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Research paper

Grundy Number of Some Chordal Graphs

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Abstract

For a given graph *G* and integer *k*, the Coloring problem is that of testing whether *G* has a *k*-coloring, that is, whether there exists a vertex mapping $c : V \rightarrow \{1, 2, . . .\}$ such that $c(u) \neq c(v)$ for every edge $uv \in E$. For proper coloring, colors assigned must be minimum, but for Grundy coloring which should be maximum. In this instance, Grundy numbers of chordal graphs like Cartesian product of two path graphs, join of the path and complete graphs and the line graph of tadpole have been executed

Keywords: *Grundy coloring; Grundy number: Ladder graph: Tadpole graph: wheel graph: Fan graph;Double fan graph.*

1. Introduction

Graph theory is a study of Mathematical structures of a graph which deals with various life problems such as electrical networks, mobile communications, logistics etc. A graph is a non-empty set of vertices and an arc connected by the given vertices is called edges. It is said to be simple if it contains no loops or multiple edges and connected if all its vertices are connected by edges. A Graph is said to be perfect if the chromatic number of its induced sub graph is equal to its clique number. A chordal graph is an example of a perfect graph which has a chord with end vertices inside the cycle C which is not an edge in the cycle.

Douglas B West in his book [6] defines graph and graph coloring in the following manner:

Let G = [V,E] be a graph where V is a set of vertices or nodes and E is a set of arcs or Edges. A k-coloring of G is a labeling f: $V(G)\rightarrow \{1,2,\ldots k\}$.The labels are colors; the vertices with color i are a color class. A k-coloring f is proper if $x\leftrightarrow y$ implies $f(x) \neq f(y)$.A graph G is k-colorable if it has a proper k-coloring. b-coloring of a graph G by k-colors is a proper vertex coloring such that there is a vertex in each color class, which is adjacent to at least one vertex in every other color class as in page.no.14 [1].

Jensen.et.al state that Grundy coloring of order k of a graph G is a k-coloring of G with colors 1,2…k such that for each vertex x the color of x is the smallest positive integer not used as a color on any neighbor of x in G. The Grundy number $\Gamma(G)$ is the largest integer k for which G has a Grundy coloring of order k.is defined by Christen and selkow as in page no.170 in [11].

Chromatic and the achromatic numbers of a fall coloring are bounded by using Grundy number which was introduced by Christen and selkow in 1979. Recently, bounds for Grundy number of power graphs of various graphs [17] were interpreted by Germain and Kheddouci. Parks.et.al determined the Grundy coloring of chessboard graphs [18].Inequalities and results for Grundy number of line graphs, Bipartite graphs and tree are executed by zaker.et.al[16] bounds for the Grundy number of lexigographic, direct,strong and Cartesian product of graphs were explored in[19] by Victor Campus.et.al. Exact value of this parameter for *n*-dimensional meshes are explored by Brice effantin.et.al [3].Also an algorithm has been presented to generate all graphs for a given Grundy number.

2. Preliminaries

Definition2.1

The Cartesian product $G_1 \times G_2$ of two graphs G_1 and G_2 is the simple graph with $V_1 \times V_2$ .As its vertex set and two vertices (u_1, v_1) and (u_2, v_2)are adjacent in $G_1 \times G_2$ if, and only if , either $u_1 = u_2$ and v_1 *is adjacent to* v_2 in G_2 , u_1 *is adjacent to* u_2 in G_1 and $v_1 = v_2$.[10]

Definition2.2

Let G_1 and G_2 be vertex disjoint graphs. Then the join, $G_1 \vee G_2$ of G_1 andG_2 is the sub graph of G_1+G_2 in which each vertex of $G_1$is adjacent to every vertex of G_2.[2];

Definition.2.3 [2]:

Two vertices that are not adjacent in a graph G are said to be independent.

A set S of vertices is independent set if no two vertices of S are not adjacent in G.

The vertex independence number or simply the independence number, of a graph G, denoted by $\alpha(G)$ is the maximum cardinality among the independent sets of vertices of G

Definition.2.4 [2]:

A clique in a Graph G is a complete sub graph of G

Definition.2.5: [2]

The clique cover number $\theta(G)$ is the minimum number cliques needed to cover V(G).

