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Research paper

Electrical insulation components reliability assessment and practical Bayesian estimation under a Log-Logistic model

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Abstract

This paper deals with the “physical reliability models” assessment and estimation for electrical insulation components. It is well known that the reliability model identification and estimation of most of the modern power system components, such as insulation components, may be better achieved, instead that using limited lifetime data, by the knowledge of the degradation mechanisms. Such mechanisms, which are responsible for component aging and failure, are indeed well established in the field of electrical insulation: this is also the case of the so called “Stress-Strength” models. In particular, the “Log-logistic” model, deduced by a suitable Weibull stress-strength probabilistic model, has found valid applications to the reliability assessment of the insulation components. In the framework of the estimation of such reliability model, a new Bayesian approach, based upon the “Odds Ratio” of the Log-logistic model is developed in this paper, based upon the properties that such information, being proportional to the reliability function, is available to the engineer on the basis of past data; moreover, being proportional to the Weibull scale parameter, allows to exploit known features of its conjugate prior Inverse Gamma distribution. Numerical examples and the results of extensive Monte Carlo simulations demonstrate the feasibility and efficiency of the proposed procedure.

Keywords: *Bayesian Statistics; Electrical Insulation; Log-Logistic Distribution; Stress-Strength Models; Weibull Distribution*

1. Introduction

Due to the high reliability values of modern components the classical “direct reliability assessment”, i.e. a reliability assessment via statistical fitting directly from in-service failure data of components, is rapidly becoming out of date. This is true, e.g., for most of the power system components, such as electrical insulation components, which play a crucial role in the whole electrical system. As pointed out by most of the modern literature on the subject [1-5], practical aids for reliability assessment can be given by the knowledge of the degradation mechanisms, which are responsible for component aging and failure. In the field of electrical insulation [6-14], such mechanisms were well established also by virtue of the availability of “accelerated tests” [15-18]. These aging and life models, when inserted in a probabilistic framework, lead to “physical reliability models”. In this respect, a key role is played by “Stress-Strength” models [5,6,19,20], which allow for evaluating the reliability of a given component in terms of the probability that its “Strength” is higher than the “Stress”. In the case of electrical insulation components, strength is the electrical endurance and the stress is the voltage surge amplitude.

A “Stress-Strength” model which has found some valid applications to electrical insulation components reliability assessment is the “Log-logistic” (LL) model. Its genesis can be deduced by a Stress-Strength model in which both Stress and Strength possess a Weibull distribution, whose validity in the field of electrical insulation is witnessed by many authoritative studies , such as [21,22].

It is recalled that the reliability function (RF) of a given component is defined as [4,5] the probability of the event $(T > t)$, where the random variable (RV) T is the component’s lifetime, and time t (a deterministic value) is the component’s age or service time:

$$R(t) = P(T > t) \quad (1)$$

In the case of the LL mode, the RF can be written as function of time $t > 0$, by [5]:

$$R(t) = 1/(1 + zt^{\beta}) \quad (2)$$

with parameters $z > 0$ and $\beta > 0$. The cumulative density function (cdf) and probability density function (pdf), when LL is used are given by, respectively:

$$F(t) = 1 - R(t) = zt\beta/[1 + zt^{\beta}] \quad (3)$$

$$f(t) = dF(t)/dt = \beta zt^{\beta-1}/[1 + zt^{\beta}]^2 \quad (4)$$

The LL model did not receive much attention in survival data analysis. In [5], the authors also discuss the similarity between the LL and the Weibull model (Appendix A), apart their hazard rate function (HRF). In the LL model, the HRF, $h(t)$, is given by:

$$h(t) = \frac{f(t)}{R(t)} = \beta zt^{\beta-1}/[1 + zt^{\beta}] \quad (5)$$

which is always decreasing with t if $\beta \leq 1$; when $\beta > 1$, $h(t)$ first increases, then decreases with time. In particular, in the latter case $h(t)$ starts from $h(0) = 0$, then reaches a maximum and goes to zero as t diverges.

It has to be pointed out that also another, more popular model features these properties of the HRF function, i.e. the Lognormal (LN) model [5,23,24], which was recently found adequate for lifetime characterization of insulation in transformers [3]. In fact, the LL distribution – as discussed in [5,6] – shares many properties with the LN distribution. The LL model is simpler analytically than the LN, but, on the other hand, it appears to be more difficult to estimate. In fact, methods such as the Maximum Likelihood (ML) allows an easy assessment of the parameter estimates for any given Lognormal RV Y, by using classical and well established methods for the Normal model of the RV $Z = log(Y)$.

It can be shown that the "Skewness Coefficient" [23] of the LL model is positive and always larger than the corresponding Weibull one, possessing the same coefficient of variation (CV) value. Thus, the LL model possesses generally larger "tails" than the Weibull one with the same central parameters and this may lead to underestimate the upper quantiles of the lifetime if a Weibull model is fitted to data generated in fact from a LL model.

In this paper, the proposed Bayes Statistical Estimation (BSE) is based upon the "Odds Ratio" (OR), Q, here defined by the ratio $F(t)/R(t)$, which by virtue of (3) and (4) is expressed by:

$$Q = F(t)/R(t) = zt^{\beta} \quad (6)$$

In the statistical literature, the OR is the ratio of the probability that the event of interest, here the component's failure before time t (given by $F(t)$) , occurs to the probability that it does not (given by $R(t)$). In this paper, the parameter β is considered known, while the parameter z is an RV according to the Bayes approach [6]. However, it is not easy to express a prior knowledge on the parameter z, which has not a direct physical or mathematical meaning, while it is easy to express a prior knowledge on the RF at a certain time t_0, $R(t_0)$, which is related to the Odds ratio Q at t_0 since:

$$Q(t_0) = (1 - R(t_0))/ R(t_0) = 1/R(t_0) - 1 \quad (7)$$

or:

$$1/R(t_0) = 1 + Q(t_0) \quad (8)$$

In the following, being t_0 a specified time value, $R(t_0)$ and $Q(t_0)$ will be simply denoted as R and Q. Of course, $1/R > 1$, so $Q > 0$, and a prior knowledge on the parameter R implies a prior knowledge on the parameter Q which can be described by any positive RV on $(0, \infty)$, such as the Gamma or Inverted Gamma (IG, Appendix B) or the above recalled LN RV. This implies in turn a prior knowledge on the LL parameter z, which is related to the parameters of the Weibull model of the stress, as discussed in the following section. So, as it will be shown, the proposed estimation procedure only requires a prior guess on the RF and some statistical data from the stress distribution; this is a realistic assumption since the stress data are easily observable, while, as above pointed out, there is a practical difficulty in obtaining component failure data. In fact, the scarcity of data for high-reliability components and some a priori technological knowledge about the physical degradation process of insulating materials does motivate the use of BSE in this context, as discussed in [6].

The rest of the paper is organized as follows: in Section 2, the deduction of the LL model is briefly illustrated. In Section 3, the analytical determination of prior and posterior distributions of reliability and other related parameters, and their Bayes point and interval estimates, are illustrated considering suitable prior distributions for the OR. In Section 4, an application of the proposed Bayesian methodology is shown, while in Section 5 an evaluation of the efficiency of the methodology is provided, by considering a large set of Monte Carlo simulations in which the parameters estimates are compared with those obtained by the ML estimate method. Such evaluation should includes an adequate "robustness analysis", as hinted at in the conclusions.

2. A "Log-Logistic" Model deduced from a Probabilistic Stress-Strength Model

In this section, it is briefly recalled how the LL model can be deduced in the case of the insulation components of the power system, when overvoltages occur. Overvoltage surges are assumed as random events described by a RV X, i.e. the peak value of electric stress affecting the insulation during the surge.

