International Journal of Engineering & Technology, 7 (3.3) (2018) 138-143

International Journal of Engineering & Technology

Website: www.sciencepubco.com/index.php/IJET

Research paper

On A Subclass of Harmonic Univalent Functions Associated with the Differential Operator

B. Ravindar[1]*, R. Bharavi Sharma[2]

[1]*S R Engineering College, Warangal, Telangana State, India*
[2]*Department of Mathematics, Kakatiya University, Warangal, Telangana State, India*
**Corresponding author E-mail:ravi.boini@gmail.com*

Abstract

In this paper, a new subclass of harmonic univalent functions in the unit disk $U = \{z \in C : |z| < 1\}$ is introduced using a differential operator. Also the coefficient estimates, convolution conditions, extreme points and convex combinations are obtained.

Keywords: *Harmonic function; Univalent functions; Differential operator.*

1. Introduction

Let D be a simply connected domain in the complex plane C and let $f = u + iv$ be harmonic in D. We can write $f = h + \overline{g}$, where h and g are analytic in D. Here h and g are called as the analytic part and co-analytic part of *f* respectively. A necessary and sufficient condition for f to be locally univalent and orientation-preserving in D is that $|h'(z)| > |g'(z)|$ in D [1].

In 1984, the class S_H was introduced by Clunie and Sheil-Small [1] and their study gave insight of few coefficient bounds. Since then, several papers were published related to S_H and its subclasses. In fact, by investigating new subclasses, Sheil-Small [18], Jahangiri [15], Silverman and Silvia [9] and Ahuja [2] presented a unified and systematic study of harmonic univalent functions.

Let S_H denote the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense- preserving in the unit disk $U = \{z \in C : |z| < 1\}$ for which $f(0) = h(0) = f_z(0) - 1 = 0$.

We may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \; g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \qquad (1.1)$$

The differential operator $D^n_{\eta,\mu}(\lambda, w)\ (n \in N_0)$ was initiated by Bucur et al. [10].

For $f = h + \overline{g}$, given by (1.1), Sahsene Altinkaya and Sibel Yalcin [5] defined the operator $D^n_{\eta,\mu}(\lambda, w)$ as

$$D^n_{\eta,\mu}(\lambda, w)\ f(z) = D^n_{\eta,\mu}(\lambda, w)\ h(z) + (-1)^n \overline{D^n_{\eta,\mu}(\lambda, w)\ g(z)} \qquad (1.2)$$

where $D^n_{\eta,\mu}(\lambda, w)\ h(z) = z + \sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda} - \eta) + k]^n a_k z^k$

and $D^n_{\eta,\mu}(\lambda, w)\ g(z) = \sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda} - \eta) + k]^n b_k z^k$,

where

$n, \eta \in N_0$, $\mu, \lambda, w \geq 0$, $0 \leq \eta \leq \mu w^{\lambda}$, $D^n_{\eta,\mu}(\lambda, w)\ f(0) = 0$.

Recently S. Porwal and Shivam Kumar [11] and A.L. Pathak et al. [12] have defined and studied a subclass of harmonic univalent functions using differential operator. They have obtained the coefficient and distortion bounds, extreme points, convolution and convex combinations.

Motivated by aforementioned work, in this paper we define a subclass $B_H(n, \eta, \alpha, \rho, \lambda, w)$ of harmonic univalent functions and study the coefficient bounds, convolution, extreme points and convex combinations.

Definition 1.1: We define a new subclass $B_H(n, \eta, \alpha, \rho, \lambda, w)$ of functions of the kind (1.1) that satisfy the condition

$$\mathrm{Re}[(1+\rho e^{ir})\frac{D^{n+1}_{\eta,\mu}(\lambda, w) f(z)}{D^n_{\eta,\mu}(\lambda, w) f(z)} - \rho e^{ir}] > \alpha,\ 0 \leq \alpha < 1, r \in R, \rho \geq 0, n, \eta \in N_0 \qquad (1.3)$$

where $D^n_{\eta,\mu}(\lambda, w)\ f(z)$ is defined by (1.2).