Definition.2.6: [6]
The clique number of a graph G, written ω(G), is the maximum size of the clique in G.

Definition.2.7. [8]
Ladder graph: A ladder graph L_n of order n is a Cartesian product of a path P_n and complete graph K_1.

Definition.2.8:
Fan graph[20] If $G=P_n$ is a path graph with n vertices and $H=K_1$ is a complete graph with one vertex then

$P_n \vee K_1=F_{1,n}$ is a fan graph.

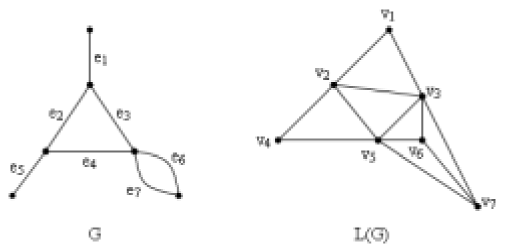
Definition.2.9:[20]
Double fan graph F_{n+2} is obtained by joining P_n and K_2 as in Page.no.61

Definition.2.10:
Tadpole graph $T_{n,k}$ is a graph obtained by joining a cycle graph C_n to a path of length k.[21].

Definition.2.11.Line graph:[2]

Line graph L(G) can be constructed in the following way:
The vertex set of L(G) is in 1-1 correspondence with the edge set of G and two vertices of L(G) are joined an edge if and only if the corresponding edges of G are adjacent in G.

Example1.
The following figure shows a graph and its line graph:



3. Main Results

3.1 Grundy Number of Ladder Graph

Theorem.3.1.1
If G be a ladder graph L_n of order 2n where $n \in N$, then the Grundy number is four.

Proof:
Let$G=L_n$ for $n \in N$, $V(G) = \{ v_1,v_2,\ldots v_n,u_1,u_2\ldots u_n\}$, $n(G) = 2n$.

We show that $\Gamma(G) = 4$ by considering the following cases.
Case.1: when n = 6k-3,
For Grundy coloring, let us define a map $f:V(G) \rightarrow \{0,1,2,3\}$ such that $f(u_i) = 0$ when i is odd,
$f(u_i) = 1$ when i is even,
$f(v_i) = 3$ when i is even.
Here $\Gamma(G) = 4$.
Case.2: when n = 6k+2,
Let us define a coloring $f: V(G) \rightarrow \{0,1,2,3\}$ such that $f(u_i) = 0$ when i is odd,
$f(u_i) = 1$ when i is even,
$f(v_i) = 2$ when i is odd,
$f(v_i) = 3$ when i is even.
Hence $\Gamma(G) = 4$.Since the ladder graph is the Cartesian product of K_2 and path graph P_n, the conjecture explored by Balogh.et.al goes as $\Gamma(K_2 \square H) \leq 2\,\Gamma(H) \leq 2.\Gamma(P_n)$ where H is the path P_n. According to Brice Effantin.et.al Grundy number of P_n is 3 when $n \geq 4$.Hence the Grundy number of Ladder graph is less than or equal to 6.But the upper bound for Grundy number is its maximum degree plus one. Hence the Grundy number of Ladder graph is 4.

Theorem.3.1.2:
Grundy number of Ladder graph is equal to its girth.

Proof:
Let G be a Ladder graph. By Theorem.3.1, $\Gamma(G) = 4$.

Adjacency matrix of the Graph G=
$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Here the girth of G=4.
Hence the grundy number of ladder graph is equal to its girth.

3.2 Grundy Coloring of Tadpole Graph:

Theorem.3.2.1
Let G=T(n,m) for m,n>5,then , $\Gamma(G) = 4$.