Insulation fails occurs when an overvoltage amplitude exceeds the component residual impulse strength: this is assumed as a RV referred to as Y. Then, the reliability assessment problem is dealt with in the framework of probabilistic Stress-Strength models [5,6,18-20], i.e. by writing the RF, for a given mission time t, as:

$$R = P(X < Y) \quad (9)$$

where X ("Stress") is the peak value of the switching voltage surge, Y ("Strength") is the insulation electric strength. Note that, as generally accepted for the applications under study, both X and Y are random variables here assumed as statistically independent. Furthermore, they share the same physical dimension which is, in the considered case, the dimension of an electric field. Denoting with $f(y)$ ($F(y)$) the pdf (cdf) of strength Y, and with $g(x)$ ($G(x)$) the pdf (cdf) of stress X, the RF of the components under study is given by:

$$R = \int_0^{\infty} g(x)P(X < Y|X = x)dx = \int_0^{\infty} g(x)(1 - F(x)dx \quad (10)$$

It is recalled that the Weibull model it is by far the most adopted one in the field of electrical insulation, for both Stress and Strength. Indeed, it has kept proving over the years one the most adequate for the statistical fitting of both stress and strength data, and some literature [5,6] illustrates the fact that it possesses some physical background and motivation for such applications. So, under reasonable hypotheses [6], let X and Y be Weibull RVs with equal shape parameter γ and with scale parameters θ for the stress X, and α for the strength Y, i.e. let the cdf of X and Y be given, respectively, by:

$$G(x) = 1 - exp[-(x/\theta)^{\gamma}]$$
$$F(y) = 1 - exp[-(y/\alpha)^{\gamma}] \quad (11)$$

The common shape parameter γ is constant with aging time, while the scale parameter of Y, $\alpha(t)$, varies with time. As for time dependence of $\alpha(t)$, it is reasonable to consider the following "Inverse power" characterization of the Strength scale parameter α with time t, in which k and m are positive constants:

$$\alpha = \alpha(t) = k/t^m \quad (12)$$

Such a model was proposed in [5,6], as motivated by studies on insulation [21,22].

Indeed, since the expectation of Y is proportional to α - it is recalled from Appendix that $E[Y] = \alpha\Gamma(1 + 1/\gamma)$ - relationship (12) implies that the expected value of Y decreases with time t as a power function of t, a popular model in such kind of analyses [5,6]. So, it can be easily shown [5,6] that, the RF can be easily derived in closed form from eqns. (10)-(11):

$$R(t) = 1/\{1 + [\theta/\alpha(t)]^{\gamma}\} = 1/[1 + zt^{\beta}] \quad (13)$$

which is the above introduced LL model [5,6] with parameters:

$$\beta = m\gamma; \; z = (\theta/k)^{\gamma}$$

The positive parameters, γ, m, k are assumed as known[1], so that the only unknown parameter is z, or (equivalently) θ.
Recalling the LL hazard rate expression, as pointed out in [6], on the basis of the above Stress-Strength model, a reasonable explanation of the seemingly strange decreasing hazard rate for large values of time is given. Such property has been sometimes observed indeed for insulating materials, and has until now been discussed in theoretical reliability literature in relation with heterogeneity of materials [25, 26].

3. Illustration of the proposed Bayesian estimation procedure

Adopting a Bayesian approach [27-30], the unknown quantity z is characterized as an RV, Z. Consequently, also the RF and other reliability parameters (e.g. failure time percentiles) are defined as RV, described by appropriate distributions.
As above hinted, the proposed methodology starts from assigning a prior pdf to the RF of the LL model [5,6] here recalled, and restated in terms of the RV Z:

$$R(t|Z) = 1/[1 + Zt^{\beta}] \tag{14}$$

where: $\beta = m\gamma$; $Z = (\theta/k)^{\gamma}$.

Since the parameters k, m, γ are considered known (thus, also the parameter $\beta = m\gamma$ is considered known), the randomness of Z implies univocally the randomness of θ, i.e. the scale parameter of the Weibull stress pdf, here written as conditional pdf, $g(x|\theta)$, of x (generic stress value) conditional to the value θ of the RV to be estimated.

$$g(x|\theta) = (\gamma/\theta)(x/\theta)^{\gamma-1}exp[-(x/\theta)^{\gamma}] \qquad (x > 0) \tag{15}$$

For purpose of mathematical convenience, it is opportune to adopt a new parametrization of the above pdf in terms of the new positive parameter $\eta = \theta^{\gamma}$, so that the Weibull stress pdf is expressed as follows:

$$g(x|\eta) = (\gamma/\eta)x^{\gamma-1}exp[-(x/\eta)^{\gamma}] \qquad (x > 0) \tag{16}$$

A straightforward BSE on η can be accomplished if, as here assumed, the stress values are observable, and the assumption of an "Inverted Gamma" (IG) pdf (App. B) for the Weibull parameter η is made. It is recalled that BSE [27-30] starts assuming a prior pdf of the parameter $\eta, p(\eta)$.
Once a data set D is observed, i.e. a random sample from n stress values is obtained, namely:

$$D = (X_1, \ldots X_n) \tag{17}$$

such information is used to derive the posterior pdf of $\eta, p(\eta|D)$, via the well-known Bayes formula, i.e.:

$$p(\eta|D) = p(\eta)L(D|\eta)/C \tag{18}$$

where $L(D|\eta)$ is the Likelihood Function of the data (i.e., in this case, the product of the n Weibull pdfs of the X_j, conditional to η) and C is a constant (with respect to η, but function of the given sample D) which has the following expression:

$$C = \int_0^{\infty} L(D|\eta)p(\eta)d\eta \tag{19}$$

[1]Such values, being characteristic of the electrical insulation's strength, may be deduced Q by data analyses provided by accelerated tests [15].

Then, as well known, once the posterior pdf of η, $p(\eta|D)$, has been obtained, the best Bayes estimate - in the mean square error sense - of a given function $\tau = \tau(\eta)$ is given by the posterior mean:

$$\tau^0 = \int_0^{\infty} \tau(\eta)p(\eta|D)d\eta \tag{20}$$

As per the prior pdf of η, the above said IG model is rather flexible, and it is a "conjugate" pdf: i.e., the posterior pdf of η, with updated parameter values, is again IG, as it well known in BSE, and recalled in App. B: let an Inverted Gamma be the prior pdf on η, denoted as $p(\eta) = p(\eta; \nu, \delta)$, with positive parameters (ν, δ):

$$p(\eta) = \frac{1}{\eta^{(\nu+1)}\delta^{\nu}\Gamma(\nu)} \exp\left(-\frac{1}{\delta\eta}\right) \tag{21}$$

which is denoted IG pdf $(\eta;\nu,\delta)$, where ν and δ are the shape and scale parameter, respectively. The mean value and variance of the distribution are finite only if $\nu > 1$ (mean) and $\nu > 2$ (variance). They are:

$$E[\eta] = \frac{1}{\delta(\nu-1)} \tag{22}$$

$$Var[\eta] = \frac{1}{[\delta^2(\nu-1)(\nu-2)]} \tag{23}$$

Such prior is converted into a posterior pdf of η which is again an IG pdf, with updated parameters:

$$p(\eta|D) = \text{IG pdf}\,(\eta; \nu_1, \delta_1), \tag{24}$$

where:

$$\nu_1, = \nu + n, \delta_1 = \delta/(1 + U\delta) \tag{25}$$

and

$$U = U(D) = \sum_{j=1}^{n} X_j^{\gamma} \tag{26}$$

So, the prior Bayes estimate of η is given by above prior mean of the IG pdf, while the posterior Bayes estimate is:

$$E[\eta|D] = \frac{1}{\delta_1(\nu_1 - 1)} \tag{27}$$

An Inverted Gamma prior pdf on η implies an "Inverted Generalized Gamma" (App. B) distribution on θ ($\theta = \eta^{1/\gamma}$), which is quite equivalent in view of estimation, but more complicated analytically: that is the reason that the parameter η was chosen as the basic parameter. However, uncertainty about the random parameter η is rather difficult to assess, while the RF, i.e. the probability that the lifetime is higher than a given value t, has a more direct interpretation, and should be easily available to the engineer on the basis of past life data, experience on similar components, specific databases, or expert's opinion. So, as above discussed, it appears preferable to express a prior knowledge on the RF $R(t_0)$ at a certain time t_0, which is closely related to the Odds ratio Q (we omit the given time value t_0 assumed for the BSE), since:

$$Q = 1/R - 1 \tag{28}$$

So a prior pdf on R implies a prior pdf on Q, and thus on Z, given that:

$$Q = (1-R)/R = Zt_0^{\beta} \tag{29}$$

Then, by virtue of the above relations, a prior pdf on R implies a prior pdf on η:

$$\eta = k^{\gamma} Z = \chi Z \tag{30}$$

where $\chi = k^{\gamma}$. Finally:

$$Q = (\eta/k^{\gamma})t_0^{\beta} \tag{31}$$

As reported in formula (28), apart from the constant -1, the uncertainty on Q is the same as for R, since of course the prior knowledge on R implies an equivalent prior knowledge on $1/R$.

It is remarked that each of the previous relations (between R and Q, Z and, R and Z), is a one-to-one relation, thus allowing a simple deduction of any implied pdf above discussed.

The proposed approach uses established methods to transform the prior knowledge in terms of a RV X into the one for another RV Y related to X by a given one-to-one transformation: $Y = h(X)$. Once the prior pdf is chosen, the simplest way to assess its parameters is to assess its mean and variance or other moments, or the values of some given percentiles (two percentiles are sufficient for most of the prior pdf, as those here adopted.)

Anticipating the numerical illustration of sec. 5, let us assume that $k = 1$, with no loss of generality (since from the above equations it is readily seen that a changing on time scale can always transform the dimensional constant k to the value 1), so:

$$Q = \eta t_0^{\beta} \tag{32}$$

Whatever the values of the known parameters, the Odds ratio Q is proportional to η, so assessing the uncertainty (i.e., the prior pdf) on Q (or, equivalently, on R) is directly transformed into an equivalent information (i.e., the "implied" prior pdf [28]) of η. In order to exploit the above said features of the IG pdf, it is convenient to adopt an IG prior pdf for Q[2]. Given, for instance, a prior guess on R in terms of mean and variance, or the values of two given percentiles, the corresponding parameters of $Q = (1/R - 1)$ can be obtained (in analytical form or by suitable approximation), and then the corresponding IG pdf of Q , denoted as $p(q) = IGpdf(q; \nu_q, \delta_q)$.

Finally, by virtue of (31), the corresponding IG pdf of η is the $IGpdf(\eta; \nu, \delta)$ with the same shape parameter $\nu = \nu_q$ and scale parameter $\delta = \delta_q t_0^{\beta}$.

So, as will be done in the following sections, the BSE procedure can be illustrated in terms of the Bayes estimates of Q, which are equivalents to those of η, and also allow to obtain the corresponding pdf and Bayes estimates of R, related to Q by the relation (26).

Indeed, by known rules of RV transformations [24], the pdf p_R (be it the prior or the posterior pdf) of R is easily expressed in terms of p_Q, which is the pdf of Q, by means of the transformation:

$$R = 1/(1+Q) \tag{33}$$

resulting in the following general expression:

$$p_R(r) = \frac{1}{r^2}\, p_Q\left(\frac{1}{r} - 1\right) \tag{34}$$

for $0 < r < 1$ and $p_R(r) = 0$ elsewhere.

It is remarked that the above pdf of R doesn't depend on the particular form of the pdf of Q and, in primis, it is easily computable analytically, whatever be the pdf of Q. Finally, the availability of the posterior pdf $p_R(r|D)$ allows, possibly by numerical methods, to obtain the Bayes estimates of R at any given instant t, using the concepts behind (20). The problem of obtaining the estimates of R will not pursued here for sake of brevity, being the object of future studies, but in any case the above integration poses no problem, However, some valid approximate value of such estimates of R are easily obtained analytically, as it will be shown in short.

In practice, for what above discussed, it is the prior of R which is transformed into a prior pdf of the OR, and for such purpose a Beta pdf is adopted, as generally done in such cases [28] and illustrated is sec. 4. Moreover, it will be shown at the end of the paper that the same prior information can be assessed on terms of lifetime percentiles, instead of Q or R.

4. A numerical illustration of the proposed Bayesian estimation approach

The BSE procedure described above is briefly illustrated here by means of an application to typical data of distribution cable, as in [6], based upon a typical sample of stress values and a prior guess o the RF. Only for sake of illustration, in order to highlight the procedure in its basic features, and with no loss of generality (since all the following parameters are known), the following parameter values are assume from now on:

k= 1; m =0.5, γ =5 (thus, β=mγ =2.5)

so: $Q = \eta t^{\beta} = \eta t^{2.5}$

As discussed in sec. 3, the BSE procedure can be stated in terms of the prior pdf of $R = R(t_0)$, a parameter on which some degree of information is generally available to the engineer. It is observed that the RF of the LL model, $R(t) = 1/\,(1 + zt^{\beta})$, can be expressed in the alternative form as $R(t) = 1\,/(1 + (t/\tau)^{\beta})$ with the obvious parameter transformations:

$$\tau = (1/z)^{(1/\beta)} \tag{35}$$

This shows that the above model has median value τ (i.e.: $R(\tau) = 0.5$, whatever the value of β). Let us suppose, for sake of illustration, that the true median value is τ =25 years, which is a typical value for such kind of components (the corresponding value of z is 3.2e-04) [6]. In order to illustrate the key features of the BSE procedure, two examples are worked out with different prior pdfs and the same data sample.

4.1. Example 1

Let us suppose that the engineer, on the basis of her/his past experience, expresses a prior guess on the R value R=R(t_0), where t_0=25 years, which is correct in its prior mean value, i.e.: E[R]=0.5. Moreover, she/he assesses her/his uncertainty on the above prior value by means of a SD value which is 10% of the mean value, i.e. the value of the coefficient of variation (CV) of $R = R(t_0)$ is:

$$CV[R] = SD[R]/\,E[R] = 0.10$$

The most practical and flexible model to express a prior pdf for a RV confined in the (0,1) interval such as R is with no doubts the Beta pdf [30], which has the following form, in which r is a generic value of the RV R, and a and b are positive parameters:

[2] As hinted at the end of the paper, also different prior pdf have been chosen, in order to test the robustness of the method with respect to the assumed prior pdf.

$$g(r;a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} r^{a-1}(1-r)^{b-1}, \quad 0<r<1 \tag{36}$$

Inverting the analytical expressions relating the parameters (a,b) to the mean ad SD of R [23], one obtains the following parameter values: a=b=49.5, and the corresponding prior pdf of R is depicted in Fig. 1.

It is observed that an equivalent way to express the above uncertainty would be to assess the values of two percentiles. In such case, e.g. the 5th and 95th percentiles are respectively 0.4177 and 0.5823. In other words, such values form the 90% "Bayesian Confidence Interval" [30], say (R_1, R_2), for $R = R(t_0)$, such as:

$$P(R_1 < R < R_2) = 0.90,$$

And this interval is also symmetrical with respect to the prior mean value 0.5 of R, indeed: (0.4177 + 0.5823)/2=0.5000.
It is possible to show that the corresponding prior pdf of the OR, $Q = Q(t_0)$,:

$$Q = (1 - R)/R \tag{37}$$

may be well approximated by a suitable IG pdf, with parameter values obtained by the mean and SD of Q, which can be in turn deduced by the above parameters of R. For sake of simplicity of illustration, here the deduction of such prior pdf of Q from the one of R is shown by means of Monte Carlo simulation. In Fig. 2, the histogram of n=10^4 simulated values of Q corresponding to the prior pdf of R in Fig. 1 is depicted[3].