Let $\overline{B}_H(n, \eta, \alpha, \rho, \lambda, w)$ denote the subclasses of $B_H(n, \eta, \alpha, \rho, \lambda, w)$ which contains harmonic functions $f_n = h + \overline{g_n}$ such that h and g_n are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \; g_n(z) = (-1)^n \sum_{k=1}^{\infty} |b_k| z^k \qquad (1.4)$$

2. Coefficient Bounds

We start with a sufficient coefficient condition for functions in $B_H(n, \eta, \alpha, \rho, \lambda, w)$.

Theorem 2.1: Let $f = h + \overline{g}$ be given by (1.1). If

$$\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)-(1+\rho)((k-1)(\mu w^{\lambda}-\eta)+k)]|a_k|$$

$$+\sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)+(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)]|b_k|$$

$$\leq 1-\alpha \tag{2.1}$$

where $n,\eta \in N_0$, $\rho \geq 0, 0 \leq \alpha < 1$, $\mu,\lambda,w \geq 0, 0 \leq \eta \leq \mu w^{\lambda}$, then f is harmonic univalent, orientation-preserving in U and $f \in B_H(n,\eta,\alpha,\rho,\lambda,w)$.

Proof: If $z_1 \neq z_2$, then

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|$$

$$=1-\left|\frac{\sum_{k=1}^{\infty} b_k(z_1^k-z_2^k)}{(z_1-z_2)+\sum_{k=2}^{\infty}a_k(z_1^k-z_2^k)}\right|$$

$$>1-\frac{\sum_{k=1}^{\infty}k|b_k|}{1-\sum_{k=2}^{\infty}k|a_k|}$$

$$\geq 1-\frac{\dfrac{\sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)+(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)]|b_k|}{1-\alpha}}{1-\dfrac{\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)-(1+\rho)((k-1)(\mu w^{\lambda}-\eta)+k)]|a_k|}{1-\alpha}}$$

≥ 0, which demonstrates univalence.

Note that f is sense-preserving in U. This is because

$$|h'(z)| \geq 1-\sum_{k=2}^{\infty}k|a_k|z^{k-1}$$

$$>1-\frac{\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)-(1+\rho)((k-1)(\mu w^{\lambda}-\eta)+k)]|a_k|}{1-\alpha}$$

$$>\frac{\sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)+(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)]|b_k||z|^{k-1}}{1-\alpha}$$

$$\geq \sum_{k=1}^{\infty}k|b_k|z^{k-1}$$

$$\geq |g'(z)|.$$

Using the fact that $\operatorname{Re}(w) > \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$.

$$w=\frac{(1+\rho e^{ir})D_{\eta,\mu}^{n+1}(\lambda,w)f(z)-\rho e^{ir}D_{\eta,\mu}^{n}(\lambda,w)f(z)}{D_{\eta,\mu}^{n}(\lambda,w)f(z)}=\frac{A(z)}{B(z)}$$

It suffices to show that

$$|A(z)-(1+\alpha)B(z)|-|A(z)+(1-\alpha)B(z)|\leq 0. \tag{2.2}$$

Consider

$$|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)|$$

$$\geq 2(1-\alpha)|z|+$$

$$2\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}[(1+\rho)((k-1)(\mu w^{\lambda}-\eta)+k)-(\alpha+\rho)]|a_k||z|^{k}-$$

$$2\sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}[(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)+(\alpha+\rho)]|b_k||\overline{z}|^{k}$$

$$>2(1-\alpha)|z|\left\{1-\frac{1}{1-\alpha}\left(\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)-(1+\rho)((k-1)(\mu w^{\lambda}-\eta)+k)]|a_k||z|^{k-1}\right)\right.$$

$$\left.-\frac{1}{1-\alpha}\left(\sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)+(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)]|b_k||z|^{k-1}\right)\right\}$$

This is non-negative by (2.1), and so the proof is complete.