Proof.
Let $G = V[T(n, m)] = \{v,v_1,v_2,\ldots v_{n-1},u_1,u_2\ldots u_{m-1}\}$ and cardinality of $V[T(n,m)]= m+n-1$.Let us define a map $f:V(G) \rightarrow \{0,1,2,3\}$ for Grundy coloring which can be explored in the following cases;.
Case.1 Let n=6k-1,m=6k+1
To assign the colors to the vertices of a graph,a map $f:V(G) \rightarrow \{0,1,2,3\}$ can be defined such that
$f(v)=0$
$f(v_i) =1$ for i=1 to n-1 where n is even
$f(v_i) =2$ for i=1 to n-1 where n is odd
$f(u_i) =1$ for i=1 to m-1 where m is even
$f(v_i) =3$ for i=1 to m-1 where m is odd
Hence $\Gamma(G) =4$.
Suppose $\Gamma(G) =3$.As the graph G is obtained by joining the cycle and path graph, its Grundy number is 3 according to the results in page No.8 in [3]but its maximum degree is three and the upper bound for Grundy number is $\Delta(G)$+1.Hence $\Gamma(G) =4$.
Case.2 Let n=6k+1,m=6k+2
For Grundy coloring of the graph G, a map $f: V(G) \rightarrow \{0,1,2,3\}$ can be defined such that $f(v)=0$
$f(v_i) =1$ for i=1 to n-1 where n is even
$f(v_i) =2$ for i=1 to n-1 where n is odd
$f(u_i) =1$ for i=1 to m-1 where m is even
$f(v_i) =3$ for i=1 to m-1 where m is odd
Hence $\Gamma(G) =4$.

Theorem.3.2.2
Tadpole graph G=T(n,m) needs four colors for Grundy coloring if and only if it is a planar graph.

Proof:
According to Theorem.3.3, Tadpole graph G= T(n,m) needs four colors for Grundy coloring. Since G contains a cycle and path, a cycle is the block of the given
graph G which is planar. According to the Theorem on page no.154 in the book [2] written by Balakrishnan.et.al, and $\delta(G) = 1$,the graph G is planar.
Conversely, the given graph G does not contain $K_{3,3}$ [Douglas B west]Hence it is a planar graph. As every planar graph is four colorable., tadpole graph is four colorable.

Theorem.3.2.3

For m,n >3,Let G =$P_n□P_m$ and H=C_n U P_{m-1},then Γ(G) = Γ(H) =4.

Proof:

Let G be the Cartesian product of the paths P_n and P_m which has a girth of length 4 and a clique of order 2 and H be the union of the cycle C_n and path P_{m-1} have a girth of length n. Since a triangle-free graph has a girth $\geq$ 4,δ(G)$\leq$ 2,Δ(G)=3 and it does not have a cycle and induced sub graph of order 3, both G and H are triangle-free graphs. According to Grotezsch Theorem, the chromatic number triangle-free and planar graphs are three. But Grundy color is maximum and the upper bound for the Grundy number is its maximum degree plus one. Hence the proof.

3.3 Grundy Coloring of Line Graph of Tadpole Graph

Theorem.3.3.1

Grundy number of line graph of Tadpole graph is four colorable.

Proof:

Let G =T(n,m) be a Tadpole graph. The graph G can be constructed by joining a cycle C_n and a path P_{m-1} having a vertex in common whereas the line graph of tadpole graph can be constructed by considering the edges of the graph as vertices and associating the vertices which are common. Line graph of tadpole graph forms a cycle with a chord with vertices w_i for i = 1 to n which is connected with a path with x_j vertices for j=1 to m-2 .n[L(G)]= n+m-2, ω(G)=3, Δ(G)=3.In accordance with the proposition.2. in [9],the grundy number of cycle is 3 if n≥4 and the grundy number of the path is 3 if n≥4. In the case of line graph of a graph G, Grundy number of L(G) is 4 as it is a chordal graph and the Grundy number of chordal graph exceeds its clique number.

Color set {0,1,2,3} is assumed for Grundy coloring, vertices of the clique are assigned by 0,1,2 and the neighboring vertex of the common vertex which connects the cycle and the path is assigned by3.Remaining vertices in the graph are assigned by the existing colors according to the procedure of proper coloring.

Hence the proof.

4. Conclusion

Grundy number of Chordal and Planar graphs are discussed here. Also grundy number of some specified graphs which have the properties of chordal graphs are investigated. Line graph of planar graph has been discussed. Grundy coloring of Cartesian product of two path graphs is being executed in accordance with its Girth.

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