International Journal of Engineering & Technology, 7 (4) (2018) 6351-6356

International Journal of Engineering & Technology

Website: www.sciencepubco.com/index.php/IJET
doi: 10.14419/ijet.v7i4.16094
Research paper

Valuation Dimension of Ring $\mathbb{Z}_n$ Using Python

Arifin S., Garminia H. [1*]

[1] *Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jalan Ganesha No.10, Bandung, Jawa Barat, Indonesia 40132,*
E-mail address: samsul.arifin@s.itb.ac.id, garminia@math.itb.ac.id
[*] *Corresponding author E-mail: samsul.arifin@s.itb.ac.id*

Abstract

We counting valuation dimension of the ring $\mathbb{Z}_n$. Recently, Ghorbani and Nazemian introduced the notion of a valuation dimension of a commutative rings that measures of how far a commutative ring deviates from being valuation. They have proven that an Artinian ring has finite valuation dimension. Therefore, the ring $\mathbb{Z}_n$ has finite valuation dimension. In this article, some methode and a tool to determine the valuation dimension of the ring $\mathbb{Z}_n$ will be provided using Python.

Keywords: *Uniserial dimension, valuation dimension, ring $\mathbb{Z}_n$, Python.*

1. Introduction

Valuation rings can be identified through a collection of all ideals that are totally ordered by inclusion. The notion of valuation ring was studied by Atiyah [1], Faith [5], Faith [6], Kaplansky [10] and Manis [12]. A measure of how far a commutative ring deviates from being valuation is called valuation dimension. The notion of valuation dimension was introduced by Ghorbani [8].

In module theory, uniserial module can be identified through a collection of all submodules that are totally ordered by inclusion. The properties of uniserial module has been studied by Facchini [4], Eisenbud [3], Lam [11] and Warfield [16]. A measure of how far a module deviates from beeing uniserial is called uniserial dimension. The notion of uniserial dimension was first introduced by Nazemian [13].

Properties of rings with valuation dimension and modules with uniserial dimension have been studied. Nazemian [13] show that for any ring R and ordinal number α, there exists R-module with uniserial dimension α. Furthermore, a commutative ring R is Noetherian if and only if any finitely generated R-module has uniserial dimension. In other literature, Ghorbani [8] show that every Noetherian ring has valuation dimension. Furthermore, every Artinian ring, which also Noetherian ring, has finite valuation dimension. However, not all commutative rings with finite valuation dimension are Artinian. In fact, any ring that has finite valuation dimension are semiperfect (see Ghorbani [8]). But the converse is not always true. They gave the counter example, that is there exist semiperfect Noetherian ring with infinite valuation dimension.

The above facts lead to a question what is the valuation dimension of ring $\mathbb{Z}_n$, and it is the question beeing adressed by this paper. A review of the ring with valuation dimension begin in Session 2. The main result, the methods and a Python's code to determine the valuation dimension of ring $\mathbb{Z}_n$ in Session 3. Examples and discussion about the output of program is in Session 4.

2. Review of Rings With Finite Valuation Dimension

In this paper, all ring are assumed that commutative with unity. The development of the notion of valuation dimension of rings can not be separated from the notion of uniserial dimension of modules (see Nazemian [13]). This notion is related to ordinal numbers which is developed by transfinite induction (see Stoll [15]). For each ordinal number $\alpha \geq 1$ and ring R, collection of R-modules ζ_α can be generated by transfinite induction, starts from ζ_1, using the collection of R-modules ζ_β for each ordinal number $\beta < \alpha$. Especially, ζ_1 contains all non-zero uniserial R-modules.

Definition 2.1. Nazemian [13]. For any ordinal number $\alpha \geq 1$ and R-module M, defined:

1. $\zeta_1 = \{M | M \, uniserial \, R-modules\}$
2. $\zeta_\alpha = \left\{ M | M \, R-module, (\forall N < M) \left(M/N \ncong M \Longrightarrow M/N \in \bigcup_{\beta<\alpha} \zeta_\beta \right) \right\}$

Referring to the above definition, the collection ζ_2 contains all R-modules M that are not uniserial and for every submodule N of M, if $M/N \ncong M$ implies M/N is uniserial. A module M is said to have uniserial dimension α if there are ordinal numbers α such that $M \in \zeta_\alpha$. It is easy to show that if $M \in \zeta_\beta$ and $\beta < \alpha$ then $M \in \zeta_\alpha$. Furthermore, a module M is said to have uniserial dimension α, denoted $u.s.dim(M) = \alpha$, if α is the smallest ordinal number such that $M \in \zeta_\alpha$. This notion is stated in the following definition.