The harmonic function

$$f(z)=z+\sum_{k=2}^{\infty}\frac{(1-\alpha)x_k z^k}{[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)-(1+\rho)((k-1)(\mu w^{\lambda}-\eta)+k)]}$$

$$+\sum_{k=1}^{\infty}\frac{(1-\alpha)\overline{y_k z^k}}{[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)+(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)]} \tag{2.3}$$

where $n,\eta \in N_0$, $\rho \geq 0, 0 \leq \alpha < 1$, $\mu,\lambda,w \geq 0, 0 \leq \eta \leq \mu w^{\lambda}$ with $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$,

shows that the coefficient bound given by (2.1) is sharp. The functions of the form (2.3) are in $B_H(n,\eta,\alpha,\rho,\lambda,w)$ because

$$\frac{1}{1-\alpha}\left(\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)-(1+\rho)((k-1)(\mu w^{\lambda}-\eta)+k)]|a_k|\right)+$$

$$\frac{1}{1-\alpha}\left(\sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)+(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)]|b_k|\right)$$

$$=\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1. \tag{2.4}$$

In the following theorem, it is shown that the condition (2.1) is also necessary for the functions $f_n = h+\overline{g_n}$, to be in the class $\overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$ where h and g_n are of the form (1.4).

Theorem 2.2: Let $f_n = h+\overline{g}_n$ be given by (1.4).

Then $f_n \in \overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$, if and only if

$$\frac{1}{1-\alpha}\left(\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}\right.$$
$$\left.[(\alpha+\rho)-(1+\rho)((k-1)(\mu w^{\lambda}-\eta)+k)]|a_k|\right)$$
$$+\frac{1}{1-\alpha}\left(\sum_{k=2}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}\right.$$
$$\left.[(\alpha+\rho)+(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)]|b_k|\right)\leq 1, \tag{2.5}$$

where $n,\eta\in N_0$, $\rho\geq 0, 0\leq\alpha<1$, $\mu,\lambda,w\geq 0, 0\leq\eta\leq\mu w^{\lambda}$.

Proof: Since $\overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)\subset B_H(n,\eta,\alpha,\rho,\lambda,w)$, we just need to prove the 'only if 'part of the theorem.

For functions f_n of the form (1.4), we notice that the condition (1.3) is correspondent to

$$\mathrm{Re}\left[(1+\rho e^{ir})\frac{D^{n+1}_{\eta,\mu}(\lambda,w)f(z)}{D^{n}_{\eta,\mu}(\lambda,w)f(z)}-\rho e^{ir}\right]>\alpha$$

$$\mathrm{Re}\left[\left((1-\alpha)-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}\right.\right.$$
$$\left.\left[(1+\rho e^{ir})\left((k-1)(\mu w^{\lambda}-\eta)+k\right)-\alpha-\rho e^{ir}\right]|a_k|z^{k-1}\right)/$$
$$\left(1-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}|a_k|z^{k-1}+\right.$$
$$\left.\frac{\overline{z}}{z}(-1)^{2n}\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}|b_k|\overline{z}^{k-1}\right)-$$
$$\left(-(-1)^{2n}\frac{\overline{z}}{z}\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}\right.$$
$$\left.\left[(1+\rho e^{ir})\left((k+1)(\mu w^{\lambda}-\eta)+k\right)+\alpha+\rho e^{ir}\right]|b_k|\overline{z}^{k-1}\right)/$$
$$\left(1-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}|a_k|z^{k-1}+\right.$$
$$\left.\left.\frac{\overline{z}}{z}(-1)^{2n}\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}|b_k|\overline{z}^{k-1}\right)\right]\geq 0 \tag{2.6}$$

The above condition (2.6) must hold for all values of z on the positive real axis, where $0\leq|z|=\gamma<1$, we must have

$$\mathrm{Re}\left[\frac{\left((1-\alpha)-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}\left[(1+\rho e^{ir})\left((k-1)(\mu w^{\lambda}-\eta)+k\right)-\alpha-\rho e^{ir}\right]|a_k|\gamma^{k-1}\right)}{\left(1-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}|a_k|\gamma^{k-1}+(-1)^{2n}\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}|b_k|\gamma^{k-1}\right)}-\frac{\left((-1)^{2n}\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}\left[(1+\rho e^{ir})\left((k+1)(\mu w^{\lambda}-\eta)+k\right)+\alpha+\rho e^{ir}\right]|b_k|\gamma^{k-1}\right)}{\left(1-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}|a_k|\gamma^{k-1}+(-1)^{2n}\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}|b_k|\gamma^{k-1}\right)}\right]\geq 0$$

