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Research paper

Equitable Power Domination Number of Certain Graphs

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Abstract

Let G be a graph with vertex set V, a set $S \subseteq V$ is said to be a power dominating set (PDS), if every vertex $u \in V - S$ is observed by some vertices in S using the following rules: (i) if a vertex v in G is in PDS, then it dominates itself and all the adjacent vertices of v and (ii) if an observed vertex v in G has $k > 1$ adjacent vertices and if $k - 1$ of these vertices are already observed, then the remaining one non-observed vertex will also be observed by v in G. The degree $d(v)$ of a vertex v in G is the number of edges of G incident with v and any two adjacent vertices u and v in G are said to hold equitable property if $|d(u) - d(v)| \leq 1$. In this paper, we introduce the notions of equitable power dominating set and equitable power domination number. We also derive the equitable power domination number of certain graphs.

Keywords: *Dominating set; Equitable dominating set; Power dominating set; Equitable power dominating set; Equitable power domination number.*

1. Introduction

All the graphs considered in this paper are simple, finite and undirected. One can refer [3, 2, and 4] regarding domination, power domination and degree equitable power domination in graphs. The concept of domination in graphs was introduced by Hedetniemi & Laskar in 1990, then the equitable domination and equitable domination number of certain graphs was studied by Swaminathan et al. In the year 1998, Haynes et al. introduced the notion of power domination in graphs and power domination number of graphs.
We first recall the definitions of domination, domination number, equitable domination, equitable domination number, power domination, power domination number. A dominating set of a graph $G = (V, E)$ is a set S of vertices such that every vertex v in $V - S$ has at least one neighbor in S. The minimum cardinality of a dominating set of G is called the domination number of G, denoted by $\gamma_d(G)$. The problem of finding a dominating set of minimum cardinality is an important problem that has been extensively studied. A dominating set $S \subseteq V$ in $G(V, E)$ is said to be equitable dominating set if for every $v \in V - S$ there exists an adjacent $u \in S$ such that the difference between degree of u and degree of v is less than or equal to 1, that is $|d(u) - d(v)| \leq 1$. The minimum cardinality of an equitable dominating set of G is called equitable domination number of G, denoted by $\gamma_{ed}(G)$. A set $S \subseteq V$ is said to be a power dominating set (PDS) of G if every vertex $u \in V - S$ is observed by some vertices in S using the following rules:

If a vertex v in G is in PDS, then it dominates itself and all the adjacent vertices of v.
If an observed vertex v in G has $k > 1$ adjacent vertices and if $k - 1$ of these vertices are already observed, then the remaining non-observed vertex will also be observed by v in G. The minimum cardinality of an power dominating set of G is called power domination number of G, denoted by $\gamma_{pd}(G)$.
In this paper, we introduce the notions of equitable power dominating set, equitable power domination number besides determining the equitable power domination number of certain graphs. We also investigate the equitable power domination number of middle graph of certain graphs.

2. Main Results

2.1. Equitable Power Domination Number of a Graph

Definition 2.1:
A power dominating set $S \subseteq V$ in $G = (V, E)$ is said to be equitable power dominating set, if for every vertex $v \in V - S$ there exists an adjacent vertex $u \in S$ such that the difference between degree of u and degree of v is less than or equal to 1, that is $|d(u) - d(v)| \leq 1$. The minimum cardinality of an equitable power dominating set of G is called the equitable power domination number of G, denoted by $\gamma_{epd}(G)$.

Note that equitable power dominating set S of a graph G is not unique.

Definition 2.2 [3]:
If any two distinct vertices of a graph G are adjacent, then G is said to be a complete graph and it is denoted by K_n.

Definition 2.3 [3]:
A Path P_n is a graph whose vertices can be listed in the order $v_1, v_2, \dots, v_n$ such that the edges are $\{v_i, v_{i+1}\}$ where $i = 1,2,\dots,n-1$.

Theorem 2.4: *Let* G *be a graph. Then* $\gamma_{pd}(G) \leq \gamma_{epd}(G)$.