Definition 2.2. Nazemian [13]. Let R be a ring and M be an R-module.

1. For $M \in \zeta_\alpha$, the minimal ordinal number α is said universal dimension from M, denote $u.s.dim(M) = \alpha$.
2. For $M = 0$, $u.s.dim(M) = 0$
3. For $M \neq 0$ but $M \notin \zeta_\alpha$ for any ordinal number $\alpha \geq 1$, we said that "$u.s.dim(M)$ is not defined" or "M has no uniserial dimension"

In the ring theory, the notion of valuation rings in commutative area, that is a ring whose lattice of ideals forms a chain. Recall that for a ring R and $a \in R$, the ideal which is generated by a, denoted $\langle a \rangle$, which means $\langle a \rangle = Ra = \{xa | for\, any\, x \in R\}$. We called R be valuation ring if $a, b \in R$ then $a \in \langle b \rangle$ or $b \in \langle a \rangle$.
In 2015, Ghorbani [8] introduced the notion of valuation dimension of a ring that measures of how far a commutative rings deviates from beeing valuation. By this definition, the class of rings that have finite valuation dimension can be viewed as a generalization of the class of valuation rings.

Definition 2.3. Ghorbani [8]. For a ring R, the uniserial dimension of the module R_R, if it exists, is called valuation dimension of R and denoted by $v.dim(R)$. If R_R does not have uniserial dimension, we say that the ring R does not have valuation dimension.

The properties of rings that has valuation dimension already found (see Nazemian [13] and Ghorbani [8]). If I is ideal of R that has valuation dimension, then R/I has valuation dimension and $u.s.dim(R/I)_R = v.dim(R/I)$. Characteristics of Noetherian rings that has valuation dimension can be seen in Ghorbani [8]. Nazemian [13] shows that if M is a module of finite length, then M has uniserial dimension and $u.s.dim(M) \leq length(M)$. They also shown that every semi-Artin rings with valuation dimension is the Artinian rings. The following lemma will be used to prove Theorem 3.4 and 3.5:

Lemma 2.4. ***Nazemian [13].*** *If M is an R-module and $u.s.dim(M) = \alpha$, then for any $0 \leq \beta \leq \alpha$, there exists a factor module M/N of M such that $u.s.dim(M/N) = \beta$.*

Lemma 2.5. ***Ghorbani [8].*** *If $I < J$ are ideals of a ring R with valuation dimension and the ring R/I is not valuation, then $v.dim(R/J) < v.dim(R/I)$. In particular if R is a non-valuation ring with valuation dimension and J is an ideal of R, then $v.dim(R/J) < v.dim(R)$.*

The valuation dimension of principal ideal domain has been shown in Arifin [14], that is, if R is principal ideal domain that is not field, then $v.dim(R) = 1$ or $v.dim(R) = \infty$. Therefore, for ring of the integer $\mathbb{Z}$, $v.dim(\mathbb{Z}) = \infty$. From this result and the literature, the methods how to counting valuation dimension of ring $\mathbb{Z}_n$ still unknown and will be discussed in the next session.

3. Methods

In this section, the methods to counting the valuation dimension of ring $\mathbb{Z}_n$ will be studied. We will see this method in Corollary 3.8, which is the main result of this section. Recall that any Artinian rings has finite valuation dimension (see Ghorbani [8]). Therefore, the ring of integer modulo n, $\mathbb{Z}_n$ has finite valuation dimension. We can determine the valuation dimension of $\mathbb{Z}_n$ using three of the following theorem. But, we will need the following lemma first to get all cyclic submodules of $\mathbb{Z}_n$, that is the set generated by all factors of n.

