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Research paper

A method for creating twelve-node finite element meshes to find the cutoff wave number for polygonal and circular waveguides

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Abstract

The numerical solution of the Helmholtz equation-driven electromagnetic waveguide eigenvalue issue is presented using the finite element method. This work utilized a 2D automated 12-noded mesh generator, run with Maple 13, to produce a highly efficient, straightforward, and accurate higher-order technique for the current work. A transcendence automated discretization is constructed. Meshes with quadrilateral elements are used for wave-guiding structures that are square, L-shaped, and unit-circular regions, but this explanation of the finite element approach is sufficient for the purposes at hand. In any numerical simulation that uses the finite element approach, meshing procedures are extremely important, the approach is shown for several waveguide configurations, and the results are compared to the most reliable numerical or analytical results. Since there is no curvature loss, the results demonstrate that the proposed methodology is precise and effective for producing finite element models of complex structures, this article provides a cutoff frequency determination using the Maple program and commercial software analysis results are taken into account for the comparison, demonstrating that the computation results for electromagnetic applications, this process can be used to obtain the most efficient energy transmission.

Keywords: *Cut Off Frequency; Helmholtz Equation; 12-Noded Mesh; Wave-Guide.*

1. Introduction

In the realm of wireless technology for next-generation applications, waveguides are a crucial component. Compared to other transmission methods, a waveguide offers higher transmission efficiency from a magnetron to a microwave oven chamber. The characteristics of the waveguide are frequently employed as high-power transmission elements to provide an antenna for propagation (Cho et al., 2001; Bernabeu et al., 2017; Shivaram et al., 2018; Rasekhmanesh et al., 2022). Currently, wireless power transmission and communications both make substantial use of radio waves. Since there is now no single direct empirical technique accessible. Numerical approaches are crucial for understanding the propagation of these waves. Numerous methods have been used for numerical eigen-analysis in recent years, including the boundary element method (BEM) (Sarkar, et al., 2002; Kouroublakis et al., 2023) and the finite element method (FEM) (Wang, 2016; Cho et al., 2002). to determine the cutoff wave numbers of conducting wave guides with an arbitrary cross section by using an iterative process known as Muller's technique to solve the integral equation (Swaminathan et al., 1990). other waveguide analysis techniques have also been presented recently. For example, a surface integral equation method was tested to ascertain the cut-off wave numbers of TE and TM modes using the most popular waveguide types by super elements were also created using the FEM for Laplacian eigenvalue computations across domains and these were then used to calculate the L-shaped, Square and unit circular waveguides in TE and TM modes (Kai Wang et al., 2022), addressing waveguide eigenvalue issues using the extended finite difference approach in both eccentric circular waveguides and conventional waveguides in various modes (Xu et al., 2022) and the Helmholtz general equation was solved using a boundary integral method (Shivaram et al., 2019) out of an analytical regularization approach for coaxial rectangular waveguides, double ridge waveguides and arbitrary polygonal cross-section waveguides.

Combining boundary elements and finite elements (BE-FE) is a technique that was developed to numerically investigate fluid-structure interaction issues (Reddy, 2005). Most recently, the two-dimensional Helmholtz equation was numerically solved using a cubic order sub-parametric finite element method to determine the eigenvalues of arbitrarily formed waveguide devices, for computers. Waveguides with an arbitrarily treated cross-section have proven to be a challenging problem. There are now many methods for dealing with this problem in the literature. Despite the fact that there are numerous methods available today. They are not sufficiently efficient in terms of simplicity and speed for the simulation of cross-section waveguides, according to the literature on this subject. In many scientific applications, mesh auto-generation and the FEM are commonly utilized as research and CAD tools. MATLAB was used to automatically generate meshes for the numerical solution of the Helmholtz problem (Berardisensale et al., 2008) using a superior sub-parametric FE code. To facilitate the modeling of subsequent calculations for a range of applications. We seek to develop meshes that accurately relate the geometric inputs containing high-quality finite elements in a variety of fields, including FEM, image processing, computer graphics, cartography, video processing, computational materials science, computational electromagnetics, photogrammetric fields, medical simulation, material engineering, etc. The output of the suggested method can be applied to more effectively solve a variety of problems using computational fluid dynamics and FEM. Sub-parametric transformation combined with higher-order FEM helps reduce calculation time and increase solution accuracy. But the solution isn't quick or easy to use. Following the discretization of the L-shaped, square, and circular waveguides into a family of 12-noded quadrilaterals, the best numerical integration technique is used for these quadrilaterals, when 12-noded quadrilaterals in the regions is increased and compared to the ones that currently exist.