Since $\mathrm{Re}\,(e^{ir})=|e^{ir}|=1,$ the above inequality reduces to

$$\left\{\left((1-\alpha)z-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}\left[(1+\rho)\left((k-1)(\mu w^{\lambda}-\eta)+k\right)-\alpha-\rho\right]|a_k|\gamma^{k-1}\right)-\left(\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}\left[(1+\rho)\left((k+1)(\mu w^{\lambda}+\eta)+k\right)+\alpha+\rho\right]|b_k|\gamma^{k-1}\right)\right\}$$
$$\times\left\{1-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}|a_k|\gamma^{k-1}+\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}|b_k|\gamma^{k-1}\right\}^{-1}\geq 0 \tag{2.7}$$

In the event that the condition (2.5) does not hold, at that point the numerator in (2.7) is negative for γ adequately near 1. Henceforth there exist a, $z_0=\gamma_0$ in $(0,1)$ for which the quotient in (2.7) is negative. This negates the condition for $f_n\in\overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$ and thus the verification is finished.

3. Convolution

In this section, we prove that the class $\overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$ is closed with reference to convolution.
For harmonic functions,

$$f_n(z) = z - \sum_{k=2}^{\infty} | a_k | z^k + (-1)^n \sum_{k=1}^{\infty} | b_k | \overline{z}^k \text{ and}$$

$$F_n(z) = z - \sum_{k=2}^{\infty} | A_k | z^k + (-1)^n \sum_{k=1}^{\infty} | B_k | \overline{z}^k .$$

The convolution of f_n and F_n is given by

$$(f_n * F_n)(z) = f_n(z) * F_n(z) =$$

$$z - \sum_{k=2}^{\infty} | a_k A_k | z^k + (-1)^n \sum_{k=1}^{\infty} | b_k B_k | \overline{z}^k \quad (3.1)$$

Theorem 3.1: For $n, \eta \in N_0$, $\rho \geq 0$, $\mu, \lambda, w \geq 0$, $0 \leq \eta \leq \mu w^\lambda$, $0 \leq \beta \leq \alpha < 1$,

Let $f_n \in \overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$ and $F_n \in \overline{B}_H(n,\eta,\beta,\rho,\lambda,w)$.

Then $(f_n * F_n) \in \overline{B}_H(n,\eta,\alpha,\rho,\lambda,w) \subset F_n \in \overline{B}_H(n,\eta,\beta,\rho,\lambda,w)$.

Proof: We desire to demonstrate that the coefficient $f_n * F_n$ satisfy the condition given in theorem (2.2).

For $F_n \in \overline{B}_H(n,\eta,\beta,\rho,\lambda,w)$, we notice that $| A_k | \leq 1$ and $| B_k | \leq 1$. Now, for the convolution function $f_n * F_n$, we obtain

$$\frac{1}{1-\alpha} \sum_{k=2}^{\infty} \left[(k-1)(\mu w^\lambda - \eta) + k\right]^n$$

$$\left[(\alpha+\rho) - (1+\rho)\left((k-1)(\mu w^\lambda - \eta) + k\right)\right] | a_k || A_k |$$

$$+ \frac{1}{1-\alpha} \sum_{k=2}^{\infty} \left[(k+1)(\mu w^\lambda - \eta) + k\right]^n$$

$$\left[(\alpha+\rho) + (1+\rho)\left((k+1)(\mu w^\lambda - \eta) + k\right)\right] | b_k || B_k |$$

$$\leq \frac{1}{1-\alpha} \sum_{k=2}^{\infty} \left[(k-1)(\mu w^\lambda - \eta) + k\right]^n$$

$$\left[(\alpha+\rho) - (1+\rho)\left((k-1)(\mu w^\lambda - \eta) + k\right)\right] | a_k |$$

$$+ \frac{1}{1-\alpha} \sum_{k=2}^{\infty} \left[(k+1)(\mu w^\lambda - \eta) + k\right]^n$$

$$\left[(\alpha+\rho) + (1+\rho)\left((k+1)(\mu w^\lambda - \eta) + k\right)\right] | b_k |$$

≤ 1.