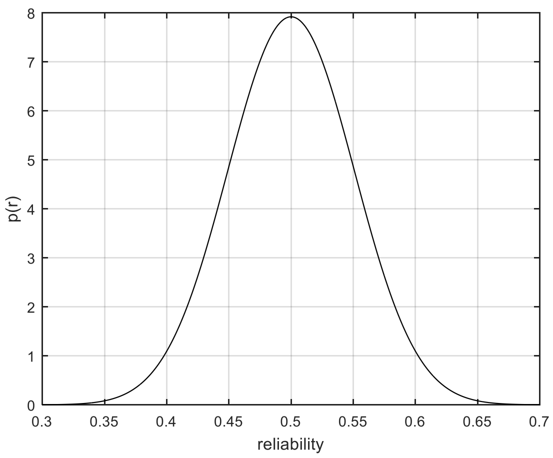

Fig. 1 Prior pdf of R

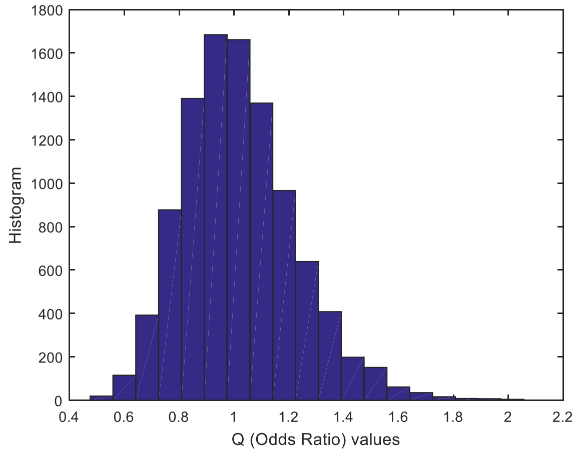

Fig. 2. Histogram of simulated values of Q corresponding to the prior pdf of R in Fig. 1

The mean and SD values of the above distribution, 1.0200 and 0.2098 respectively, are in accordance with those which can be evaluated numerically, e.g. by means of Taylor expansion [27]. So, inverting the relations expressing the mean and SD as functions of the IG parameters (App. B; see also second example for details), the corresponding IG pdf of Q is obtained as:

$$p(q) = IGpdf(q; \nu_q, \delta_q)$$

with: ν_q= 25.6337 δ_q = 0.0398[4]
The excellent degree of approximation (or "goodness of fit") of the above IG pdf is shown in Fig. 3. The prior mean of Q, 1.02, roughly corresponds to the prior mean of R, 0.5, since $Q = 1/R - 1$, so that $E[Q] \cong E[1/R] - 1 = 2 - 1 = 1$ (in fact, by Jensen inequality: $E[1/R] > 1/E[R] = 2$, so that $E[Q] > 1/E[R] - 1 = 1/0.5 - 1 = 1$).
Such prior pdf is characterizes by a slight degree of asymmetry with positive skewness [23], indeed its mode (0.9434) is smaller than its median (0.9931), which is smaller than its mean (1.2000).
The prior pdf is then been updated on the basis of a sample of 7 elements of stress X, randomly generated from the assumed Weibull distribution, with Z (or, equivalently, Q) generated by the above Inverted Gamma pdf.

The sample values of X, i.e. the data D of equation (17), are:

D = (0.1769 0.2592 0.1145 0.1456 0.0984 0.1745)

The value of the sufficient statistics U, formula (26), is given by U =0.0017 (we recall that γ is known: γ=5). The other sufficient statistics is n=7 (sample size). Such values are all is needed to deduce the posterior pdf of Q, which is an IG pdf:

$$p(q|D) = IGpdf\ (q; \nu_1, \delta_1)$$

whose updated parameters, after easy computations, are:

$$\nu_1 = \nu + n = \nu + 7 = 32.6337$$
$$\delta_1 = \delta/[(1 + U\,\delta t^{\beta}] = 0.0329$$

[3] The random values q_k of Q where generated using the Monte Carlo simulation for generating random values r_k of R by means of the function "*betarnd*" of Matlab © software, and then applying, from (37): q_k =(1-q_k)/q_k

[4] In the following, the symbols (ν,δ) will be used instead of (ν_q,δ_q) for the prior parameters of Q, while (ν_1,δ_1) will denote the posterior parameters of Q.

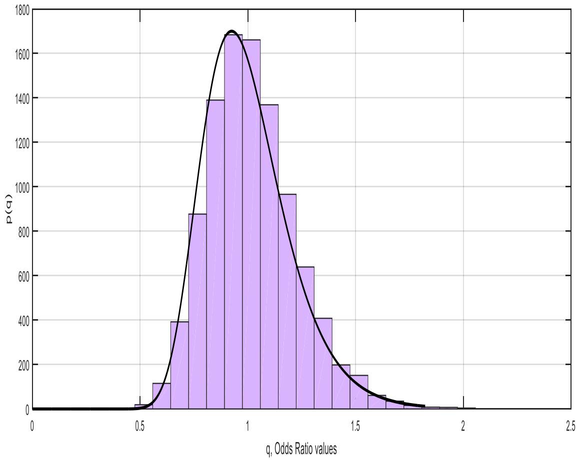

Fig. 3 IG prior pdf of Q superimposed to the Histogram in Fig. 2

The above prior pdf and the posterior pdf of Q are shown in Fig. 4. As above discussed, the BE, say $Q°$, of Q is the posterior mean:

$$Q° = E[Q|D] = 1/[\delta_1(\nu_1 - 1)] = 0.9622$$

The estimate of the RF requires numerical integration of the pdf of R in (34), but can be roughly approximated by: $R° = 1/(1 + Q°) = 1/(1 + 0.9622) = 0.5096$ (while the prior mean was 0.5000). The posterior pdf of R, which is of course an interesting information to estimate, but has a more cumbersome expression with respect to the one of Q, will be discussed through the percentiles of R.

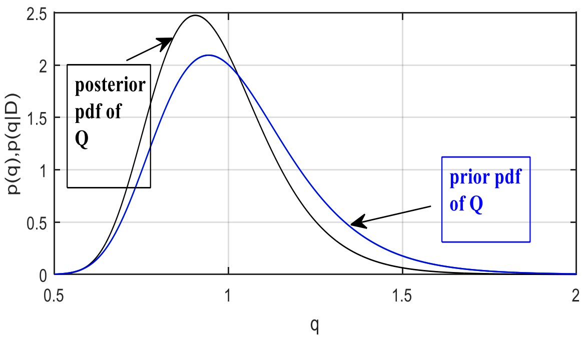

Fig. 4 Prior pdf of Q (blue line) and posterior pdf of Q (black line)

The point estimate is only a parameter of the whole posterior pdf of Q, about which following main comments can be made (numerical values reported afterwards can be easily obtained by the above equations regarding the prior and posterior IG pdf):

1) The CV of the G pdf of Q assumes the value 0.2057 in the prior model, 0.1807 in the posterior model, thus denoting a "shrinkage" of the distribution. This is confirmed, e.g., by the symmetrical 90% Bayesian Confidence Interval of Q at $t = 25$, (Q_1,Q_2), which is (0.7285,1.4028) for the the prior model. Instead, for the posterior model, the same interval becomes: (Q'_1,Q'_2)=(0.7151,1.2768), with a decrease of 17% about in amplitude.