Proof
Let G be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and for any $k < n$, let S be a power dominating set with $\gamma_{pd}(G) = k$. Let $G - S = \{w_1, w_2, \dots, w_t\}$. When constructing an equitable power dominating set S' of G, the following two cases arise:
Case (i): When $|d(v_i) - d(w_j)| \leq 1$ for $1 \leq i \leq n$ and $1 \leq j \leq t$
It is clear from the definition that $|S| = |S'|$, hence $\gamma_{pd}(G) \leq \gamma_{epd}(G)$.
Case (ii): When $|d(v_i) - d(w_j)| > 1$ for some $i, j, 1 \leq i \leq n$ and $1 \leq j \leq t$
Then one must choose w_j to be in S'. Hence $|S| < |S'|$. Therefore $\gamma_{pd}(G) \leq \gamma_{epd}(G)$.

Remark 1:
Let P_n be a Path. Then $\gamma_{pd}(P_n) = \gamma_{epd}(P_n)$ for $n \geq 1$.

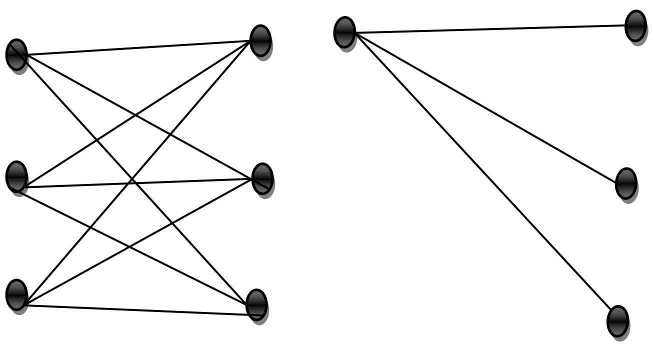
Remark 2:
Let C_n be a cycle. Then $\gamma_{pd}(C_n) = \gamma_{epd}(C_n)$ for $n \geq 3$.

Remark 3:
Let K_n be a complete graph. Then $\gamma_{pd}(K_n) = \gamma_{epd}(K_n)$ for $n \geq 1$.

Theorem 2.5:
Let G *be a graph. If* $|d(u) - d(v)| \geq 2$ *for every pair of adjacent vertices* u and v *in* G, *then* $\gamma_{epd}(G) = n$.

Proof
Let G be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ Let $d(v_1), d(v_2), \dots, d(v_n)$ denote the degrees of $v_1, v_2, \dots, v_n$ respectively. Without loss of generality, let $v_1 \in G$ with $d(v_1) = k$ be in an equitable power dominating set S of G with $|S| \neq \phi$. As all the adjacent vertices of v_1 are either of degree $\geq k+2$ or $\leq k-2$, the equitable property does not hold good. This is because $|d(v_1) - d(v_i)| \geq 2$, for some i, $2 \leq i \leq n$. Therefore these vertices $v_i's$, $2 \leq i \leq n$ which are adjacent to v_1 must be in S. If all the vertices of G are observed, then we are through. If not, let v_k be the next non-observed vertex of G with degree $d(v_k) = m$ to be added in S. Now all the other adjacent vertices of v_k are either of degree $\geq m+2$ or $\leq m-2$ and the equitable property does not hold good as it was the case of vertex v_1. Hence all the adjacent vertices of v_k will be in S. Continuing the same process for the remaining non-observed vertices of G, we obtain that $|S| = n$. Thus $\gamma_{epd}(G) = n$.

Remark 4:
Let G be a graph with $\gamma_{epd}(G) = k$ and H be a subgraph of G. Then equitable power domination number of H need not be less than or equal to k.
For example see Fig.1, equitable power domination number of the complete bipartite graph $G = K_{3,3}$ and its sub graph $H = K_{1,3}$.



(a) $\gamma_{epd}(G) = 2$ **(b)** $\gamma_{epd}(H) = 4$

Fig.1: Equitable power domination number of G and its subgraph H

2.2. Equitable Power Domination Number of Certain Classes of Graphs

In this section we determine the equitable power domination number of certain well-known graphs such as paths, cycles, complete graphs etc.

Definition 2.4 [1]:
A complete bipartite graph, denoted $K_{m,n}$, is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y.

Theorem 2.6:
Let C_n, $n \geq 3$ *be a cycle. Then* $\gamma_{epd}(C_n) = 1$.