2. Constructing a quadrilateral mesh with twelve nodes

Quadrilateral elements are commonly used in 2D simulations. Physical phenomena or structures are broken down into more manageable components to speed up numerical calculations. Because of their computing efficiency and simplicity, in many applications, creating a mesh is a necessary initial step, including finite element modelling or finite element analysis. The term "12-noded" refers to an element that has twelve nodes, or vertices, per quadrilateral, which is defined in this context as a four-sided polygonal shape. Typically, the "12-noded" quadrilateral element consists of four nodes at each corner and eight additional nodes are the points on the line joining the edges. This type of finite element mesh is used in numerical simulations and finite element analysis, it is feasible to more accurately describe the geometry and deformation within the element when comparing this node distribution to lower-order elements. The process of creating a 12-noded quadrilateral mesh, which is widely used in numerical simulations for structural and mechanical analysis, involves discretizing a physical domain into quadrilateral elements, each with twelve nodes.

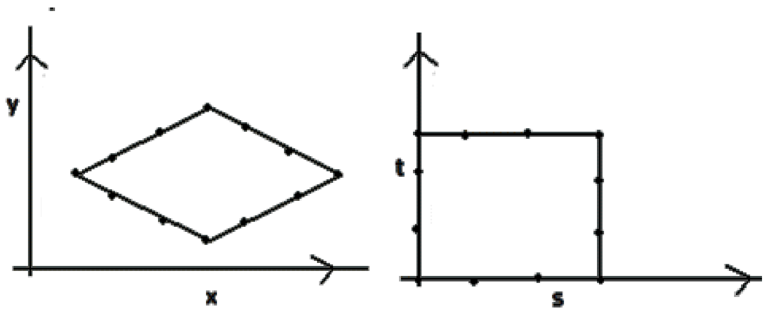

Fig. 1: 12-Noded Rectangular Element Mapped to 12-Noded Standard Square Element

A set of nodal points is frequently used in finite element analysis to construct a quadrilateral element. After that, the geometry within the element is interpolated using the values assigned to these nodal points. The values assigned to the twelve nodal points and the values of s and t, all of which change from -1 to 1, are used to calculate the fluctuations of the physical quantities within the element using interpolation functions, also referred to as form functions.
The following shape functions are commonly provided for a 12-noded quadrilateral element

$$n_1 = \frac{1}{32}(1-s)(1-t)(-10+9(s^2+t^2))$$

$$n_2 = \frac{9}{32}(1-t)(1-3s)(1-s^2)$$

$$n_3 = \frac{9}{32}(1-t)(1+3s)(1-s^2)$$

$$n_4 = \frac{1}{32}(1+s)(1-t)(-10+9(s^2+t^2))$$

$$n_5 = \frac{9}{32}(1+s)(1-3t)(1-t^2)$$

$$n_6 = \frac{9}{32}(1+s)(1+3t)(1-t^2)$$

$$n_7 = \frac{1}{32}(1+s)(1+t)(-10+9(s^2+t^2))$$

$$n_8 = \frac{9}{32}(1+t)(1+3s)(1-s^2)$$

$$n_9 = \frac{9}{32}(1+t)(1-3s)(1-s^2)$$

$$n_{10} = \frac{1}{32}(1-s)(1+t)\left(-10+9(s^2+t^2)\right)$$

$$n_{11} = \frac{9}{32}(1-s)(1+3t)\left(1-t^2\right)$$

$$n_{12} = \frac{9}{32}(1-s)(1-3t)\left(1-t^2\right) \quad (1)$$

$$J = \frac{\partial(x,y)}{\partial(s,t)} = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}$$