Lemma 3.1. ***Gallian [7].*** *For each positive divisor k of n, the set $\left\langle \frac{n}{k} \right\rangle$ is the unique subgroup of $\mathbb{Z}_n$ of order k. Morover, these are the only subgroups of $\mathbb{Z}_n$.*

Lemma 3.2. ***Huang [9].*** *The following statements are true:*

1. *Let m and n be positive integers. If gcd(m,n) = 1 (i.e. m and n are relative prime), then $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and is isomorphic to $\mathbb{Z}_{mn}$, and $(1,1)$ is a generator of $\mathbb{Z}_m \times \mathbb{Z}_n$.*
2. *The group $\prod_{i=1}^{n} \mathbb{Z}_{m_i}$ is cyclic and is isomorphic to $\mathbb{Z}_{m_1 \ldots m_n}$ if and only if the numbers m_i for $i = 1, \ldots, n$ are pairwise relative prime, that is, the gcd of any two of them is 1.*
3. *If a positive integer n is factorized as a product of powers of distinct prime numbers: $n = p_1^{n_1} \ldots p_r^{n_r}$ then $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{n_1}} \times \ldots \times \mathbb{Z}_{p_r^{n_r}}$.*

Recall that if $\mathbb{Z}_n$ be the integer modulo n with $n = pq$ and p, q are two distinct prime, $\langle p \rangle$ be any subgroup generated by p, then we will have:

$$\begin{aligned} \mathbb{Z}_n / \langle p \rangle &= \{0 + \langle p \rangle, \ldots, (n-1) + \langle p \rangle\} \\ &= \{0 + \langle p \rangle, \ldots, (p-1) + \langle p \rangle\} \\ &= \{\overline{0}, \ldots, \overline{p-1}\} \\ &\cong \mathbb{Z}_p \end{aligned}$$

This properties will be used in Theorem 3.4 and 3.5. First type of the theorem are as follows.

Theorem 3.3. *Let $\mathbb{Z}_n$ be the integer modulo n with $n = p^n$ and p are prime and $n \in \mathbb{Z}^+$. Then $v.dim(\mathbb{Z}_n) = 1$.*

Proof. Let $M = \mathbb{Z}_n$ be a module over $\mathbb{Z}_n$. If $n = p^n$, then all submodules of $\mathbb{Z}_n$ are of the form $N_1 = \langle p \rangle$,..., $N_n = \langle p^n \rangle$ where $N_1 \supseteq \ldots \supseteq N_n$. For any two submodules I, J of M with the form $I = \langle p^{n_1} \rangle$ and $J = \langle p^{n_2} \rangle$. If $n_1 > n_2$ then $I \subseteq J$ and $n_1 < n_2$ then $I \supseteq J$. Therefore M is uniserial, or $M \in \xi_1$. We can say that $v.dim(\mathbb{Z}_n) = 1$. ■
The lattice of submodules of M in Theorem 3.3 can be described on the following diagram:

$$
\begin{array}{c}
M = \mathbb{Z}_n \\
\uparrow \\
\langle p \rangle \\
\uparrow \\
\langle p^2 \rangle \\
\uparrow \\
\ldots \\
\uparrow \\
\langle 0 \rangle = \langle p^n \rangle
\end{array}
$$

The following are the second type of the theorem.

Theorem 3.4. *Let $\mathbb{Z}_n$ be the integer modulo n with $n = p_1...p_k$ and p_i are distinct prime, $i = 1,...,k$. Then $v.dim(\mathbb{Z}_n) = k$.*

Proof. Let $M = \mathbb{Z}_n$ be a module over $\mathbb{Z}_n$ with $n = p_1...p_k$ are distinct primes. The proof is by induction on k. The case $k = 1$ is clear by Teorema 3.3. Now let $k = l$, then $n_l = p_1...p_l$. Therefore by hypothesis, $v.dim(\mathbb{Z}_{n_l}) = l$. It means $\mathbb{Z}_{n_l} \in \xi_l$ and $\mathbb{Z}_{n_l} \notin \xi_{l-1}$. We have to show that for $k = l+1$ the statement is true. Let $n_{l+1} = p_1...p_{l+1} = (p_1...p_l)\, p_{l+1}$. Let N be any proper submodule of $M_{l+1} = \mathbb{Z}_{n_{l+1}}$ and $a \in N$, $a \neq 0$ where $a = p_1...p_{(l+1)-1} = p_1...p_l$. Without lost the generality, there exist prime element $\overline{p} \in \mathbb{Z}_{n_{l+1}}$, $gcd(a,p) = 1$ such that $b = pa = p(p_1...p_l)$. So, there exist a submodule $\langle b \rangle = pa$ of M_{l+1} such that $\langle b \rangle \subseteq N \subseteq M_{l+1}$ and $M_{l+1}/\langle b \rangle \cong M_l$. By hypothesis, $M_{l+1}/\langle b \rangle \in \xi_l$ and $M_{l+1}/\langle b \rangle \notin \xi_{l-1}$. Note that $M_{l+1}/N \cong (M_{l+1}/\langle b \rangle)/(N/\langle b \rangle) = M_l/(N/\langle b \rangle)$. By Lemma 2.4, $M_{l+1} \in \xi_{l+1}$ and $M_{l+1} \notin \xi_{(l+1)-1}$, so $v.dim(M_{l+1}) = l+1$, which means that for $k = l+1$ the statement is true. Now, we can say that $v.dim(\mathbb{Z}_n) = k$ for $n = p_1...p_k$, $k \in \mathbb{Z}^+$.■

