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Research paper

Gutman Index and Harary Index of Unitary Cayley Graphs

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Abstract

In this paper, we determine the Gutman Index and Harary Index of Unitary Cayley Graphs. The Unitary Cayley Graph X_n is the graph with vertex set $V(X_n) = \{u|u \in Z_n\}$ and edge set $\{uv|gcd(u-v,n) = 1 \text{ and } u,v \in Z_n\}$, where $Z_n = \{0,1,\ldots,n-1\}$

Keywords: *Complete Graph; Gutman Index; Harary Index; Topological index; Unitary Cayley Graphs.*

1. Introduction

The general concepts of graph theory can be viewed in [2]. Here, we consider the Unitary Cayley Graph $X_n = Cay(Z_n;U_n)$ where Z_n is the additive group of intergers modulo n and U_n is the multiplicative group of its units $(n > 1)$. Therefore, its vertex set comprises of elements u in $\{0,1,\ldots,n-1\}$ and u, v are adjacent if and only if $gcd(u-v,n) = 1$. X_n is $\phi(n)$-regular where $\phi(n) = |U_n|$. Also, it is complete when n is prime p and complete bipartite when n is a prime power p^t (for the properties of unitary Cayley graphs, see [4]).
A topological index, also known as graph-theoretic index, is graph invariant and is a type of molecular descriptor [3]. Several distance-based and degree-based topological indices have been defined. Among them, we choose 2 distance based topological indices− Gutman Index and Harary Index for the computation of the respective indices of Unitary Cayley graphs.
Gutman proposed the idea of Gutman Index Gut(G) (Schultz index of the 2^{nd} kind) of a connected undirected graph G in 1994 [1]and it is defined as

$$Gut(G) = \sum_{u,v \in V(G)} d(u)d(v)d_G(u,v). \tag{1}$$

Plavšić et.al introduced the Harary index [5] of a graph G on n vertices in 1993 and it is defined as

$$H(G) = \sum_{u,v \in V(G)} \frac{1}{d_G(u,v)} \tag{2}$$

In both defintions, the summation goes over all unordered pairs of vertices of G, $V(G)$ represents the vertex set of graph G and $d_G(u,v)$ denotes the number of edges in a shortest path connecting vertices u and v. Also, $d(u)$ and $d(v)$ denote the degrees of vertices u and v.
In this paper, the folowing two lemmas (for the proof, see [4]) are applied for the computation.
Lemma 1.1: *The Unitary Cayley graph $X_n, n \geq 2$, is bipartite if and only if n is even.*

Lemma 1.2: *For integers $n \geq 2$, a and b, denote by $F_n(a-b)$ the number of common neighbours of distinct vertices a, b in the Unitary Cayley graph X_n is given by*

$$F_n(a-b) = n\prod_{p/n}(1-\frac{\varepsilon(p)}{p}), \tag{3}$$

$$\text{where } \varepsilon(p) = \begin{cases} 1, & \text{if p divides (a-b)} \\ 2, & \text{if p doesnot divide (a-b)} \end{cases} \text{(p is prime).}$$

2. Gutman Index of Unitary Cayley Graphs

In this section, Gutman Index of the Unitary Cayley graphs is determined.
Theorem 2.1: Let X_n be the Unitary Cayley graph on n vertices. Then for an integer $n \geq 2$, we deduce:

1. if n is prime, $Gut(X_n) = \frac{n(n-1)^3}{2}$.
2. if $n = 2^r$ and $r > 1$, $Gut(X_n) = \frac{n^3(3n-4)}{16}$.
3. if n is odd but not prime, $Gut(X_n) = \frac{n\phi(n)^2[2(n-1)-\phi(n)]}{2}$.
4. if n is even and has an odd prime divisor, $Gut(X_n) = \frac{n\phi(n)^2[5n-4(\phi(n)+1)]}{4}$.

Proof: Let X_n be the Unitary Cayley graph and X_n is $\phi(n)$-regular.

1. Suppose n is prime p.
   Then $X_p = K_p$, a complete graph.
   Therefore, by definition of $Gut(G)$ and $H(G)$,

$$\begin{aligned} Gut(X_n) &= \underbrace{\phi(n)^2 + \phi(n)^2 + \cdots + \phi(n)^2}_{\frac{n(n-1)}{2}} \\ &= (n-1)^2 \cdot [\frac{n(n-1)}{2}] \\ &= \frac{n(n-1)^3}{2}. \end{aligned} \tag{4}$$

2. Suppose $n = 2^r$ and $r > 1$.
Then $X_n = K_{n/2,n/2}$, a complete bipartite graph with $V(X_n) = V(AUB)$; $A = \{0,2,...,n-2\}$, $B = \{1,3,...,n-1\}$.
Therefore, by applying lemma 1.2, we obtain the distance between $(n/2)^2$ pairs of vertices as 1 and distance between $\frac{n(n-2)}{4}$ pairs of vertices as 2.

$$\begin{aligned} \text{So, } Gut(X_n) &= \sum_{u,v\in V(X_n)} d(u)d(v)d_G(u,v) \\ &= \sum_{u,v\in V(X_n)} d(u)d(v) + 2\sum_{u,v\in V(X_n)} d(u)d(v) \\ &= (n/2)^2\sum 1 + (n/2)^2\sum 2 \\ &= (n/2)^2\cdot n^2/4 + 2(n/2)^2\cdot\frac{n(n-2)}{4} \\ &= \frac{n^3(3n-4)}{16}. \end{aligned} \tag{5}$$