2) The posterior pdf of Q is slightly "shifted" to the left with respect to the prior pdf, thus implying larger values of $R = 1/(1 + Q)$. In practice, however, the prior and posterior pdf appear to yield substantially the same information. Indeed, both the mode (0.9050) of the posterior pdf of Q is smaller than the prior one (0.9434), and also the posterior mean (the above computed value 0.9622) is smaller than the prior mean (1.0200). However, such differences are rather small, evidencing that the prior information is substantially confirmed, but with less uncertainty, as the diminution of the CV shows. This could be expected, as long as sample data was generated through the assumed prior information.

The first property – the one of a "shrinkage" of the distribution - is a general one, being typical of the Bayesian inference: the posterior distribution is more concentrated, thus the OR estimate is less uncertain after a data sample has been obtained, due to the gain in information. The same happens of course to the reliability.
It was already reported that the symmetrical 90% Bayesian Confidence Interval of reliability at $t = 25$, (R_1,R_2), was (0.4177, 0.5823) for the prior model. Instead, for the posterior model, the same Interval, being $R = 1/(1 + Q)$ a monotone decreasing function of Q, is evaluated as follows[5], in terms of the corresponding 90% Interval of Q:

$$R'_1 = 1/(1 + Q_2) = 1/(1 + 1.2768) = 0.4392$$
$$R'_2 = 1/(1 + Q_1) = 1/(1 + 0.7151) = 0.5831$$

So the interval of R decreases of 13% about in amplitude.
The above figures confirm that also the posterior distribution of R is more concentrated, and shifted to the right (towards larger values of R) with respect to the prior pdf, in accordance with the increase of the BE of R from the prior mean (0.5000) to the posterior prior (0.5096). This is also confirmed if the posterior pdf of R is examined, although it has a cumbersome expression, which can nonetheless be easily obtained through (34) in which p_q is the posterior pdf of Q.
It is however remarked that the two RV, Q and R, carry the same information, as above discussed, so the way by which to assess such information is more or less a matter of convenience.
Ultimately, it is pointed out that - both for Q and for R - the discrepancies between the posterior and the prior distribution in terms of uncertainty measures (CV, Confidence Interval amplitudes) is of course limited, given that only seven data are observed, nonetheless it is noticeable in view of the small sample size.

4.2. Example 2

In this second example, a different, prior distribution, is adopted, supposing that the engineer assumes a slightly "wrong" prior distribution. The example, illustrated only numerically for sake of brevity, serves to highlight another key feature of BSE, i.e. that a possibly "wrongly specified" prior distribution may be easily corrected into realistic posterior distributions. Taking also in account the results of example 1, this is in accordance with the useful property often found for BSE, the one of "improving the prior information when it is poor, or substantially confirming it when it is good" [15], even in the presence of a small sample.
Here, the same small sample D of seven elements is assumed (generated through simulation according to the "right" prior distribution), but a more pessimistic prior is assumed, i.e. one with the following prior parameter:
$E[Q] = 1.2000$
which is higher than the previous value 1.0200; $E[Q] = 1.2000$ implies a corresponding lower value of the mean of R, roughly:
$E[R] \cong 1/(1 + E[Q]) = 0.4545$
For sake of comparison, the same prior CV as in the 1st example is assumed, $CV[Q] = 0.2057$, so the same shape prior parameter of Q is obtained, as it is easy to verify:

$$\nu_q = (1/CV^2 + 2) = 25.6337$$

The SD of Q is evaluated as SD[Q]=CV[Q] · E[Q]= 0.2057·1.2 = 0.2468. The scale parameter is:

[5] Also this is a "first order" approximation, analogous to the one of the expected value: $E[R] \cong 1/(1 + E[Q])$

$\delta_q = 1/\{E[Q]\cdot(\nu_q - 1)\} = 1/[1.2\cdot(25.6337-1)] =$
$= 0.0338$

Omitting simple numerical details, since the values of the sufficient statistics U and n are still (being the sample invariant[6]) given by $U = 0.0017$ and $n = 7$ (sample size), it is easy to deduce the posterior pdf parameters of Q:
$\nu_1 = \nu_q + 7 = 32.6337$
$\delta_1 = 0.0287$

The results imply that:

1) The posterior mean, i.e. the BE of Q, is 1.1015, roughly halfway between 1.0 (the "true" value corresponding to $R = 0.5$) and 1.2 (the wrongly assumed prior estimate). The corresponding BE of R, is about: $E[R|D] \cong 1/(1 + E[Q|D]) = 0.4759$,which is nearer to the "right" value 0.5000[7].

2) The same diminution of the CV of Q occurs as in example 1, from the value CV=0.2057 in the prior model, to the new CV= 0.1807 in the posterior model, denoting the already observed "shrinkage" of the distribution.

3) The symmetrical 90% Bayesian Confidence Interval of Q at t=25, (Q_1,Q_2), is (0.8571, 1.6503) for the the prior model. Instead, for the posterior model, the same Interval is: (Q'_1,Q'_2) = (0.8193,1.4628), with a decrease of 19% about in amplitude.

4) Analogous considerations hold for the symmetrical 90%Bayesian Confidence Interval of R at t=t_0=25, (R_1,R_2), which is for the new prior model of this second example:

$$R_1 = 1/(1 + Q_2) = 1/(1 + 1.6503) = 0.3773$$
$$R_2 = 1/(1 + Q_1) = 1/(1 + 0.8571) = 0.5385$$

Instead, for the posterior model, the same interval is evaluated as follows, with a decrease of 11% about in amplitude.

$$R'_1 = 1/(1 + Q'_2) = 1/(1 + 1.4628) = 0.4060$$
$$R'_2 = 1/(1 + Q'_1) = 1/(1 + 0.8193) = 0.5497$$

The halfway point of the above posterior interval, i.e. (0.4060 + 0.5497)/2 = 0.4723 is close to the BE of R, above evaluated as 0.4759.

Briefly, such results confirm the above recalled BSE property of "improving the prior information when it is poor"; indeed, as above evaluated, the posterior estimate of R, 0.4759, is roughly halfway between 0.5000 (the "true" value) and 0.4545[8] (the wrongly assumed prior estimate). It is again remarked that this improvement occurs even after observing a small sample.

5. Evaluation of the Efficiency of the Proposed Bayes Estimation

In order to assess the overall performance of the proposed Bayes estimation - and its merits with respect to the traditional ML one - a large series of Monte Carlo simulations have been carried out, relevant to different sample sizes. Here, a small sample of the results obtained when evaluating the efficiency of the proposed Bayes estimation is reported for sake of illustration. A thorough analysis has been accomplished by assuming various typical values for the mean and SD of the basic random parameters of interest. For each set of the above values, N random samples of size n of stress values X_j have been generated according to the assumed Weibull pdf of X, with random parameter η which was, in turn, generated according to its prior IG distribution.

For each sample size n, a number N=10000 of replications has been performed; let us refer to each considered couple of values of n and N as a single simulation case study; different simulation case studies were considered, by testing different sample sizes. Here in particular – in view of the previously discussed need to deal with a limited amount of data - the results for small (n = 1 ,n= 3, n= 5) or moderate (n= 15, n= 20) sample sizes are reported in the following Tables, among the many more performed. With reference to the generic parameter ζ to be estimated, it will be denoted as $\zeta_j°$ the estimate of the "true" value ζ_j of ζ relevant to the particular n-sized sample generated at the j^{th} simulation cycle. The basic statistics - estimated at the end of each simulation case study - which describe the efficiency of the proposed estimates, are:

- *MSEB*: Mean Square Error of the Bayes estimator;
- *MSEL*: Mean Square Error of the ML estimator;
- *REFF = MSEL/MSEB*: relative efficiency of the Bayes estimator with respect to the ML estimator.

