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Research paper

Numerical Solution of Time Fractional Gas Dynamics Equation

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Abstract

In this paper, we propose new technique for solving time fractional gas dynamics equation using Laplace-Adomian Decomposition method coupled with fractional complex transform. It is found that the tested examples reveal the present method is more reliable and does not require strong assumptions.

Keywords: *Gas Dynamics equation, Laplace-Adomian decomposition method, Fractional derivative, Fractional complex transform.*

1. Introduction

In this paper, we consider the time fractional nonlinear gas dynamics equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = -\mathrm{u}u_x + \mathrm{ku}(1-\mathrm{u}) + \mathrm{g}(x,t) \qquad \mathrm{t} > 0, \qquad 0 < \alpha \le 1 \quad (1)$$

Subject to the initial condition

$$u(x,0) = f(x) \quad (2)$$

Where $\mathrm{g}(x,t)$, the source term and k is suitable constant. The different types of gas dynamics equations in mathematical physics have been handled by Evans et al [1] and Elizarova [2]. Numerical solution of nonlinear gas dynamics equations was studied by many researchers. Das and Kumar [4] have applied differential transform method (DTM) to solve the nonlinear gas dynamics equations. Later, the same type of problems was solved by applying fractional homotopy analysis transform method by Rashidi et al [5]. In 2013, Jagdev Singh et al [19] have given the numerical solution of (1) by homotopy perturbation method coupled with Sumudu transform. Aminikanth et al [23] have solved the same type of problem by new homotopy perturbation method via Laplace transform. Recently, Tamsir et al [6] solved the similar type of problems by fractional reduced differential transform method (FRDTM).
For the past two decades, many researchers have paid attention in studying the coupling technique i.e., combination of two analytic methods to solve nonlinear problems The Laplace Adomian-Decomposition method is one such technique introduced by Khuri[16] used to solve the class of nonlinear differential equations. Later many researchers have handled this technique to investigate many nonlinear problems. For example, Wazwaz [13] used Adomian decomposition method coupled with Laplace transform technique for handling nonlinear Volterra integro-differential equations. The similar numerical technique was investigated by Susmita Paul et al [12] for solving Lokta-Volterra Prey Predator model which is dynamic in nature. Ongun [10] and Nurettin Dogan[11] proposed this technique for solving the mathematical model for HIV infection of $CD4^+$Cells. Later the same method has been applied for solving nonlinear fractional diffusion wave equation by Jafari [14] and nonlinear partial differential equation by Arun kumar et al [15].

2. Brief Analysis of Complex Fractional Transform

2.1. Jumaries Fractional Derivatives

Jumaries fractional derivative is a modified Riemann-Liouvilles derivative of order $'\alpha'$ defined as

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_0^t (t-\zeta)^{-\alpha-1}[f(\zeta)-f(0)]d\zeta & \alpha < 0 \\ \frac{1}{\Gamma(-\alpha)}\frac{d}{dt}\int_0^t (t-\zeta)^{-\alpha-1}[f(\zeta)-f(0)]d\zeta, & 0<\alpha<1 \\ \left[f^{(\alpha-m)}(t)\right]^{(m)}, & m \le \alpha \le m+1, \quad m \ge 1 \end{cases} \quad (3)$$

where $f: R \to R$ is a continuous function.
We list some important properties for the modified Riemann-Liouvilles derivatives as follows:

$$(i) D_t^\alpha(c) = 0, \alpha > 0, c \text{ is a constant} \quad (4)$$

$$(ii) D_t^\alpha[cf(t)] = cD_t^\alpha f(t), \alpha > 0 \quad (5)$$

$$(iii) D_t^\alpha t^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha}, \quad \beta > \alpha > 0 \quad (6)$$

$$(iv) D_t^\alpha[f(t)g(t)] = [D_t^\alpha f(t)]g(t) + f(t)[D_t^\alpha g(t)] \quad (7)$$

$$(v) D_t^\alpha[f(h(t))] = f_h'(h(t))D_t^\alpha h(t) \quad (8)$$

2.2 Fractional Complex Transform Method

The fractional complex transform is the simplest approach in fractional calculus which converts the fractional differential equation