3. Suppose n is odd but not prime.
i.e., $n = (p_1)^{\alpha_1}(p_2)^{\alpha_2}\cdots(p_r)^{\alpha_r}$; $p_i \neq 2$ and $1 \le i \le r$. Therefore, we can infer that to every pair of distinct vertices, there exists a common neighbour by lemma 1.2.
Then distance between $\frac{n\phi(n)}{2}$ pairs of vertices is 1 and distance between $\frac{n[n-(\phi(n)+1)]}{2}$ pairs of vertices is 2.

$$\begin{aligned} Gut(X_n) &= \sum_{u,v\in V(X_n)} d(u)d(v)d_G(u,v) \\ &= \sum\phi(n)^2 + 2\sum\phi(n)^2 \\ &= \phi(n)^2\cdot[\frac{n\phi(n)}{2}] + 2n\phi(n)^2\cdot[\frac{n-(\phi(n)+1)}{2}] \\ &= \frac{n\phi(n)^2[2(n-1)-\phi(n)]}{2}. \end{aligned} \tag{6}$$

4. Suppose n is even and has an odd prime divisor p. Then X_n is bipartite with vertex partition $A = \{0,2,...,n-2\}$ and $B = \{1,3,...,n-1\}$. Also, $d(u) = d(v) = \phi(n)$ since X_n is $\phi(n)$-regular.
Claim: Calculate $d_G(u,v)$
To obtain $d_G(u,v)$, consider 2 cases.
Case 1: Consider $u \in A$.
Taking $v \in A$, we obtain a common neighbour by lemma 1.2. Thus $\mathrm{d}_G(u,v) = 2$. Taking $v \in B$, we obtain $d_G(u,v) = 1$ and $d_G(u,v) = 3$ by considering B as the union of 2 sets B_1 and B_2 comprising of elements adjacent to u and non-adjacent to v respectively.
Case 2: Consider $u \in B$.
Similarly, we obtain $d_G(u,v)$ as 1, 2 and 3 when $v \in B_1$, $v \in A$ and v_2 respectively.

$$\begin{aligned} Gut(X_n) &= \sum_{u,v\in V(X_n)} d(u)d(v)d_G(u,v) \\ &= \sum_{u,v\in V(X_n)} d(u)d(v) + 2\sum_{u,v\in V(X_n)} d(u)d(v) + 3\sum_{u,v\in V(X_n)} d(u)d(v) \\ &= \frac{\phi(n)^2\cdot n\phi(n)}{2} + 2\frac{\phi(n)^2\cdot(n^2-2n)}{4} + 3\frac{\phi(n)^2[n/2-\phi(n)]n}{2} \\ &= \frac{n\phi(n)^2[5n-4(\phi(n)+1)]}{4}. \end{aligned} \tag{7}$$

3. Harary Index of Unitary Cayley Graphs

We determine Harary Index of Unitary Cayley graphs in this section.
Theorem 3.1: For the Unitary Cayley graph X_n $(n > 1)$,
the Harary Index ,

$$H(X_n) = \begin{cases} \frac{n(n-1)}{2}, & n \text{ is prime} \\ \frac{n(3n-2)}{8}, & n = 2^r \text{ and } r > 1 \\ \frac{n(\phi(n)+n-1)}{4}, & n \text{ is odd but not prime} \\ \frac{n[5n+2(4\phi(n)-3)]}{24}, & n \text{ is even and has an odd prime divisor} \end{cases} \tag{8}$$

Proof: For n is prime, we get a complete graph X_n. So by definition, $H(X_n) = \underbrace{1+1\cdots+1}_{\frac{n(n-1)}{2}} = \frac{n(n-1)}{2}$.
For $n = 2^r$ and $r > 1$, we get a biclique X_n with vertex partition. Thus $H(X_n) = \frac{n(3n-2)}{8}$.
For n is odd but not prime, we get $d_G(u,v)$ as 1 and 2 (using lemma 1.2) respectively.

$$\begin{aligned} \text{Thus, } H(X_n) &= \sum_{u,v\in V(X_n)} \frac{1}{d_G(u,v)} \\ &= \frac{n\phi(n)}{2} + 1/2\cdot\frac{n[n-(\phi(n)+1)]}{2} \\ &= \frac{n[\phi(n)+n-1]}{4}. \end{aligned} \tag{9}$$

For n is even and has an odd prime divisor, we get a bigraph X_n. Then it can be easily understood from theorem 2.1 that $d_G(u,v)$ is 1, 2 and 3 respectively.

$$\begin{aligned} \text{Thus, } H(X_n) &= \sum_{u,v\in V(X_n)} \frac{1}{d_G(u,v)} \\ &= \sum\frac{1}{1} + \sum\frac{1}{2} + \sum\frac{1}{3} \\ &= \frac{n\phi(n)}{2} + \frac{n^2-2n}{8} + \frac{n[n/2-\phi(n)]}{6} \\ &= \frac{n[5n+2(4\phi(n)-3)]}{24}. \end{aligned} \tag{10}$$

4. Conclusion

In this paper, terminologies used were discussed as well. Moreover, the Gutman Index and Harary index of Unitary cayley graphs X_n were deduced for an integer $n \geq 2$.

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