The above-defined quantities are based on the concept of "MSE". Given an estimator[9], ζ°, of the parameter ζ (in the present case, the only parameter to be estimated is Q) its MSE is – as well known - defined as:

$$MSE = \mathrm{E}[(\zeta° - \zeta)^2] \tag{38}$$

The MSE is here evaluated at the end of each simulation case study by means of the ordinary large-sample estimator [27] as:

$$MSE = \frac{1}{N}\sum_{j=1}^{N}\left(\zeta°_j - \zeta_j\right)^2 \tag{39}$$

The more the ratio REFF exceeds unity, the more efficient is the Bayes estimate as compared to the ML estimate.

Six sets of simulation cases, among the many more performed, are shown in Tables I-VI, with reference to the examples of Sec. 4. The first three (shown in Tables I-III), denoted as "Case A" as a whole, assume the "right" prior pdf of Q, with prior mean E[Q]=1.02 , of example 1, but with different values of prior coefficient of variation (the ratio CV between standard deviation and mean value of Q), giving rise to three different sub-cases (A1, A2, A3) as follows:

- Case A1: E[Q]=1.02 , CV= CV_0= 0.2057 (the same of example 1);
- Case A2: E[Q]=1.02, CV= CV_0/2;
- Case A3: E[Q]=1.02, CV= CV_0/3.

The three different cases are chosen in order to show different degrees of prior uncertainty (a larger prior CV value implies a higher degrees of prior uncertainty, so the BSE results are expected to be less efficient, as will be verified by the following results).

The following three more cases (shown in Tables IV-VI), denoted as "Case B" as a whole, assume the "wrong" prior pdf of Q, with prior mean E[Q]=1.2 , of example 2, and the same three different prior CV values as above for the following three different sub-cases (B1, B2, B3).

- Case B1: E[Q]=1.20, CV= CV_0=0.2057 (the same of example 2);
- Case B2: E[Q]=1.20, CV= CV_0/2;
- Case B3: E[Q]=1.20, CV= CV_0/3.

Moreover, also two more simulation sets are shown in Tables VII and VIII, in which the stress sample D was generated by a distribution which is different from the Weibull model assumed so far, as will be illustrated later in the framework of a "robustness analysis".

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[6]It is remarked that the above sample D used for the BE is generated according the "right" prior model of example 1, so it is not in accordance with the prior model assumed in this example.

[7]Of course, the terms *right* and *wrong* should be interpreted with caution, since, in the real practice, it is not known which the "right value" is. Here, it is meant as the value according which the simulated data D have been generated.

[8] The midpoint of the above interval is 0.4773.

[9]In the following, for a given parameter ζ, ζ° will denote its Bayes estimate, while ζ∗ will denote its ML estimate

In each Table, the sample sizes are reported in the first column. In order to facilitate the reproduction of simulations by part of the reader, the prior IG parameters (ν,δ) of the Odds Ratio Q[10] are reported in the second and third columns respectively. Then, the MSEB, MSEL and REFF indexes are reported in the fourth, fifth and sixth columns respectively.

TABLE I: NUMERICAL RESULTS OF ESTIMATION EFFICIENCY
CASE A1

sample size	*ν*	*δ*	*MSEB*	*MSEL*	*REFF*
1	25.63	0.0398	0.0405	1.0258	25.33
3	25.63	0.0398	0.0386	0.3806	10.00
5	25.63	0.0398	0.0365	0.2190	6.000
15	25.63	0.0398	0.0276	0.0734	2.659
20	25.63	0.0398	0.0236	0.0543	2.301

TABLE II NUMERICAL RESULTS OF ESTIMATION EFFICIENCY
CASE A2

sample size	*ν*	*δ*	*MSEB*	*MSEL*	*REFF*
1	96.53	0.0103	0.0109	1.0207	93.64
3	96.53	0.0103	0.0110	0.3398	30.89
5	96.53	0.0103	0.0104	0.2145	20.62
15	96.53	0.0103	0.0094	0.0700	7.436
20	96.53	0.0103	0.0091	0.0503	5.528

TABLE III NUMERICAL RESULTS OF ESTIMATION EFFICIENCY
CASE A3

sample size	*ν*	*δ*	*MSEB*	*MSEL*	*REFF*
1	214.7	0.0046	0.0064	1.0528	164.5
3	214.7	0.0046	0.0049	0.3524	71.92
5	214.7	0.0046	0.0047	0.2095	44.57
15	214.7	0.0046	0.0047	0.0070	14.89
20	214.7	0.0046	0.0045	0.0517	11.49

TABLE IV NUMERICAL RESULTS OF ESTIMATION EFFICIENCY
CASE B1

sample size	*ν*	*δ*	*MSEB*	*MSEL*	*REFF*
1	25.63	0.0338	0.0600	1.4767	24.62
3	25.63	0.0338	0.0540	0.4987	9.235
5	25.63	0.0338	0.0500	0.2911	5.822
15	25.63	0.0338	0.0378	0.0979	2.590
20	25.63	0.0338	0.0331	0.0729	2.2024

TABLE V NUMERICAL RESULTS OF ESTIMATION EFFICIENCY
CASE B2

sample size	*ν*	*δ*	*MSEB*	*MSEL*	*REFF*
1	96.53	0.0087	0.0150	1.4560	97.07
3	96.53	0.0087	0.0144	0.4756	33.03
5	96.53	0.0087	0.0147	0.2936	19.97
15	96.53	0.0087	0.0132	0.0958	7.258
20	96.53	0.0087	0.0126	0.0732	5.809

TABLE VI NUMERICAL RESULTS OF ESTIMATION EFFICIENCY
CASE B3

sample size	*ν*	*δ*	*MSEB*	*MSEL*	*REFF*
1	214.7	0.0039	0.0064	1.3831	216.1
3	214.7	0.0039	0.0066	0.4803	72.77
5	214.7	0.0039	0.0067	0.2883	43.03
15	214.7	0.0039	0.0064	0.0982	15.34
20	214.7	0.0039	0.0063	0.0738	11.80

The results reported in Tables I through VI point out the efficiency of the proposed Bayesian approach, evidencing that the Bayes estimate errors, in terms of mean square error, (as measured by the *MSEB* index) are *per se* reasonably limited. Moreover, the relative efficiency (as measured by the *REFF* index) of the MSE with respect to the ML estimate is always larger than 1; in particular:

- examining the results within each Table, the REFF index is much larger for small sample sizes, for which the ML estimates are outperformed by the Bayes ones: this latter feature is particularly useful for the kind of application examined here, where – as discussed before – very few data may be expected;

- examining the results across the various Tables, the REFF index, as expected, becomes much and much larger as the prior CV diminishes (i.e. from Table I to Table III, and from Table IV to Table VI);

- the results relevant to "Case B" (in Tables IV through VI), in which a "wrong" prior pdf of Q is assumed, are very satisfactory and very similar to those in case A (Tables I through III), and in some cases even better.

The last aspect confirms the known "robustness property" of the BSE [6,28], which assures that the influence of the prior PDF - which is based upon expert or subjective judgment - doesn't alter the adequacy of the Bayes estimation in terms of relative efficiency, at least for small departures from the reference prior model. In fact, the robustness of the proposed methodology with respect to the prior distributions has been verified also by means of many other simulations assuming also very different (e.g. Lognormal and Uniform) prior pdf for Q, instead of the hypothetically assumed Inverted Gamma pdf[11].

Also a second kind of robustness analysis, i.e. a "model robustness analysis", has been performed in which it is hypothesized that the basic model distribution so far used as input for the BSE, i.e. the Weibull models of stress, is substituted by a different model, the Lognormal one, which is also widely adopted in power system applications for its great flexibility [31].

In practice, in this case, denoted as "case C", a Lognormal model (with the same values of mean and SD as the former Weibull model) was assumed as the true model generating the stress values, while the computations are performed as if the Weibull were the true distribution. In such analysis the same conjugate IG pdf of Q, with the same parameters values as before illustrated for cases A1and B1 (as explained afterwards) was adopted.