into integer order differential equation making the solution procedure extremely simple [19, 20, 21, 22].The fractional complex transform was
first proposed by Zeng Biao Li et al [20] and is defined as

$$T = \frac{at^{\alpha}}{\Gamma 1+\alpha} \quad (9)$$

$$X = \frac{bx^{\beta}}{\Gamma 1+\beta} \quad (10)$$

$$Y = \frac{cy^{\gamma}}{\Gamma 1+\gamma} \quad (11)$$

$$Z = \frac{dz^{\delta}}{\Gamma 1+\delta} \quad (12)$$

Where a, b, c, d are unknown constants and $0 \le \alpha, \beta, \gamma, \delta \le 1$

3. Basic Idea of Laplace-Adomian Decomposition Method

In this section, we discuss the general procedure for numerical treatment of initial value problem of fractional gas dynamics equation (1)
Applying the complex transformation [31], we get the following nonlinear differential equation:

$$\frac{\partial u}{\partial T} = -\mathrm{u}u_x + \mathrm{ku}(1-\mathrm{u}) + \mathrm{g}(x,T) \quad T > 0, 0 < \alpha \le 1 \quad (13)$$

Applying Laplace transform on both sides of (13) and using differential property, we have

$$L(u) = \frac{1}{s}L(ku^2 - uu_x) - \frac{k}{s}L(u) + \frac{1}{s}L[g(x,T) + f(x)] \quad (14)$$

According to Adomian decomposition method, the solution u (x, T) can be expanded as an infinite series

$$u(x,,T) = \sum_{n=0}^{\infty} u_n(x,T) \quad (15)$$

Where u_n for $n \ge 0$ can be computed recursively and $ku^2 - uu_x$ is decomposed as

$$ku^2 - uu_x = \sum_{n=0}^{\infty} A_n \quad (16)$$

where $A_{n,}$ $n = 0,1,2$ ... are Adomian polynomials of $u_0, u_1, u_2, ...$ defined by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[k\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)^2 - \left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\left(\sum_{i=0}^{\infty}\lambda^i u_{i_x}\right)\right]_{\lambda=0} \quad (17)$$

The first few Adomian polynomials are given by

$$A_0 = ku_0^2 - u_0 u_{0_x} \quad (18)$$

$$A_1 = 2ku_0u_1 - u_0u_{1_x} - u_1u_{0_x} \quad (19)$$

$$A_2 = 2ku_0u_2 + ku_1^2 - u_0u_{2_x} - u_1u_{1_x} - u_2u_{0_x} \quad (20)$$

$$A_3 = 2ku_0u_3 + 2ku_1u_2 - u_0u_{3_x} - u_1u_{2_x} - u_2u_{1_x} - u_3u_{0_x} \quad (21)$$

And so on.
Now, applying inverse Laplace transform to both sides of (14) and using (15) and (16), we have

$$\sum_{n=0}^{\infty} u_n = L^{-1}\left[\frac{1}{s}L\left(\sum_{n=0}^{\infty} A_n\right)\right] - kL^{-1}\left[\frac{1}{s}L\left(\sum_{n=0}^{\infty} u_n\right)\right] + F(x,T) \quad (22)$$

Where $F(x,T) = L^{-1}\left[\frac{1}{s}L[g(x,T) + f(x)]\right]$
The Adomian Decomposition presents the recursive relation

$$u_0 = F(x,T) \quad (23)$$

$$u_{n+1} = L^{-1}\left[\frac{1}{s}L(A_n)\right] - kL^{-1}\left[\frac{1}{s}L(u_n)\right] n = 0,1,2,... \quad (24)$$

Utilizing the Adomian polynomials (18) - (21) in to the recursive relation (24), we get

$$u_1 = L^{-1}\left[\frac{1}{s}L(A_0)\right] - kL^{-1}\left[\frac{1}{s}L(u_0)\right] \quad (25)$$

$$u_2 = L^{-1}\left[\frac{1}{s}L(A_1)\right] - kL^{-1}\left[\frac{1}{s}L(u_1)\right] \quad (26)$$

$$u_3 = L^{-1}\left[\frac{1}{s}L(A_2)\right] - kL^{-1}\left[\frac{1}{s}L(u_2)\right] \quad (27)$$