Where

$$\frac{\partial x}{\partial s} = \sum_{r=1}^{12} x_r \frac{\partial n_r}{\partial s} \ , \ \frac{\partial x}{\partial t} = \sum_{r=1}^{12} x_r \frac{\partial n_r}{\partial t}$$

And

$$\frac{\partial y}{\partial s} = \sum_{r=1}^{12} y_r \frac{\partial n_r}{\partial s} , \ \frac{\partial y}{\partial t} = \sum_{r=1}^{12} y_r \frac{\partial n_r}{\partial t} \quad (2)$$

3. 12-noded Mesh Generation in polygonal and circular regions

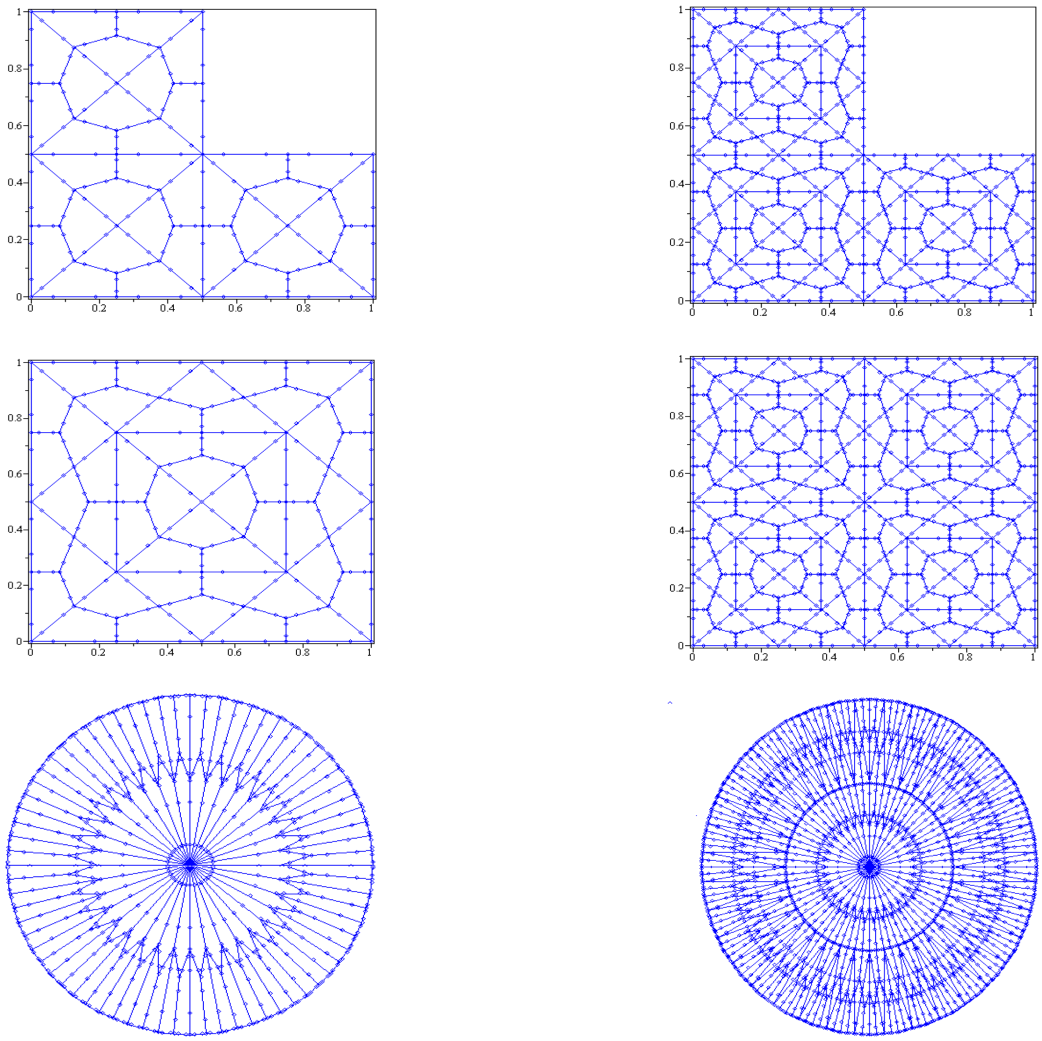

Fig. 2: L, Square and Circular Wave Guide Discretized Into 12-Noded Quadrilateral Meshes.

A kind of mesh that uses quadrilateral elements, each defined by twelve nodes, is called a 12-noded quadrilateral mesh. Finite element analysis frequently uses this kind of mesh to solve a variety of scientific and engineering challenges. The following are the essential features and procedures for creating a 12-noded quadrilateral mesh.

- Every element consists of twelve nodes and four sides.

- Higher precision is achieved because the form functions interpolated within the element are more sophisticated than those of lower-order elements.
- Determine and specify the domain's geometry, which calls for the mesh. This could be an intricate shape derived from experimental data or determined by mathematical formulas.
- Within the domain, create nodes. The discrete points where an approximation of the numerical solution will be made are represented by these nodes. To precisely capture the geometry, make sure the nodes are distributed properly.
- Describe the nodes' connectedness to create quadrilateral elements. Each of the twelve nodes that make up a quadrilateral element will be defined by a set of twelve nodes, the node order is essential for accurate interpolation.
- Analyze the created mesh's quality by looking at metrics like skewness, aspect ratios, and other quality indicators. A high-quality mesh enhances the numerical solution's stability and accuracy.