This constitutes a heavy departure from the assumed model, since such prior pdf is no more conjugate for the above Lognormal stress model and, most of all, the assumed equations for the posterior parameters become wrong: nonetheless, the robustness still holds, as shown in Tables VII and VIII.

The results of Tables VII and VIII are referred to the right (E[Q]=1.02) and wrong (E[Q]=1.20) prior pdf respectively, denoted as "C1" and "C2". For sake of brevity, the results are reported only in the most unfavorable case of larger CV, i.e. the same CV

[10]It is remarked that the prior IG parameters depend only on the he mean and CV of Q, i.e. only on prior information(of course), so that they remain the same in any Table; instead, the IG posterior parameters are function on the sample D and also on each sample size n. So, any simulation is characterized by a different couple of posterior parameters.

[11]The results are not shown here for sake of brevity, but are available on request from the authors.

value of Tables I and IV, i.e. cases A1 and B1 ($CV= CV_0= 0.2057$). So, two different sub-cases (C1, C2) are reported in Tables VII and VIII.

- Case C1: Lognormal stress, IG pdf for Q with E[Q]=1.02 , CV= CV0= 0.2057 (Table VII);
- Case C2: Lognormal stress, IG pdf for Q with E[Q]=1.20 , CV= CV0= 0.2057(Table VIII).

They results of Tables VII and VIII show clearly the good performances of the Bayes estimates also in this case. Also in such cases the Bayes estimates achieve sometimes even better results than in the "standard" case of Tables I through VI. The IG parameters of cases C1 and C2 are the same as in cases A1 and B1 respectively, since they depend only on the prior mean and CV of Q.
It is remarked that case C2 examines, to a certain extent, both the above said "model robustness analysis", and also the robustness property with respect to the prior distribution, since a "wrong" prior is assumed for Q. It is so remarkable that results of cases C1 and C2 are comparable.
Although the *MSEB* and the *REFF* index are definitively the most adopted measures of efficiency of the BSE, other significant quantities were evaluated, in order to assess the performances of the estimates, such as the "Relative Average Bias" and "Maximum Relative Error", which measure the "precision " of the BSE [6]. They all showed very satisfactory results, reinforcing the above judgments of adequacy, efficiency and robustness of the proposed methodology.

TABLE VII NUMERICAL RESULTS OF ESTIMATION EFFICIENCY CASE **C1**

sample size	ν	δ	*MSEB*	*MSEL*	*REFF*
1	25.63	0.0398	0.0480	3.3741	70.29
3	25.63	0.0398	0.0572	1.1945	20.88
5	25.63	0.0398	0.0614	0.6528	10.63
15	25.63	0.0398	0.0774	0.2726	3.522
20	25.63	0.0398	0.0777	0.2107	2.712

TABLE VIII NUMERICAL RESULTS OF ESTIMATION EFFICIENCY CASE **C2**

sample size	ν	δ	*MSEB*	*MSEL*	*REFF*
1	25.63	0.0338	0.0644	3.3932	52.69
3	25.63	0.0338	0.0741	1.2127	16.37
5	25.63	0.0338	0.0782	0.6696	8.563
15	25.63	0.0338	0.0953	0.2913	3.057
20	25.63	0.0338	0.0947	0.2272	2.399

Finally, the proposed methodology allows an easy estimation of any parameter of the LL lifetime distribution, such as the percentile of failure time; it is recalled that the $100p^{th}$ -percentile, T_P, (or "quantile p") of a given RV T with cdf F(t) is the value T_P such that $F(T_P) = p$, or $R(T_p) = 1-p$. In other words, the 100p% of the observed value of the RV T should fall, theoretically, in the interval $(0,T_P)$. E.g., the 50th percentile of failure time is the $T_{0.50}$ value such that $P(T>T_{0.50}) =R(T_{0.50})= 0.50$, i.e. it is the "median" value of (35) which is again reported here:

$$\tau = (1/z)^{(1/\beta)} \tag{40}$$

Then, it is easy to show, inverting the relation $R(T_p) = (1-p)$ in which R(t) is the RF of the LL model (equation (2)) - that the $100p^{th}$ percentile, T_P, is simply expressed as a function of τ and $R = 1-p$:

$$T_P = \tau[p/(1-p)]^{1/\beta} = \tau[R/(1-R)]^{1/\beta} \tag{41}$$

So, it is equivalently and easily expressible in terms of the OR, Q=(1/R-1):

$$T_P = \tau/Q^{1/\beta} \tag{42}$$

The assumed Inverted Gamma distribution for Q imply an "Inverted Generalized Gamma" (App. B) distribution on T_P, which is a useful analytical result. This is another practical aspect of the method, since it is a realistic assumption that some prior information on the lifetime percentiles is known (e.g.., the engineer may know that 50% of the components survive after 25 years, or that 90% of the components survive after 15 years, and so on). This fact may have some practical aspect, since often the main target of a reliability estimation procedure is the establishment of adequate maintenance actions to be taken when the reliability value falls below a given threshold. As discussed in [6], this implies that one is in such cases more interested in the mission times corresponding to given reliability values, rather than in the reliability function at a given mission time, $R(t)$.
Summarizing the method, it can be stated that it allows a practical way to assess the prior information on the model to be estimated in an analytically simple and flexible way. Such information can be expressed in terms of the Odds Ratio, or the reliability function or the percentiles, which are all available from experience or experts' opinion, maintaining in each case the same high degrees of efficiency and robustness.

6. Conclusion

The assessment and the estimation of components' life models in the case of high degrees of reliability is made easier by means of the knowledge of the degradation mechanisms, which provide a mathematical model of the physics of the ageing and failure. In almost all applications it is not possible to obtain an accurate deterministic mathematical model of degradation; therefore, a probabilistic approach can be a valuable tool in the estimation of components' reliability. In this framework, the paper studies the Log-logistic model, deduced by a suitable probabilistic model, which has found valid applications to the reliability assessment of the insulation components. The most important reason of the failure of an electrical components is the damage of the insulation. The probabilistic approach to the insulation failure is based upon the use of a Stress-Strength model, which determines the reliability through the comparison between the electrical endurance of the insulation and the voltage surge amplitude applied. This paper develops a new probabilistic approach to the evaluation of insulation reliability obtained by the use of BSE. This procedure allows the possibility to perform efficient estimations of the insulation reliability using only a prior distribution for the reliability and some statistical data on the insulation stress, which are easily obtainable. An important feature of the method is the one of allowing a simple analytical computation of the prior and posterior pdf and their key parameters, in primis the posterior mean, i.e. the Bayes estimate of the OR, while the Bayes estimate of the RF requires simple numerical integration. Using the Monte Carlo simulation, the results of some numerical applications carried by means of the Matlab © software are reported in the final part of paper. A deep analysis of this results shows the analytical feasibility and the efficiency of the proposed method. Finally, also the robustness analysis (i.e. using prior distributions or models different from the ones chosen here) yielded very satisfactory results. Although the proposed method takes its motivations from the reliability modeling of electrical insulation, it is deemed to be a valid tool for any applied reliability modelling characterized by a Log-logistic distribution

Appendix A. The Weibull model

The Weibull model is probably the most popular model in reliability applications – since its birth, in 1939, for application in mechanical engineering (e.g., fatigue life of steel). Its popularity is due to two basic features:
1) its flexibility (e.g. : the Gamma, Normal and the LN models can be often well approximated by a suitable Weibull pdf. The HRF may be increasing, decreasing or constant);