Since $0 \leq \beta \leq \alpha < 1$, and $f_n \in \overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$.

Therefore

$(f_n * F_n) \in \overline{B}_H(n,\eta,\alpha,\rho,\lambda,w) \subset F_n \in \overline{B}_H(n,\eta,\beta,\rho,\lambda,w)$.

4. Convex Combination

In this section, we illustrate that the class $\overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$ is closed with regard to convex combination of its members.

Let the functions $f_{n_i}(z)$ be defined, for $i = 1,2,.......,m$ by

$$f_{n_i}(z) = z - \sum_{k=2}^{\infty} | a_{k,i} | z^k + (-1)^n \sum_{k=1}^{\infty} | b_{k,i} | \overline{z}^k \quad (4.1)$$

Theorem4.1: Let the functions $f_{n_i}(z)$ defined by (4.1) be in the class $\overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$ for every $i = 1,2,.....,m$. Then the functions $t_i(z)$ defined by

$t_i(z) = \sum_{i=1}^{\infty} c_i f_{n_i}(z),\ 0 \leq c_i \leq 1,$ are also in the class $\overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$, where $\sum_{i=1}^{\infty} c_i = 1$.

Proof: As indicated by the definition of t_i, we can write

$$t_i(z) = z - \sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty} c_i | a_{k,i} |\right) z^k + (-1)^n \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} c_i | b_{k,i} |\right) \overline{z}^k .$$

Further, since $f_{n_i}(z)$ are in $\overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$, for every $i = 1,2,.......,m$, then

$$\sum_{k=2}^{\infty} \left[(k-1)(\mu w^\lambda - \eta) + k\right]^n \left[(\alpha+\rho) - (1+\rho)\left((k-1)(\mu w^\lambda - \eta) + k\right)\right] \left(\sum_{i=1}^{\infty} c_i | a_{k,i} |\right)$$

$$+ \sum_{k=1}^{\infty} \left[(k+1)(\mu w^\lambda - \eta) + k\right]^n \left[(\alpha+\rho) + (1+\rho)\left((k+1)(\mu w^\lambda - \eta) + k\right)\right] \left(\sum_{i=1}^{\infty} c_i | b_{k,i} |\right)$$

$$\leq \sum_{i=1}^{\infty} c_i (1-\alpha) \leq (1-\alpha).$$

5. Extreme Points

In this section, we attain extreme points for the class $\overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$.

Theorem5.1: Let f_n be given by (1.4).

Then $f_n \in \overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$,

if and only if $$f_n(z) = \sum_{k=1}^{\infty}\left(X_k h_k(z) + Y_k g_{n_k}(z)\right), \quad (5.1)$$

where

$h_1(z) = z$,

$h_k(z) =$

$$z - \left[\frac{(1-\alpha)z^k}{\left[(k-1)(\mu w^\lambda - \eta) + k\right]^n \left[(\alpha+\rho) - (1+\rho)(k-1)(\mu w^\lambda - \eta) + k\right]} \right]$$

$g_{n_k}(z) =$

$$z + \left[\frac{(-1)^n (1-\alpha)\overline{z}^k}{\left[(k+1)(\mu w^\lambda - \eta) + k\right]^n \left[(\alpha+\rho) + (1+\rho)(k+1)(\mu w^\lambda - \eta) + k\right]} \right]$$

$k = 2, 3, \ldots\ldots$, and $k = 1, 2, \ldots\ldots$, respectively, with

$$\sum_{k=1}^{\infty} (X_k + Y_k) = 1, X_k \geq 0, Y_k \geq 0 .$$

Especially, the extreme points of $\overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$ are $\{h_k\}$ and $\{g_{n_k}\}$.