4. Numerical Applications

Example 4.1. Consider the time fractional gas dynamics Eqn. (1) with $g(x,t) = 0, k = 1$ and $u(x,0) = f(x) = e^{-x} = u_0$
we utilizing the eqns.(23) and (25) – (27), we can obtain the first few components of series solution:

$$u_0 = e^{-x} \quad (28)$$

$$u_1 = Te^{-x} \quad (29)$$

$$u_2 = \frac{T^2}{2!}e^{-x} \quad (30)$$

$$u_3 = \frac{T^3}{3!}e^{-x} \quad (31)$$

Substituting the above components in to the Eqn. (15), we have

$$u(x,T) = e^{-x} + e^{-x}\left(\frac{t^{\alpha}}{\Gamma 1+\alpha}\right) + \frac{e^{-x}}{2}\left(\frac{t^{\alpha}}{\Gamma 1+\alpha}\right)^2 + \frac{e^{-x}}{6}\left(\frac{t^{\alpha}}{\Gamma 1+\alpha}\right)^3 + \cdots \quad (32)$$

That converges to the exact solution
$u(x,t) = e^{-x+t}$ when α = 1
which is the same solution as obtained by DTM[4], FHATM[5], HPTM[7], HPSTM[8], HAM [9], RDTM[17] and LTNHPM [23] .
Example 4.2. Consider the nonhomogeneous fractional gas dynamics equation (1) with $g(x,t) = -e^{t-x}$, $k = 1$ and $u(x,0) = f(x) = 1 - e^{-x}$
Utilizing the eqns. (23) and (25) – (27), we can obtain the first few components of series solution:

$$u_0 = 1 - e^{-x+T} \quad (33)$$
$$u_1 = u_2 = u_3 = \cdots = 0$$

Substituting the above components in to the Eqn. (15), we have

$$u(x,t) = 1 - e^{-x+\left(\frac{t^{\alpha}}{\Gamma 1+\alpha}\right)} \quad (34)$$

that converges to the exact solution
$u(x,t) = e^{-x+t}$ when α = 1
which is exactly the same as the result obtained by
DTM [4], FRDTM [6], RDTM[17]and LTNHPM [23].
Example 4.3. Consider the nonlinear fractional gas dynamics equation (1) with $k = loga$
$g(x,t) = 0$and $u(x,0) = f(x) = a^{-x}$
By utilizing the Eqns. (13) and (15) – (18), we can obtain the first few components of series solution given by

$$u_0 = a^{-x} \tag{35}$$

$$u_1 = Ta^{-x}loga \tag{36}$$

$$u_2 = \frac{a^{-x}(T\ loga)^2}{2!} \tag{37}$$

$$u_3 = \frac{a^{-x}(T\ loga)^3}{3!} \tag{38}$$

And so on
Finally, substituting the above components in to the Eqn. (15), we Have

$$u(x,T) = a^{-x} + a^{-x}loga\left(\frac{t^\alpha}{\Gamma 1+\alpha}\right) + \frac{a^{-x}}{2!}(loga)^2\left(\frac{t^\alpha}{\Gamma 1+\alpha}\right)^2 + \frac{a^{-x}}{3!}(loga)^3\left(\frac{t^\alpha}{\Gamma 1+\alpha}\right)^3 + \cdots \tag{39}$$

that converges to the exact solution
$u(x,t) = e^{-x+t}$ when $\alpha = 1$
which is exactly the same as the result obtained by FHATM [5], FRDTM [6], HPTM [7] and RDTM[17].

5. Conclusion

In this paper, Laplace-Adomian decomposition method has been successfully applied to solve time fractional gas dynamics equation via fractional complex transform. The present technique shows great potential to handle fractional nonlinear differential in simple manner, does not require linearization, discretization and small perturbation.

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