The following are the third type of the theorem.

Theorem 3.5. *Let $\mathbb{Z}_n$ be the integer modulo n with $n = p^s q$, $s \in \mathbb{Z}^+$ and p,q are distinct primes. Then $v.dim(\mathbb{Z}_n) = s+1$.*

To proof of the theorem 3.5, the following theorem will be needed.

Theorem 3.6. *Let $\mathbb{Z}_n$ be the integer modulo n with $n = pq$ and p,q are two distinct prime. Then $v.dim(\mathbb{Z}_n) = 2$.*

Proof. Let $M = \mathbb{Z}_n$ be a module over $\mathbb{Z}_n$. All proper submodules of $\mathbb{Z}_n$ are $I = \langle p \rangle$ and $J = \langle q \rangle$. Since p,q are distinct primes, then M/I and M/J are simple, so that uniserial. Therefore, $M \in \xi_2$. On the other hand, $I \not\subseteq J$ and $J \not\subseteq I$, that means $M \notin \xi_1$. Now, we can say that $v.dim(\mathbb{Z}_n) = 2$. ■

The lattice of submodules of M in Lemma 3.6 can be described on the following diagram:

$$
\begin{array}{ccccc}
 & & M = \mathbb{Z}_{pq} & & \\
 & \nearrow & & \nwarrow & \\
\langle p \rangle & & \uparrow & & \langle q \rangle \\
 & \nwarrow & & \nearrow & \\
 & & \langle 0 \rangle = \langle pq \rangle & &
\end{array}
$$

Now, Theorem 3.5 can be proof by using induction on the index that is prime factors of n.

Proof Of Theorem 3.5. Let $M = \mathbb{Z}_n$ be a module over $\mathbb{Z}_n$ with $n = p^s q$, $s \in \mathbb{Z}^+$ and p,q are distinct primes. The proof is by induction on s. The case $s = 1$ is clear by Teorema 3.6, since $v.dim(\mathbb{Z}_{pq}) = 2 = 1+1$.

Now let $s = l$, then $n_l = p^l q$. Therefore by hypothesis, $v.dim(\mathbb{Z}_{n_l}) = l+1$. It means $\mathbb{Z}_{n_l} \in \xi_{l+1}$ and $\mathbb{Z}_{n_l} \notin \xi_l$. We have to show that for $s = l+1$ the statement is true. Let $n = p^{l+1} q$. Let N be any proper submodule of $M_{l+1} = \mathbb{Z}_{n_{l+1}}$ and $\overline{a} \in N$, $\overline{a} \neq 0$ where $a = p^{(l+1)-1} q = p^l q$. Without lost the generality, there exist prime element $c \in \mathbb{Z}_{n_{l+1}}$, $gcd(a,c) = 1$ such that $b = ca = p(p^l q)$. So, there exist a submodule $\langle b \rangle = ca$ of M_{l+1} such that $\langle b \rangle \subseteq N \subseteq M_{l+1}$ and $M_{l+1}/\langle b \rangle \cong M_l$. By hypothesis, $M_{l+1}/\langle b \rangle \in \xi_l$ and $M_{l+1}/\langle b \rangle \notin \xi_{l-1}$. Note that $M_{l+1}/N \cong (M_{l+1}/\langle b \rangle)/(N/\langle b \rangle) = M_l/(N/\langle b \rangle)$. By Lemma 2.4, $M_{l+1} \in \xi_{l+1}$ and $M_{l+1} \notin \xi_{(l+1)-1}$, which means that for $k = l+1$ the statement is true. Now, we can say that if $n = p^s q$, $s \in \mathbb{Z}^+$ and p,q are distinct primes, then $v.dim(\mathbb{Z}_n) = s+1$.■

Using commutativity of the integers, we will also get that if $\mathbb{Z}_n$ be the integer modulo n with $n = pq^t$, $t \in \mathbb{Z}^+$ and p,q are distinct primes. Then $v.dim(\mathbb{Z}_n) = 1+t$. Therefore, we will get the following result.