Proof: For the functions f_n of the kind (5.1), we have

$$f_n(z) = \sum_{k=1}^{\infty} X_k h_k(z) + Y_k g_{n_k}(z)$$

$$= \sum_{k=1}^{\infty} X_k \left\{ z - \left(\frac{1-\alpha}{\left[(k-1)(\mu w^\lambda - \eta) + k\right]^n \left[(\alpha+\rho) - (1+\rho)\left((k-1)(\mu w^\lambda - \eta) + k\right)\right]} \right) z^k \right\}$$

$$+ \sum_{k=1}^{\infty} Y_k \left\{ z + \left(\frac{(-1)^n (1-\alpha)}{\left[(k+1)(\mu w^\lambda - \eta) + k\right]^n \left[(\alpha+\rho) + (1+\rho)\left((k+1)(\mu w^\lambda - \eta) + k\right)\right]} \right) \overline{z}^k \right\}$$

$$= z - \sum_{k=2}^{\infty} \left[\frac{1-\alpha}{\left[(k-1)(\mu w^\lambda - \eta) + k\right]^n \left[(\alpha+\rho) - (1+\rho)\left((k-1)(\mu w^\lambda - \eta) + k\right)\right]} \right] X_k z^k$$

$$+ (-1)^n \sum_{k=1}^{\infty} \left[\frac{1-\alpha}{\left[(k+1)(\mu w^\lambda - \eta) + k\right]^n \left[(\alpha+\rho) + (1+\rho)\left((k+1)(\mu w^\lambda - \eta) + k\right)\right]} \right] Y_k \overline{z}^k$$

Therefore

$$\frac{1}{1-\alpha} \sum_{k=2}^{\infty} \left(\frac{\left[(k-1)(\mu w^\lambda - \eta) + k)\right]^n}{\left[(\rho+\alpha) - (1+\rho)\left((k-1)(\mu w^\lambda - \eta) + k\right)\right]} \right) |a_k|$$

$$+ \frac{1}{1-\alpha} \sum_{k=2}^{\infty} \left(\frac{\left[(k+1)(\mu w^\lambda - \eta) + k)\right]^n}{\left[(\rho+\alpha) + (1+\rho)\left((k+1)(\mu w^\lambda - \eta) + k\right)\right]} \right) |b_k|$$

$$= \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k$$

$= 1 - X_1 \leq 1$, and also $f_n \in \overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$.

Conversely, assume that $f_n \in \overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$.

Setting,

$$X_k = \frac{1}{1-\alpha} \left(\frac{\left[(k-1)(\mu w^\lambda - \eta) + k\right]^n}{\left[(\alpha+\rho) - (1+\rho)\left((k-1)(\mu w^\lambda - \eta) + k\right)\right]} \right) |a_k|,$$

$0 \leq X_k \leq 1, k = 2, 3, \ldots..$

$$Y_k = \frac{1}{1-\alpha} \left(\frac{\left[(k+1)(\mu w^\lambda - \eta) + k\right]^n}{\left[(\alpha+\rho) + (1+\rho)\left((k+1)(\mu w^\lambda - \eta) + k\right)\right]} \right) |b_k|,$$

$0 \leq Y_k \leq 1, k = 1, 2, \ldots\ldots$

and $X_1 = 1 - \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k$.

Therefore f_n can be written as,

$$f_n(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k| \overline{z}^k$$

$$= z - \sum_{k=2}^{\infty} \frac{(1-\alpha)X_k}{\left(\left[(k-1)(\mu w^\lambda - \eta) + k\right]^n \left[(\alpha+\rho) - (1+\rho)\left((k-1)(\mu w^\lambda - \eta) + k\right)\right] \right)} z^k +$$

$$(-1)^n \sum_{k=1}^{\infty} \frac{(1-\alpha)Y_k}{\left(\left[(k+1)(\mu w^\lambda - \eta) + k\right]^n \left[(\alpha+\rho) + (1+\rho)\left((k+1)(\mu w^\lambda - \eta) + k\right)\right] \right)} \overline{z}^k$$

$$= \sum_{k=2}^{\infty} h_k(z) X_k + \sum_{k=1}^{\infty} g_{n_k}(z) Y_k + zX_1$$

$= \sum_{k=1}^{\infty} (h_k(z)X_k + g_{n_k}(z)Y_k)$, as required.