Proof.
Let C_n be a cycle on n vertices with vertex set $V(C_n) = \{v_1, v_2, \dots, v_n\}$. Without loss of generality, let for any $i, 1 \leq i \leq n$ let $S = \{v_i\}$. As C_n being the 2-regular graph, it is easy to see that v_i equitably power dominates v_{i-1} and v_{i+1}. Moreover v_{i-1} and v_{i+1} equitably power

dominates its neighbors namely v_{i-2}. and v_{i+2}, respectively. This process continues until all the vertices v_i, $1 \le i \le n$ are observed. Therefore $\gamma_{epd}(C_n) = 1$.

Definition 2.5 [10]:

Any two distinct vertices of a graph G are adjacent then G is said to be complete graph and it is denoted by K_n.

Theorem 2.7:

For a complete graph K_n, $\gamma_{epd}(K_n) = 1$.

Proof.

Let K_n be a complete graph on n vertices with vertex set $V(K_n) = \{v_1, v_2, \dots, v_n\}$. It is clear from Definition 2.5 that $d(v_i) = n-1$ and also $|d(v_i) - d(v_{i-1})| < 1$ for all $1 \le i \le n$. Therefore it is enough to choose any one of the vertices of K_n to be in equitable power dominating set S of G. This is because the chosen vertex, say v_1, equitably power dominates all the other vertices of K_n. Hence $\gamma_{epd}(K_n) = 1$.

Theorem 2.8:

Let $K_{m,n}$, $m, n \ge 2$ be a a complete bipartite graph. Then

$$\gamma_{epd}(K_{m,n}) = \begin{cases} m+n & if\ |m-n| \ge 2 \\ 2 & if\ |m-n| < 2 \end{cases};$$

Proof.

Let $K_{m,n}$ be the given complete bipartite graph with vertex set $V(K_{m,n}) = V_1 \cup V_2$, where $V_1 = \{u_1, u_2, \dots, u_n\}$ and $V_2 = \{v_1, v_2, \dots, v_n\}$ are the two partition sets of $K_{m,n}$. Then the following two cases arise:

Case 1:
$|m-n| \ge 2.$

It is clear from Definition 2.4 that $d(u_i) = n$ for every u_i in V_1 and $d(v_j) = m$ for every v_j in V_2. Since $|m-n| \ge 2$, equitable property does not hold good between the vertices of partition V_1 and V_2. Then all the vertices of V_1 and V_2 must be chosen to get the desired equitable power domination set S.

Case 2
: $|m-n| < 2$

Without loss of generality, let $S = \{u_1, v_1\}$. Choosing one vertex from each partition is well enough to get the required equitable power dominating set S. Therefore, $\gamma_{epd}(K_{m,n}) = 2$.

Definition 2.6 [10]:

The wheel graph with n spokes, $W_{1,n}$ is the graph that consists of a cycle C_n and one additional vertex, say u, that is adjacent to all the vertices of the cycle C_n.
It is interesting to note that $\gamma_{epd}(W_{1,3}) = \gamma_{epd}(W_{1,4}) = 1$ as choosing any one rim vertex or the central vertex in $W_{1,n}$ gives the required equitable power dominating set S. For $n \ge 5$, we have the following theorem.

Theorem 2.9:

Let $W_{1,n}$ be a wheel graph. Then $\gamma_{epd}(W_{1,n}) = 2$ for $n \ge 5$.

Proof.

Let $W_{1,n}$ be the given wheel graph with the vertex set $V(W_{1,n}) = \{v_0, v_1, v_2, \dots, v_n\}$ where v_0 is the central vertex and v_i, $1 \le i \le n$ are the rim vertices. Since the degree of the central vertex v_0 is n and the degree of each vertex in the rim is 3, it is clear that $|d(v_0) - d(v_i)| > 2$ for $1 \le i \le n$. Hence the central vertex v_0 and anyone of the rim vertices must be chosen to form an equitable power dominating set S. This is because a vertex from the rim, say v_1, paves a way to equitably power dominates all the vertices in the rim using the similar argument that of Theorem 2.6 and v_0 dominates itself. Therefore $\gamma_{epd}(W_{1,n}) = 2$.

Definition 2.7 [12]:

The gear graph G_n is a graph obtained from the wheel graph $W_{1,n}$ by subdividing each edge of the outer n −cycle of $W_{1,n}$ just once. One can easily obtain that $\gamma_{epd}(G_3) = \gamma_{epd}(G_4) = 1$, for $n \ge 5$.