Corollary 3.7. Let $\mathbb{Z}_n$ be the integer modulo n with $n = p^s q^t$, $s,t \in \mathbb{Z}^+$ and p,q are distinct primes. Then $v.dim(\mathbb{Z}_n) = s+t$.

From Theorem 3.4 and Corollary 3.7, we will get the following corollary. This is the main result of counting the valuation dimension of ring $\mathbb{Z}_n$ where $n = p_1^{l_1}...p_k^{l_k}$, $l_i \in \mathbb{Z}^+$ and p_i are distinct primes, $i = 1,...,k$.

Corollary 3.8. Let $\mathbb{Z}_n$ be the integer modulo n with $n = p_1^{l_1}...p_k^{l_k}$, $l_i \in \mathbb{Z}^+$ and p_i are distinct primes, $i = 1,...,k$. Then $v.dim(\mathbb{Z}_n) = l_1 + ... + l_k$.

We will combine between Corollary 3.8 and prime factorization of the integer (see Salsabiela [2]) to make a program using Python 2.7.14 to determine the valuation dimension of ring $\mathbb{Z}_n$. The reader can copy and paste the code into Python, and press F5 to execute the program. To see the lattice of submodules of any regular modules $\mathbb{Z}_n$, we can use Wolfram CDF Player and download the file from their official website (see Wolfram [17]). The main program is as below.

```
print "========================================="
print "Determine Valuation Dimension of Ring Z_n"
print "-----------------------------------------"
x = input ("Input n:")
def faktorPrima(x) :
    a = []
    b = []
    hasil = 0
    bil = x
    prima =True
    for i in range(2,x):
        prima = True
        for u in range(2, i) :
            if i % u == 0 :
                prima = False
        if prima :
            a.append(i)
    idx = 0
    while bil > 1 :
        try:
            if (bil%a[idx]) == 0 :
                hasil = bil/a[idx]
                bil = hasil
                b.append(a[idx])
            else :
                idx = idx + 1
        except IndexError :
            break
    return b
print "Prime factorization of ",x,":", faktorPrima(x)

h = faktorPrima(x)
ph = set(h)

if len(ph) == 1:
    vdimku = len(ph)
else:
    vdimku = len(h)
print "Therefore, v.dim(Z_",x, ") =", vdimku
```

4. Results and Discussion

In this section, the results of the valuation dimension of ring $\mathbb{Z}_n$, that obtained from Theorem 3.4 and Corollary 3.8 will be studied. The discussion will ended by the output of the Python program above. Recall that to get the valuation dimension of any ring $\mathbb{Z}_n$, we need to find all proper submodule of regular module $\mathbb{Z}_n$, so we can discribed the lattice of them in one diagram.
From Theorem 3.4, we can get Example 4.1 below.

Example 4.1. For $M=\mathbb{Z}_n$ where $n=pqr$ and p,q,r are any distinct primes, we will show that $v.dim\,(\mathbb{Z}_n)=3$. Now let $M=(\mathbb{Z}_n)_{\mathbb{Z}_n}$. Since any module M contains submodul $\{0\}$ and M, then the lattice of all submodules of M are:

$$
\begin{array}{ccccc}
 & & \mathbb{Z}_{pqr} & & \\
 & \nearrow & \uparrow & \nwarrow & \\
\langle p\rangle & & \langle q\rangle & & \langle r\rangle \\
\uparrow & \times & & \times & \\
\langle pq\rangle & & \langle pr\rangle & & \langle rq\rangle \\
 & \nwarrow & \uparrow & \nearrow & \\
 & & \langle 0\rangle & &
\end{array}
$$

From the lattice above, we get $v.dim\,(\mathbb{Z}_n)=3$ by Theorem 3.4.