This completes the proof the theorem 5.1.

References

[1] J. Clunie and T. Sheil – Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser. Al. Math., 9 (1984), no. 3, 3-25.

[2] O.P. Ahuja, Planer harmonic univalent and related mappings, J. Inequal. Pure Appl. Math, 6 (2005), no.4, 122, 1-18.

[3] J. M. Jahangiri, N. Magesh and C. Murugesan, Certain subclasses of starlike harmonic functions defined by subordination, J. Frac. Cal. Appl., 8 (2017), no.2, 88-100.

[4] K. Rajya Laxmi and R. Bharavi Sharma, Coefficient Inequalities of Second Hankel Determinants for Some Classes of Bi-Univalent Functions, Mathematics., 2016, 4, 9; doi:10.3390/math4010009

[5] S. Altinkaya and S. Yalcin, On a Class of Harmonic Univalent Functions Defined by Using a New Differential Operator, Th. Appl. of Math. And Comp. Sci. 6 (2) (2016) 125-133.
[6] J. M. Jahangiri, G. Murugusundaramoorthy and K. Vijaya, Starlikeness of Ruscheweyh type harmonic univalent functions, J. Indian Acad. Math. 26 (2004), 191-200.
[7] K. Rajya Laxmi and R. Bharavi Sharma, Second Hankel Determinant coefficients for some subclass of Bi-univalent functions, Global Journal of Pure and Applied Mathematics, vol.11(2) (2015), pp. 89-92.
[8] K. Thilagavathi, K. Vijaya and N. Magesh, Certain properties of a subclass of harmonic convex functions of complex order defined by multiplier transformations, Malaya J. Math. 4(3) (2016), 362-372.
[9] H. Silverman and E.M. Silvia, Subclasses of harmonic univalent functions, New Zeal. J. Math. 28 (1999), 275-284.
[10] R. Bucur, L. Andrei and D. Daniel, Coefficient bounds and Fekete-Szego problem for a class of analytic functions defined by using differential operator. Appl. Math. Sci. 9(2015), 1355-1368.
[11] S. Porwal and Shivam Kumar, A new subclass of harmonic univalent functions defined by derivative operator, Elect J. Math. Anal. and Appl., 5(1) (2017), 122-134.
[12] A.L. Pathak, S.B. Joshi, Preeti Dwivedi and R. Agarwal, A subclass of harmonic univalent functions associated with the derivative operator, Hacet. J. Math. and Stat., 41(1) (2012), 47-58.
[13] R. Bharavi Sharma and B. Ravindar, On a subclass of harmonic univalent functions defined by convolution and integral convolution, International Journal of Pure and Applied Mathematics, vol. 117(7) 2017, pp. 135-145.
[14] R. Bharavi Sharma and M. Haripriya, On a class of α-convex functions subordinate to a shell shaped region, The Journal of Analysis, vol.25(1) (2017), pp. 99-105.
[15] J. M. Jahangiri, Harmonic functions starlike in the unit disk, J. Math. Anal. Appl., 235, (1999), 470-477.
[16] N. Magesh and S. Mayilvaganan, On a subclass of harmonic convex functions of complex order, Int. J. Math. Sci. 2012, Article ID 496731, page 13.
[17] N. Magesh and S. Porwal, Harmonic uniformly β-starlike functions defined by convolution and integral convolution, Acta Univ. Apulensis Math. Inform, no. 32 (2012), 129-141.
[18] Sheil-Small. T, Constants for planar harmonic mappings, J. London Math. Soc. 2 (42) (1990), 237-248.
[19] R. M. Ali, S. K. Lee, V. Ravichandran, and S. Supramaniam, The coefficient functional for transforms of analytic functions, Bull. Iranian Math. Soc. 35(2) (2009), 119-142.
[20] R. Bharavi Sharma and B. Ravindar, On a subclass of harmonic univalent functions, Journal of Physics: Conf. Series 1000 (2018) 012115, doi: 10.1088/1742-6596/1000/1/012115.