- Given the nodes on the domain border, apply boundary conditions to them. When tackling issues with clearly defined boundary restrictions, this phase is fundamental.
- Use the proper solvers or simulation programs to integrate the mesh if it is going to be used for solving PDE or other physical simulations.

4. Finite element formulation procedure for solving the Helmholtz equation

Helmholtz equation gives the classic partial differential equation for figuring out the cutoff frequency of an electromagnetic wave traveling through a waveguide. The Helmholtz equation's mathematical representation is given by

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\omega_c^2 u=0 \qquad (3)$$

Where u is the scalar potential and ω_c^2 is the unknown cut-off frequency, the problem in MAPLE-13 is resolved using the finite element method with a family of 12-noded quadrilateral elements; at the boundary, the wave amplitude for the TM modes is zero, while for the TE modes, the normal derivative is zero through the use of the following procedure.

- First, cover the two-dimensional waveguide structure with a 12-noded quadrilateral mesh using the automatic mesh generators.
- The Galerkin weighted residual finite element technique must be used to express the element geometry in terms of the Lagrange shape function to derive the finite element equation.

$$[K+Q\,]_{12X12} * U_{12X1} = [0]_{12X1} \qquad (4)$$

$$K_{i,j} = \iint_\Omega \left(\frac{\partial n_i}{\partial u}\frac{\partial n_j}{\partial u}+\frac{\partial n_i}{\partial v}\frac{\partial n_j}{\partial v}\right)dxdy = K_{u,u}+K_{v,v} \qquad (5)$$

$$K_{u,u} = \int_{-1}^{1}\int_{-1}^{1}\left(\frac{\partial v}{\partial \eta}\frac{\partial n_i}{\partial \xi}+\frac{\partial v}{\partial \xi}\frac{\partial n_i}{\partial \eta}\right) * \left(\frac{\partial v}{\partial \eta}\frac{\partial n_j}{\partial \xi}+\frac{\partial v}{\partial \xi}\frac{\partial n_j}{\partial \eta}\right)\frac{1}{J}\,d\xi d\eta$$