From Theorem 3.5, we can get Example 4.2 below, which can give a better understanding of the finite valuation dimension of ring $\mathbb{Z}_n$. For general number n generated by the power of two distinct primes, that is $n=p^kq^l$, $k,l\in\mathbb{Z}^+$ and p,q are any distinct primes, the following example is very intersting.

Example 4.2. For $M=\mathbb{Z}_n$ where $n=p^kq^l$, $k,l\in\mathbb{Z}^+$ and p,q are distinct primes, we will show that $v.dim\,(\mathbb{Z}_n)=k+l$. Now let $M=(\mathbb{Z}_n)_{\mathbb{Z}_n}$. Since any module M contains submodul $\{0\}$ and M, then all submodules of M are:

$$
\begin{gathered}
\mathbb{Z}_n, \langle p\rangle, \ldots, \langle p^{k-1}\rangle, \langle p^k\rangle, \\
\langle q\rangle, \langle pq\rangle, \ldots, \langle p^{k-1}q\rangle, \langle p^k q\rangle, \\
\langle q^2\rangle, \langle pq^2\rangle, \ldots, \langle p^{k-1}q^2\rangle, \langle p^k q^2\rangle, \\
\ldots \\
\langle q^{l-1}\rangle, \langle pq^{l-1}\rangle, \ldots, \langle p^{k-1}q^{l-1}\rangle, \langle p^k q^{l-1}\rangle, \\
\langle q^l\rangle, \langle pq^l\rangle, \ldots, \langle p^{k-l}q^l\rangle, \langle p^k q^l\rangle = \{0\}.
\end{gathered}
$$

a) For $k > l, k = 4, l = 1$, we have the lattices of submodules of M as follow:

$$
\begin{tikzcd}
 & & & & \mathbb{Z}_{p^4q} & \\
 & & & \langle p\rangle \arrow[ur] & & \langle q\rangle \arrow[ul] \\
 & & \langle p^2\rangle \arrow[ur] & & \langle pq\rangle \arrow[uu] \arrow[ul] \arrow[ur] & \\
 & \langle p^3\rangle \arrow[ur] & & \langle p^2q\rangle \arrow[uu] \arrow[ul] \arrow[ur] & & \\
\langle p^4\rangle \arrow[ur] & & \langle p^3q\rangle \arrow[uu] \arrow[ul] \arrow[ur] & & & \\
 & \langle p^4q\rangle = \langle 0\rangle \arrow[uu] \arrow[ul] \arrow[ur] & & & &
\end{tikzcd}
$$

From the lattice above, we get $v.dim\left(\mathbb{Z}_{p^4q}\right) = 4 + 1 = 5$ by Corollary 3.7.

b) For $k < l, k = 2, l = 4$, we have the lattices of submodules of M as follow

$$
\begin{tikzcd}
 & & \mathbb{Z}_{p^2q^4} & & & & \\
 & \langle p\rangle \arrow[ur] & & \langle q\rangle \arrow[ul] & & & \\
\langle p^2\rangle \arrow[ur] & & \langle pq\rangle \arrow[uu] \arrow[ul] \arrow[ur] & & \langle q^2\rangle \arrow[ul] & & \\
 & \langle p^2q\rangle \arrow[uu] \arrow[ul] \arrow[ur] & & \langle pq^2\rangle \arrow[uu] \arrow[ul] \arrow[ur] & & \langle q^3\rangle \arrow[ul] & \\
 & & \langle p^2q^2\rangle \arrow[uu] \arrow[ul] \arrow[ur] & & \langle pq^3\rangle \arrow[uu] \arrow[ul] \arrow[ur] & & \langle q^4\rangle \arrow[ul] \\
 & & & \langle p^2q^3\rangle \arrow[uu] \arrow[ul] \arrow[ur] & & \langle pq^4\rangle \arrow[uu] \arrow[ul] \arrow[ur] & \\
 & & & & \langle p^2q^4\rangle = \langle 0\rangle \arrow[uu] \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

From the lattice above, we get $v.dim\left(\mathbb{Z}_{p^2q^4}\right) = 2 + 4 = 6$ by Corollary 3.7.