Theorem 2.10:

Let G_n be a gear graph. Then $\gamma_{epd}(G_n) = 2$, for $n \ge 5$.

Proof.

The proof is similar to the proof of Theorem 2.9.

Definition 2.8 [14]:

The n-ladder graph can be defined as $P_2 XP_n$ where P_n is a path. It is therefore equivalent to the $2 \times n$ grid graph.

We label the vertices of the first and second copy of P_n as $\{v_1, v_2, \dots, v_n\}$ and $\{v_1', v_2', \dots, v_n'\}$ respectively. We call a set $W = \{v_1, v_2, v_1', v_2', v_{n-1}, v_{n-1}', v_n, v_n'\}$.

Theorem 2.11:

For the $n-$ ladder graph $P_2 XP_n$, $\gamma_{epd}(P_2 XP_n) = \begin{cases} 1 & when\ S = \{v \in W\} \\ 2 & otherwise \end{cases}$.

Proof.

Let $P_2 XP_n$ be the given n-ladder graph as defined in Definition 2.8. Note that $|G| = 2n$. Now one can see that $|d(v_i) - d(v_{i+1})| \le 1$, $|d(v_i') - d(v_{i+1}')| \le 1$ for $1 \le i \le n-1$, and $|d(v_i) - d(v_i')| < 1$ for $1 \le i \le n$. Let S be the equitable power dominating set of G. Now the following two cases arise:

Case1:
$when\ S = \{v; v \in W\}$

Without loss of generality let $S = \{v_1\}$. It is quite easy to see that v_1 equitably power dominates v_2 and $v_1{}'$, $v_1{}'$ equitably power dominates $v_2{}'$. Moreover v_2 equitably power dominates v_3, v_3 equitably power dominates v_4 and so on. Also $v_2{}'$ equitably power dominates $v_3{}'$, $v_3{}'$ equitably power nates $v_4{}'$ and so on. Therefore it is enough to have only one vertex $v \in W$ in S. Thus $\gamma_{epd}(P_2 X P_n) = 1$.

Case 2:
when $S \subseteq V - W$
Consider $S = \{v_3\}$, then v_3 equitably power dominates v_2, v_4 and $v_3{}'$. Now there are two non-observed vertices for the already observed vertices v_2, v_4 and $v_3{}'$. None of them equitably power dominates any other vertices. Hence we must choose one more vertex to be in S say $v_3{}'$. Now v_3 and $v_3{}'$ equitably power dominates every vertices of $G = P_2 X P_n$. Hence $\gamma_{epd}(P_2 X P_n) = 2$.

Definition 2.9 [13]:
The n-barbell graph is a simple graph obtained by connecting two copies of a complete graph K_n by a bridge.

Theorem 2.12:
Let G be a n-barbell graph. Then $\gamma_{epd}(G) = 2$ for $n > 2$.

Proof.
Let G be the given n-barbell graph with vertex set V(G) = $\{v_1, v_2, \dots, v_n, v_1{}', v_2{}', \dots, v_n{}'\}$. By the Theorem 2.7, it is clear that $\gamma_{epd}(K_n) = 1$. Now S contains exactly one vertex from V, say v_1, then this vertex v_1 equitably power dominates all the adjacent vertices of v_1. And also v_1 equitably power dominates the vertex which is the end vertex of the bridge connecting the second copy to the first, say $v_1{}'$ which has $n - 1 \geq 2$ non-observed adjacent vertices. Now v_1' cannot equitably power dominates any of its adjacent vertices because it violates the Definition of PDS. Therefore we must choose at least one vertex from the second copy of complete graph K_n. Hence $\gamma_{epd}(G) = 2$.

Definition 2.10 [2]:
A windmill graph, denoted $W_d(k, n)$, consists of n copies of complete graph K_k connected to a common vertex of degree $n(k-1)$.

Theorem 2.13:
Let $W_d(k, n)$ be the windmill graph. Then $\gamma_{epd}(W_d(k, n)) = n + 1$ for $n, k \geq 2$.

Proof.
It is clear from the Definition 2.10 that $W_d(k, n)$ consists of n-copies of K_k connecting to the common vertex with degree $n(k-1)$, say v_0.