2) the fact that the Weibull belongs (as it was proved in 1945 by Gnedenko) to the family of extreme-values distributions, being able to represent the failure mechanisms of "chain-like" systems that fail when the weakest link is broken [4].
The cdf and pdf of a Weibull RV can be written as follows (in the paper, x denotes a stress value):

$$F(x) = 1 - exp[-(x/\theta)^{\gamma}] \qquad (x > 0) \tag{A.1}$$

$$f(x) = (\gamma/\theta)(x/\theta)^{\gamma-1} exp[-(x/\theta)^{\gamma}] \tag{A.2}$$

In the paper also the alternative parametrization of the above pdf in terms of the parameter: $\eta=\theta^{\gamma}$, is used, so that the Weibull stress pdf is expressed as follows

$$g(x\,|\,\eta) = \frac{\gamma}{\eta} x^{\gamma-1} e^{-\frac{x^{\gamma}}{\eta}} \qquad (x>0) \tag{A.3}$$

Mean and Variance of a Weibull RV are better expressed in the first parametrization form, and given by:

$$E[X] = \mu = \theta\Gamma(1 + 1/\gamma) = \theta\Gamma(1 + 1/\gamma)) \tag{A.4}$$

$$Var[X] = \theta^2\Gamma(1 + 2/\gamma) - \mu^2 \tag{A.5}$$

Appendix B. The Gamma, Inverted Gamma, and some related distributions

The Gamma model [5,28,29] is one of the most popular in applied probability, and is characterized by the following pdf:

$$f(x) = \frac{1}{\delta^{\nu}\Gamma(\nu)} x^{\nu-1} \exp\left(-\frac{x}{\delta}\right) \tag{B.1}$$

The pdf is assumed zero for negative values of x, and the same holds for all the subsequent pdf of the following Appendixes.
$\Gamma(x)$ is the Euler-Gamma special function. Such pdf is denoted as *gampdf*($\xi;\nu,\delta$),where ν and δ, positive constants, are the shape and scale parameter, respectively. Any such RV is referred to in symbols as $G(\nu, \delta)$. The mean value and variance of the distribution of a RV X following the G(ν,δ) model are:

$$E[X] = \nu\delta; \quad Var[X] = \nu\delta^2 \tag{B.2}$$

The cdf is expressed through the incomplete Gamma function $\Gamma(x, y)$:

$$F(x;\, \nu, \delta) = \Gamma(\nu, \delta x)/\Gamma(\nu) \tag{B.3}$$

Some hints also at three kinds of "transformed" Gamma RV, i.e. the "*Inverted Gamma*", "*Generalized Gamma*" and *"Inverted Generalized Gamma*" models [5,23,24] are given here. Let X be a Gamma $G(\nu, \delta)$ RV, then:

$$Y = 1/X \tag{B.4}$$

is a so-called "*Inverted Gamma*" RV.
By ordinary rules on RV transformations [27], it is well known that the pdf of Y can be expressed, for y>0, by:

$$f_Y(y) = (1/y^2)\ gampdf(1/y; \nu, \delta) \tag{B.5}$$

So, the IG model has the following pdf, with argument y:

$$f(y) = \frac{1}{y^{(\nu+1)}\delta^{\nu}\Gamma(\nu)} \exp\left(-\frac{1}{\delta y}\right) \tag{B.6}$$

The expected value (or "mean") and variance of the distribution are finite only if $\nu>1$ (mean) and $\nu>2$ (variance). They are:

$$E[Y] = 1/[\delta(\nu - 1)] \tag{B.7}$$

$$Var[X] = 1/[\delta^2(\nu - 1)(\nu - 2)] \tag{B.8}$$

Then, being X a $G(\nu, \delta)$ RV, and letting χ be a positive parameter, and defining $\beta=1/\chi$, the RV T defined as:

$$T = X^{\chi} = X^{1/\beta} \tag{B.9}$$

possesses a so called "*Generalized Gamma*" pdf. It's not difficult to realize that the pdf of T is symbolically expressed for $t > 0$ by:

$$f_T(t) = \beta t^{(\beta-1)} gampdf(t^{\beta}; \nu, \delta) \tag{B.10}$$

So, the above model has the following pdf, with argument t and three parameters (ν,δ,β).

$$f(t;\nu,\delta,\beta) = \frac{\beta t^{(\nu\beta-1)}}{\delta^{\nu}\Gamma(\nu)} \exp(-t^{\beta}/\delta) \tag{B.11}$$

Such pdf is denoted by *gengampdf*(t*;* ν, δ ,β). Finally, the RV defined by the transformation: W=1/T, where T is the *above Generalized Gamma RV*, possesses a so called *"Inverted Generalized Gamma"* pdf, which can be expressed by the above recalled transformation rules as follows:

$$f_W(w) = (1/w^2) gengampdf(1/w; \nu, \delta\,; \beta) \tag{B.12}$$

Appendix C. The Inverted Gamma pdf as a "conjugate" pdf for the Weibull parameter η

Let the scale parameter η be the only unknown in a Weibull random sample $\underline{X} = (x_1, x_2..x_n)$ where all the values in the sample are realizations of statistically independent and identically distributed RV from of a Weibull model, with common pdf:

$$f(x\,|\,\eta) = \frac{\gamma}{\eta} x^{\gamma-1} e^{-\frac{x^{\gamma}}{\eta}} \qquad (x>0) \tag{C.1}$$

For the purpose of estimatingη, let us consider the Likelihood function of the sample, which is the joint pdf of the above n RV, conditional toη, so being expressed by:

$$L(x_1, x_2..x_n\,|\,\eta) = \prod_{i=1}^{n} f_i(x\,|\,\eta) \text{ , i.e.:}$$

$$L(x,\eta) = \left(\frac{\gamma}{\eta}\right)^n \prod_i x_i^{\gamma} e^{-\frac{\sum_i x_i^{\gamma}}{\eta}} \tag{C.2}$$

Let the prior pdf on η be the Inverted Gamma model, with the following pdf, with argument η:

$$p(\eta) = \frac{1}{\eta^{(\nu+1)}\delta^{\nu}\Gamma(\nu)} \exp\left(-\frac{1}{\delta\eta}\right) \tag{C.3}$$

Then, by multiplying $p(\eta)$ by the Likelihood function, it is easy to verify that the a posteriori pdf of η is again an IG pdf, with updated parameter values, as specified in the main text.
The ML estimator of η is instead evaluated through maximization of (C.2) with respect toη.The following result is obtained:

$$\eta^* = U/n \tag{C.4}$$

where:

$$U = U(D) = \sum_{j=1}^{n} X_j^{\gamma}$$

Then, by virtue of (32) and know properties of MLE [27,28], the MLE of Q is:

$$Q^* = \eta^* t_0^{\beta} \tag{C.5}$$

List of main acronyms[12]

BE	Bayes estimate
BSE	Bayes Statical Estimation
cdf	Cumulative distribution function
CV	Coefficient of variation
$D[Y]$	Standard deviation of the RV Y
$E[Y]$	Expectation of the RV Y
$f(x), F(x)$	pdf and cdf of Stress
$g(y), G(y)$	pdf and cdf of Strength
HRF	Hazard rate function
IG	Inverted Gamma (distribution)
LL	Log-logistic (distribution)
LN	Lognormal (distribution)
ML	Maximum Likelihood
MSEB	Mean Square Error of the Bayes estimator
MSEL	Mean Square Error of the ML estimator
OR	Odds Ratio
pdf	Probability density function
REFF	Relative efficiency of the Bayes estimator
RF	Reliability function
R(t)	Reliability function at mission time t
RV	Random variable
SD, σ	Standard deviation
Var, σ^2	Variance
X	Stress RV
Y	Strength RV
$\Gamma(\,)$	Euler-Gamma function
μ	Mean value (Expectation)
ζ°	Bayes estimate of parameter ζ

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[12] Remarks: - the singular and plural of names are always spelled the same; - "log" always denotes natural logarithm; - random variables are denoted by uppercase letters.