From Theorem 3.3, clearly $v.dim\left(\mathbb{Z}_{961}\right) = 1$ since $961 = 31^2$, from Theorem 3.6, clearly $v.dim\left(\mathbb{Z}_{30049}\right) = 2$ since $30049 = 151.199$, and from Theorem 3.4, clearly $v.dim\left(\mathbb{Z}_{1155}\right) = 4$ since $1155 = 3.5.7.11$. Another examples, by using Python program above, we can give input $n = 210$ and $n = 1024$, which means what is the valuation dimension of rings $\mathbb{Z}_{210}$ and $\mathbb{Z}_{1024}$. Output of the result is as follows: $v.dim\left(\mathbb{Z}_{210}\right) = 4$ and $v.dim\left(\mathbb{Z}_{1024}\right) = 1$.

```
==========================================
Determine Valuation Dimension of Ring Z_n
------------------------------------------
Input n:210
Prime factorization of  210 : [2, 3, 5, 7]
Therefore, v.dim(Z_ 210 ) = 4
>>>
```

```
==========================================
Determine Valuation Dimension of Ring Z_n
------------------------------------------
Input n:1024
Prime factorization of  1024 : [2, 2, 2, 2, 2, 2, 2, 2, 2, 2]
Therefore, v.dim(Z_ 210 ) = 1
>>>
```

5. Conclusion

We can determine the valuation dimension of the ring $\mathbb{Z}_n$ using Python. From the program that has been created, the maximum value of n can be calculated at $1.7x10^{308}$. This value corresponds to the upper limit of integers in Python that can be checked by writing the following command.

```
import sys
int(sys.float_info.max)
```

Acknowledgement

The authors wish to thank Professor Pudji Astuti and Dr. Hanni Garminia. This work was supported with the help of facilities and infrastructure of Institut Teknologi Bandung. Thank you deeply for your support during this time.

References

[1] Atiyah, M. F. dan MacDonald, I. G., 1969, Introduction to Commutative Algebra, Addison-Wesley Publication Co., London
[2] D. H. Salsabiela, Program Python Menentukan Faktorisasi Prima Sebuah Bilangan Bulat, http://intechgram.blogspot.com/2018/03/program-python-menentukan-faktorisasi.html, 29 April 2018
[3] Eisenbud, D. dan Griffith, P., 1971, Serial Rings, J. Algebra, 17, 389-400.
[4] Facchini, A. dan Salce, L., 1990, Uniserial modules: sums and isomorphisms of subquotients, Comm. Algebra 18, 2, 499–517.
[5] Faith, C., 1984, The structure of valuation rings, J. Pure Appl. Algebra, 31, 7-27.
[6] Faith, C., 1986, The structure of valuation rings II, J. Pure Appl. Algebra, 42, 37-43.
[7] Gallian, J.A., 2017, Contemporary Abstract Algebra, 9th Edition, USA.
[8] Ghorbani, A. dan Nazemian, Z., 2015, On commutative rings with uniserial dimension, Journal of Algebra and Its Aplications, 14, 1, 1-10.
[9] Huang, H. Algebra Lecture Notes. Auburn University Press, http://www.auburn.edu/ huanghu/math5310/, 2010.
[10] Kaplansky, I., 1952, Modules over Dedekind rings and valuation rings, Trance. Amer. Math. Soc., 72, 327-340.
[11] Lam, T. Y. (1991): A First Course in Noncommutative Rings, Grad. Texts in Math., Springer, New York.
[12] Manis, M., 1969, Valuation on a commutative ring, Proc. Amer. Math. Soc, 20, 193-198
[13] Nazemian, Z., Ghorbani, A. dan Behboodi, M., 2014, Uniserial dimension of modules, J. Algebra, 399, 894-903.
[14] Samsul A, Hanni G.Y., Pudji A., 2016, Dimensi Valuasi dari Daerah Ideal Utama, Prosiding Semnas KPA Univ. Sanata Dharma, 9-16.
[15] Stoll, R. R., 1961, Set Theory and Logic, Dover Publication Inc., New York.
[16] Warfield, R. B., 1975, Serial rings and finitely presented modules, J. Algebra, 37, 187–222.
[17] WOLFRAM Demonstrations Project, Subgroup Lattices of Finite Cyclic Groups, http://demonstrations.wolfram.com/SubgroupLatticesOfFiniteCyclicGroups/, 20 Juli 2018