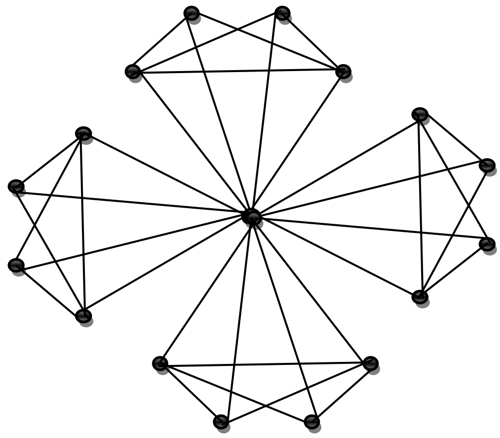


Fig. 2: Windmill graph $W_d(5,4)$

By the Theorem 2.7 it is clear that $\gamma_{epd}(K_n) = 1$. Therefore one has to choose at least one vertex for each copy of K_k to be in S. And these n vertices do not equitably power dominate the common vertex v_0 because the degree difference between the common vertex and any other vertices of n copies of K_k exceeds 2 which clearly violates the required equitable property that is $|d(v_0) - d(v) ; v \in V - v_0| \geq 2$. Thus one has to choose v_0 to be in the equitable power dominating set S to get the desired result. Therefore $\gamma_{epd}(W_d(k, n)) = n + 1$.

2.3. Equitable Power Domination Number of Middle Graph of Certain Graphs

Definition 2.11 [1]:
The middle graph $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices is adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it.

Theorem 2.14:
Let P_n be a path. Then $\gamma_{epd}(M(P_n)) = n$ for $n \geq 3$.

Proof.
Let P_n be a path on 'n' vertices $u_1, u_2, \dots, u_n$ and $n - 1$ edges namely $e_1, e_2, \dots, e_{n-1}$. The middle graph of a path P_n, denoted $M(P_n)$ is defined as follows: $V(M(P_n)) = V(P_n) \cup E(P_n) = V_1 \cup V_2$, where $V_1 = \{u_1, u_2, \dots, u_n\}$ and $V_2 = \{e_1, e_2, \dots, e_{n-1}\}$ and $E(M(P_n)) = E_1 \cup E_2$, where $E_1 = \{e_i e_{i+1}, 1 \leq i \leq n-2\}$ and $E_2 = \{u_i e_j ; 1 \leq i \leq n, 1 \leq j \leq n-1$ whenever $|i - j| \leq 1\}$. It is interesting to see that there are $n - 2$ vertices are of degree 2 in V_1 of $M(P_n)$ and the remaining two vertices namely u_1 and u_2 are of degree one. Moreover there are $n - 3$ vertices are of degree one in V_2 of $M(P_n)$ and the remaining two vertices namely e_1 and e_{n-1} are of degree 3. In order to obtain equitable power dominating set S of $M(P_n)$, the vertices u_1 and u_n must be in S as there are no adjacent

vertices of u_1 and u_2 satisfy the required equitable property. Similarly vertices u_j, $3 \leq j \leq n-2$ must be in S as there are no adjacent vertices yield the equitable property. Now one can choose either u_2 or e_1 to be in S. So we choose u_2 to be in S. Now u_2 equitably power dominates e_1, e_1 equitably power dominates e_2, e_2 equitably power dominates e_3 and so on. This is because $|d(e_i) - d(e_j)| \leq 1$. Similar process holds good for choosing u_{n-1} or e_{n-1} to be in S. We choose u_{n-1} to be in S. Hence we get $S = \{u_1, u_2, \dots, u_n\}$. Therefore $\gamma_{epd}(M(P_n)) = |S| = n$. Hence the theorem.

Theorem 2.15:
Let C_n be a cycle. Then $\gamma_{epd}(M(C_n)) = n+1$ for $n \geq 3$.