$$K_{v,v} = \int_{-1}^{1}\int_{-1}^{1}\left(\frac{\partial u}{\partial \eta}\frac{\partial n_i}{\partial \xi}+\frac{\partial u}{\partial \xi}\frac{\partial n_i}{\partial \eta}\right) * \left(\frac{\partial u}{\partial \eta}\frac{\partial n_j}{\partial \xi}+\frac{\partial u}{\partial \xi}\frac{\partial n_j}{\partial \eta}\right)\frac{1}{J}\,d\eta d\xi$$

$$Q_{i,j} = \iint_\Omega \omega_c^2\, n_i\, n_j\, dxdy = \int_{-1}^{1}\int_{-1}^{1} \omega_c^2\, n_i(\xi,\eta)\, n_j(\xi,\eta)\, J d\eta d\xi \qquad (6)$$

Assemble the element equations so that, depending on the global node numbering, the impacts of each element are considered for the entire region to obtain the global matrix equation.

$$\{K+Q\}_{N_P X N_P} * U_{N_P X1} = \{0\}_{N_P X1} \qquad (7)$$

Where Np is the total number of nodes. This turns Eq. (3) into an eigenvalue problem, from which the wave numbers. ω_C are obtained by applying the formula

$$\omega_C = \sqrt{\text{eigen value}}$$

Equation (7) produces ω_C, Which represents the TE modes' capacity to assess by reducing the Helmholtz equation to an algebraic equation's eigenvalue problem by applying boundary conditions to identify the TM mode.

$$\{K+Q\}_{mXm} * U_{mX1} = \{0\}_{mX1} \qquad (8)$$

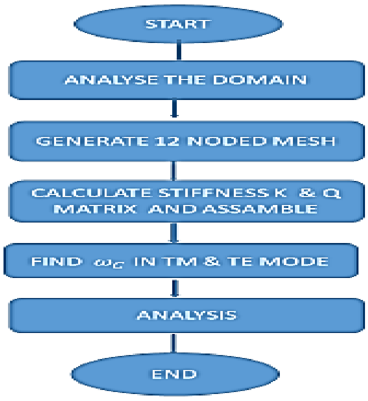

Fig. 3: Flow Chart for Finite Element Process.

To get the TM mode, compute the eigenvalues; to find the cutoff wave number, find the minimum wave number that may be obtained; This method makes use of a program designed for the FEM methodology, an efficient 12-noded mesh generator, and the quadrature method. Decrease computational time and common errors in the FEM analysis of the problems that arise with standard FEM procedure and decrease numerical and discretization of error in the FEM equation solution. As a result, numerous energy engineering applications can be resolved using the suggested method, including microwave professionals, with an accurate and efficient numerical solution.

5. Numerical examples

5.1. Waveguides in the shapes of an L-shaped, a square, and a circular region

Regarding electromagnetic wave propagation in long wave guides, the Helmholtz equation can be reduced to Maxwell's equations, which are PDEs that control electromagnetic radiation. This equation provides the cutoff wave numbers for various waveguide setups; these numbers have been derived using a range of methods documented in the literature. This method is used to resolve a variety of electromagnetic issues using any wave guide and normalizes CPU time for every analysis, The suggested method for the L, square, and circular-shaped wave guides is used in the MAPLE program to calculate the first four cutoff wave numbers. It is an efficient solution to the problem, then contrast the outcomes. 12-noded mesh using the current mesh-less technique, analyze the waveguide structure reported in references (Swaminathan et al., 1990) and (Kai Wang, 2022). Accordingly, Figure 2 shows the structured 12-noded quadrilateral mesh for the L, square and circular-shaped wave guides. Table 1-3 shows the computed cutoff wave numbers for the transverse and longitudinal modes in the L, square and circular-shaped wave guide, utilizing the 12-noded quadratic components that comprise the suggested structured automated mesh generator, one of this structure's sharp edges has a singularity, the numerical findings in Tables 1-3 demonstrate that 12-noded quadratic components outperform 3-noded triangle components for electromagnetic issues. The 12-noded automatic meshing approach considerably reduces the computing time. The proposed finite element methodology offers the most efficient and simple way to compute eigenvalues. In microwave-based applications, it is also among the most effective techniques for transmitting energy efficiently, since it provides a useful way to determine the TE and TM modes of any type of waveguide arrangement with minimal energy loss. As such, this methodology can be applied with efficacy to various energy, electromagnetic, and microwave-related issues.

