each $u \in B_q$,

$$\left\|(\Phi_i^2 u)(t) - (\Phi_i^{2,\varepsilon} u)(t)\right\| \leq \int_{t-\varepsilon}^t M_1 m_{G_2}(s)\Omega_{G_2}[(1 + K_{e_2})D_1^* q + p_2]ds \leq \epsilon[\Lambda] \to 0 \text{ as } \epsilon \to 0.$$

which are relatively compact sets arbitrarily close to the set $V_\epsilon(t), t > 0$ as a result $V_\epsilon(t)$ is relatively compact in X. From the above steps, it follows by the Krasnoselskii's fixed point theorem, we get that $\bar{\Phi}$ has at least one fixed point $u(t) \in B_h''$. With these, a fixed point of the operator $\bar{\Phi}$ is the mild solution u of the problem (1) - (3). This finishes the verification of the hypothesis.

3. Applications

To epitomize our hypothetical results, now, we consider the following INIDE with infinite delay of the structure

$$\frac{\partial}{\partial t}\left[u(t,x)\ \left\{\int_{-\infty}^{t} a_1(t,x,s-t)P_1\big(u(s,x)\big)ds + \int_0^t\int_{-\infty}^{s} K_1(s,\tau)P_2\big(u(\tau,x)\big)d\tau ds\right\}\right] = \frac{\partial^2}{\partial x^2}u(t,x)\ \left\{\int_{-\infty}^{t} a_2(t,x,s-t)Q_1\big(u(s,x)\big)ds + \int_0^t\int_{-\infty}^{s} K_2(s,\tau)Q_2\big(u(\tau,x)\big)d\tau ds + \left\{\int_{-\infty}^{t} a_3(t,x,s-t)Q_3\big(u(s,x)\big)ds + \int_0^t\int_{-\infty}^{s} K_3(s,\tau)Q_4\big(u(\tau,x)\big)d\tau ds\right\}\right\},\ X\in[0,\pi], t\in[o,b], t\neq t_k, \quad (8)$$

$$u(t,0) = 0 = u(t,\pi) = 0, t \geq 0 \quad (9)$$

$$u(t,x) = \varphi(t,x), t\in(-\infty,0], x\in[0,\pi], \quad (10)$$

$$u_i(t,x) = \int_{-\infty}^{t}\eta(t_i - s)u(s,x)ds, (t,x)\in(t_i,s_i)\times[0,\pi], \quad (11)$$

The prefixed real numbers are $0 < t_1 < t_2 < \cdots < t_n < b$ and $\varphi \in B_h$. Let $X = L^2[0,\pi]$ whose norm is $|\cdot|_{L^2}$ and determine the operator $A: D(A) \subset X \to X$ by $Aw = w''$ with the domain $D(A) = \{w\,E\,X: w, w'$ are absolutely continuous, $w'' \in X, w(0) = w(\pi) = 0\}$. Then $Aw = \sum_{n=1}^{\infty} n^2\langle w, w_n\rangle\ w_n, w \in D(A)$, in which $w_n(s) = \sqrt{\left(\frac{2}{\pi}\right)}\sin(ns)$, $n = 1,2,\ldots,$
is the orthogonal set of eigenvectors of A. $T(t)w = \sum_{n=1}^{\infty} e^{-n^2t}\langle w, w_n\rangle w_n$, for all $w \in X$, and every $t > 0$. For each $w \in X, (A)^{\frac{1}{2}}w = \sum_{n=1}^{\infty}(n)\langle w, w_n\rangle w_n$ and $\left\|(A)^{\frac{1}{2}}\right\| = 1$, $g([0,b]\times B_h\times L^2) \subseteq D(-A)^{\frac{1}{2}}$,
$||(A)^{\frac{1}{2}}G_1(t,\varphi_1,\mu_1)x - (A)^{\frac{1}{2}}G_1(t,\varphi_2,\mu_2)x \leq N_1\big[||\varphi_1-\varphi_2||_{B_h} + |\mu_1-\mu_2|\big]$ for $N_1 > 0$ and
$||(A)^{\frac{1}{2}}G_3(t,\varphi_1,\mu_1)x - (A)^{\frac{1}{2}}G_3(t,\varphi_2,\mu_2)x \leq N_2\big[||\varphi_1-\varphi_2||_{B_h} + |\mu_1-\mu_2|\big]$ for $N_2 > 0$.
The continuous functions $Q_i, i = 1,2$ are defines for each $(\theta, x) \in (-\infty,0]\times[0,\pi]$, and
$0 \leq Q_i\big(u(\theta)(x)\big) \leq \prod\left(\int_{-\infty}^{0} e^{2s}\|u(s,.)\|_{L^2}ds\right)$ and the continuous non decreasing function is defined as $\Pi: [0,\infty] \to (0,\infty)$ and we can take $\Omega_{G_2}(r) = \Pi(r)$ in (H3). Presently we can see that

$$||G_2(t,\varphi,H_2\varphi)||_{L^2} = \left[\int_0^{\pi}\left(\int_{-\infty}^{0} a_2(t,x,\theta)Q_1\big(\varphi(\theta)\big)(x)d\theta + \big(H_2\varphi(\theta)\big)(x)\right)^2 dx\right]^{\frac{1}{2}}$$

$$\leq\left\{\left[\int_0^{\pi}\big(m_1(t,x)\big)^2dy\right]^{\frac{1}{2}} + \left[\int_0^{\pi}\big(m_2(t)\big)^{\frac{1}{2}}dx\right]^{\frac{1}{2}}\right\}\Pi(\|\varphi\|_h)$$

$$\leq\left[\overline{m_1}(t) + \sqrt{\pi}\overline{m_2}(t)\right]\Pi(\|\varphi\|_h) \leq m(t)(\|\varphi\|_h)$$

Since $\Pi: [0,\infty] \to (0,\infty)$ is a continuous and non decreasing function, We can take $m(t) = \{\overline{m_1}(t) + \sqrt{\pi}\overline{m_2}(t)\}$. Along these lines the condition (H3) holds. Hence by theorem 3.1, we comprehend that the system (8)-(11) has a unique mild solution on I.

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