Proof.
Let C_n be a given cycle on n vertices $u_1, u_2, \dots, u_n$ and n edges $e_1, e_2, \dots, e_n$. The middle graph of cycle C_n, denoted $M(C_n)$ is defined as follows: $V(M(C_n)) = V(C_n) \cup E(C_n) = V_1 \cup V_2$, where $V_1 = \{u_1, u_2, \dots, u_n\}$ and $V_2 = \{e_1, e_2, \dots, e_n\}$. $E(M(C_n)) = E_1 \cup E_2$, where $E_1 = \{e_i\, e_{i+1};\ 1 \leq i \leq n-1\}$ and $E_2 = \{u_i e_j;\ 1 \leq i \leq n,$ whenever $|i-j| \leq 1\}$. Note that $|V(M(C_n))| = 2n$. Then one can observe that there are n vertices are of degree 2 and the other n vertices are of degree 4. Now to obtain the equitable power dominating set S of $M(C_n)$, we must choose u_i's, $1 \leq i \leq n$ which are of degree 2. This is because there are no adjacent vertices of u_i's, $1 \leq i \leq n$ satisfy the equitable property. Next there are n non-observed vertices are of degree 4. Choosing any one of these n vertices namely $e_1, e_2, \dots, e_n$ to be in S gives rise to the required equitable power dominating set S of $M(C_n)$. This is because any one of the $e_i's, 2 \leq i \leq n-1$ of $M(C_n)$ equitably power dominates its adjacent vertices e_{i-1} and e_{i+1}. Now e_{i-1} and e_{i+1} equitably power dominates its non-observed neighboring verrtices. Hence $S = \{u_1, u_2, \dots, u_n, e_i;\ 1 \leq i \leq n\}$; $|S| = n+1$.
Thus $\gamma_{epd}(M(C_n)) = n+1$.

Lemma 2.1:
Let $K_{1,n}$ be a star. Then $M(K_{1,n})$ contains K_{n+1} as its sub graph for all $n \geq 1$.

Proof.
Let $K_{1,n}$ be a star graph with the common vertex u and pendant vertices $u_1, u_2, \dots, u_n$. Let $e_1, e_2, \dots, e_n$ be the edges of $K_{1,n}$. Note that all the edges of $K_{1,n}$ are mutually adjacent to each other and so by the definition of middle graph there will be an edge between every pair of edges e_i's ; $1 \leq i \leq n$ of $K_{1,n}$ in $M(K_{1,n})$. Moreover the common vertex u incidents with every other edges of $K_{1,n}$. So again by the definition of middle graph, there will be an edge between the common vertex u and newly added vertices (which were originally edges of $K_{1,n}$) of $M(K_{1,n})$. Thus the common vertex u together with newly added vertices of $M(K_{1,n})$ forms a complete graph K_{1+n} which is a sub graph of $M(K_{1,n})$. Hence the lemma.

Theorem 2.16:
Let $K_{1,n}$ be a star. Then $\gamma_{epd}(M(K_{1,n})) = n+1$ for $n \geq 2$.

Proof.
Let $K_{1,n}$ be the given star graph with pendant vertices $u_1, u_2, \dots, u_n$ and the common vertex u. The edges of $K_{1,n}$ are $e_1, e_2, \dots, e_n$. The middle graph of a star graph $K_{1,n}$, denoted $M(K_{1,n})$ is defined as follows: $V\left(M(K_{1,n})\right) = V_1 \cup V_2 \cup \{u\}$, where $V_1 = \{u_1, u_2, \dots, u_n\}$ and $V_2 = \{e_1, e_2, \dots, e_n\}$. And $E(M(K_{1,n})) = E_1 \cup E_2 \cup E_3$, where $E_1 = \{e_i\, e_j;$ for all $i \neq j\}$, $E_2 = \{ue_i;\ 1 \leq i \leq n\}$ and $E_3 = \{u_i e_i;\ 1 \leq i \leq n\}$. It is interesting to note that all the pendant vertices of $K_{1,n}$ are of degree one in $M(K_{1,n})$. The remaining common vertex u together with the newly added vertices of $M(K_{1,n})$ forms a complete graph K_{1+n} by the Lemma 2.1. By Theorem 2.7, it is enough to choose any one of the vertices of K_{1+n} in $M(K_{1,n})$ to be in S. Moreover all the pendant vertices $u_1, u_2, \dots, u_n$ must be in S because there are no adjacent vertices satisfying the required equitable dominating property. Hence $S = \{u, u_1, u_2, \dots, u_n\}$ and $|S| = n+1$. Hence the proof.

3. Conclusion

In this paper the notion of equitable power domination in graphs has been introduced besides investigating the equitable power domination number of various classes of graphs. Establishing equitable power domination number of other classes of graphs is open and this is for future work.

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