Table 1: Cutoff Wavenumbers for the TM and TE Modes Over an L-Shaped Waveguide Structure

Computed in [Swaminathan]		Current approach				Current approach			
		Total number of 12-noded elements =36 CPU Time = 1				Total number of 12-noded elements =144 CPU Time = 1.3			
TM	TE	TM	Diff. %	TE	Diff.%	TM	Diff. %	TE	Diff. %
4.891	1.913	4.5255	0.07473	1.9089	0.0021	4.8835	0.00153	1.9150	0.0010
6.139	2.961	6.1043	0.00565	2.9581	0.0009	6.1201	0.00308	2.9642	0.0010
6.997	4.945	6.9310	0.00943	4.9376	0.0015	6.9754	0.00309	4.9362	0.0017
8.557	5.315	8.4472	0.01283	5.2933	0.0040	8.5604	0.00039	5.3095	0.0010

Table 2: Cutoff Wavenumbers for the TM and TE Modes Over A Square-Shaped Waveguide Structure

Computed in [Swaminathan]		Current approach				Current approach			
		Total number of 12-noded elements =48 CPU Time = 1.2				Total number of 12-noded elements =192 CPU Time = 1.5			
TM	TE	TM	Diff. %	TE	Diff. %	TM	Diff. %	TE	Diff. %
2.221	1.571	2.2132	0.0035	1.4089	0.1031	2.2254	0.0019	1.5655	0.0035
3.512	2.221	3.4904	0.0061	2.2054	0.0070	3.5033	0.0024	2.2201	0.0004
4.442	3.142	4.4033	0.0087	3.1390	0.0009	4.4364	0.0012	3.1335	0.0027
4.967	3.512	4.8905	0.0154	3.5306	0.0052	4.9671	0.0002	3.5653	0.0151

Table 3: Cutoff Wavenumbers for the TM and TE Modes Over A Circular-Shaped Waveguide Structure

Computed in [Kai Wang]		Current approach				Current approach			
		Total number of 12-noded elements =672 CPU Time = 1.3				Total number of 12-noded elements =1152 CPU Time = 2.0			
TM	TE	TM	Diff. %	TE	Diff.%	TM	Diff.%	TE	Diff.%
2.4073	1.8412	2.40057	0.0028	1.83039	0.0058	2.4065	0.0003	1.8416	0.0002
3.8317	3.0542	3.82991	0.0004	3.05115	0.0010	3.8309	0.0002	3.0540	0.0000
5.1356	3.8317	5.12088	0.0028	3.83779	0.0015	5.1348	0.0001	3.8316	0.0000
5.5200	4.2088	4.21550	0.2360	4.20712	0.0004	5.5192	0.0001	4.2088	0.0000

6. Conclusions

An automated 12-noded mesh generator in conjunction with the FEM technique was proposed in this study as a straightforward, effective, and precise numerical solution process for microwave applications. Table 1-3, which shows the numerical results for three distinct waveguides used to compute the eigenvalues and, consequently, the TE and TM modes, illustrates the applicability of the suggested method. The suggested method is easy to implement and makes effective use of computer resources. It offers the most precise approximation for figuring out the dominant TM and TE modes' cutoff wavenumbers for a range of waveguide topologies and geometries with singularities. This means that a variety of energy applications, including microwave applications, can successfully use this technology.

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